A CHARACTERISATION OF THE ELLIPSOID

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To Professor Bernhard H. Neumann on his 60th birthday

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1. Introduction

The ellipsoid is characterised among all convex bodies in n-dimensional Euclidean space, R^n, by many different properties. In this paper we give a characterisation which generalizes a number of previous results mentioned in [2], p. 142. The major result will be used, in a paper yet to be published, to prove some results concerning generalizations of the Minkowski theory of reduction of positive definite quadratic forms.

An *n-dimensional convex body K in* R^n for $n \geqq 3$ is a closed, bounded, and convex subset of R^n which contains exactly n linearly independent points. We call the sets of boundary and interior points of K respectively the *frontier* and *interior of K*. The notation 'hK' means the dilation of K about the origin in the ratio h. A *tac plane of K* at a point of the frontier (also known as a *support plane*) is a hyperplane in R^n whose intersection with K includes that point, but no interior points of K. These concepts are discussed in detail in [2]. We denote points and vectors in R^n as $\boldsymbol{a}$, $\boldsymbol{b}_1$, etcetera, and in particular $\boldsymbol{O}$ is the origin, $\boldsymbol{u}$ always denotes a unit vector, and $\boldsymbol{ab}$ is the chord joining $\boldsymbol{a}$ and $\boldsymbol{b}$.

There are exactly two distinct tac planes of K perpendicular to a vector $\boldsymbol{u}$ and we call the distance between them, the *width of K in the direction* $\boldsymbol{u}$. A concept of major importance in this paper is that of equivalence of convex bodies. Two n-dimensional convex bodies K_1 and K_2 are called *equivalent* if the ratio of the width of K_1 in the direction $\boldsymbol{u}$ to the width of K_2 in the direction $\boldsymbol{u}$ is a constant j as $\boldsymbol{u}$ varies in R^n. This is written in symbols $K_1 \sim K_2$, or more explicitly $K_1 \sim jK_2$.

The major result of this paper is now described (Theorem 1 and Corollary 2). *Let K be an n-dimensional convex body in* R^n *where* $n \geqq 3$. *Suppose that any two parallel hyperplanes 'sufficiently close' to a tac plane of K intersect K in equivalent convex bodies. Then K is an ellipsoid.* Examples of equivalent convex bodies are two bodies of constant width, or two homothetic convex bodies.

2. Equivalent convex bodies

Convex bodies can be discussed very conveniently in terms of their tac func-

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tions. The *tac function* $H : R^n \to R$ of a convex body K is defined by

$$H(\boldsymbol{v}) = \sup \cdot \boldsymbol{v} \cdot \boldsymbol{x}, \text{ where } \boldsymbol{x} \in K.$$

The properties of this function are discussed in detail in [2] under the name 'Stützfunktion'. We define the *outer normal* $\boldsymbol{u}$ of a tac plane U of K as the unit vector perpendicular to U so that $\boldsymbol{u}$ is directed away from K when $\boldsymbol{u}$ is translated to a point of contact of U with K. It easily follows that if $\boldsymbol{O}$ is in the interior of K then $H(\boldsymbol{u})$ is the distance from $\boldsymbol{O}$ to the tac plane U with outer normal $\boldsymbol{u}$, and so $H(\boldsymbol{u})+H(-\boldsymbol{u})$ is the width of K in the direction $\boldsymbol{u}$. From now on we assume $\boldsymbol{O}$ is in the interior of K.

If K_1 and K_2 are convex bodies with tac functions H_1 and H_2 then we now see that $K_1 \sim jK_2$ *if, and only if,*

$$H_1(\boldsymbol{u})+H_1(-\boldsymbol{u}) = j(H_2(\boldsymbol{u})+H_2(-\boldsymbol{u}))$$

for each $\boldsymbol{u} \in R^n$. In Lemma 1 we give another 'characterisation' of equivalence. The result of Lemma 1 is probably well-known but no reference is available, though the 'only if' part is in [6], p. 293 for example. It is easily generalized to R^n. A tac plane of K is called *regular* if it has only a single point in common with K.

LEMMA 1. *Let K_1 and K_2 be two 2-dimensional convex bodies with tac functions H_1 and H_2. Let U_1 and V_1 be tac planes of K_1 at $\boldsymbol{a}_1$ and $\boldsymbol{b}_1$ both perpendicular to $\boldsymbol{v}$ and U_2 and V_2 similarly for K_2 at $\boldsymbol{a}_2$ and $\boldsymbol{b}_2$ as shown in Diagram 1. Then for some j we have $K_1 \sim jK_2$ if, and only if, $\boldsymbol{a}_1\boldsymbol{b}_1$ is parallel to $\boldsymbol{a}_2\boldsymbol{b}_2$ for any choice of $\boldsymbol{v}$ for which U_1, V_1, U_2, and V_2 are all regular (in these cases $\boldsymbol{a}_1$, $\boldsymbol{b}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{b}_2$ are uniquely defined).*

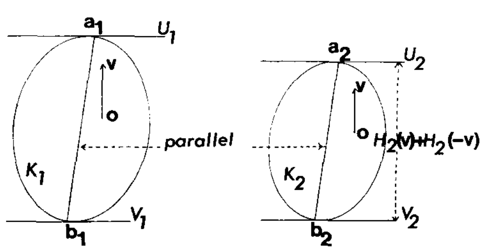

Diagram 1

PROOF. The tac planes U_1, V_1, U_2, and V_2 must all be regular for all $\boldsymbol{v}$ in R^2 except possibly a set of measure zero (on this see [2], section 9). We assume in the following that $\boldsymbol{v}$ is chosen so that the four tac planes are regular. D_i denotes the partial derivative with respect to the i'th variable.

If we denote the components of $\boldsymbol{a}_1$ by a_{1i}, for $i = 1, 2$, and similarly for $\boldsymbol{a}_2$, $\boldsymbol{b}_1$, and $\boldsymbol{b}_2$, then according to [2], p. 24, 26 the partial derivatives of H_1 and H_2 exist at $\boldsymbol{v}$, and for $i = 1, 2$ and $j = 1, 2$,

$$D_iH_j(\boldsymbol{v}) = a_{ji} \text{ and } D_iH_j(-\boldsymbol{v}) = b_{ji}. \tag{1}$$

It is clear that the width of K_j in the direction $\boldsymbol{v}$, for $j = 1, 2$, (H_j is homogeneous of degree one) is

(2) $$(\boldsymbol{a}_j - \boldsymbol{b}_j) \cdot \boldsymbol{v} = H_j(\boldsymbol{v}) + H_j(-\boldsymbol{v}).$$

Now let us assume that $\boldsymbol{a}_1 \boldsymbol{b}_1$ is parallel to $\boldsymbol{a}_2 \boldsymbol{b}_2$ whenever U_1, V_1, U_2, and V_2 are all regular. Then for the regular cases there is a function J so that

(3) $$(\boldsymbol{a}_1 - \boldsymbol{b}_1) = J(\boldsymbol{v})(\boldsymbol{a}_2 - \boldsymbol{b}_2)$$

or in components, for $i = 1, 2$,

$$(a_{1i} - b_{1i}) = J(\boldsymbol{v})(a_{2i} - b_{2i}).$$

By (1) this becomes

(4) $$D_i(H_1(\boldsymbol{v}) - H_1(-\boldsymbol{v})) = J(\boldsymbol{v}) D_i(H_2(\boldsymbol{v}) - H_2(-\boldsymbol{v})).$$

If we take the scalar product of both sides of equation (3) by $\boldsymbol{v}$ and use (2), then we find

(5) $$H_1(\boldsymbol{v}) + H_1(-\boldsymbol{v}) = J(\boldsymbol{v})(H_2(\boldsymbol{v}) + H_2(-\boldsymbol{v})).$$

This shows that J is differentiable when $\boldsymbol{v} \neq \boldsymbol{O}$ and if we take the i'th partial derivative of both sides of equation (5) then we find using (4) that

(6) $$D_i J(\boldsymbol{v}) = 0 \text{ for } i = 1, 2.$$

This result holds for all $\boldsymbol{v}$ except for a set of measure zero. Consider J to be defined for all $\boldsymbol{v} \neq \boldsymbol{O}$ by (5). A simple analysis argument using (6) and the absolute continuity of H_1 and H_2 shows that J is a constant j on any set of points of R^n which excludes a neighbourhood of $\boldsymbol{O}$ (the one-dimensional case of this result is well known and is given in [7], p. 90 for example). Hence we deduce from (5) that $K_1 \sim jK_2$ as required.

On the other hand, we now assume that $K_1 \sim jK_2$ for some constant $j > 0$, so that

(7) $$H_1(\boldsymbol{v}) + H_1(-\boldsymbol{v}) = j(H_2(\boldsymbol{v}) + H_2(-\boldsymbol{v})).$$

In the regular cases, the partial derivatives of H_1 and H_2 exist according to [2], p. 24, 26. Hence the differentiation of both sides of (7), together with (1), gives, for $i = 1, 2$,

$$a_{1i} - b_{1i} = j(a_{2i} - b_{2i}).$$

Therefore

$$\boldsymbol{a}_1 - \boldsymbol{b}_1 = j(\boldsymbol{a}_2 - \boldsymbol{b}_2),$$

so that $\boldsymbol{a}_1 \boldsymbol{b}_1$ is parallel to $\boldsymbol{a}_2 \boldsymbol{b}_2$. This completes the proof.

One further simple result on equivalence is needed.

LEMMA 2. *Let K_1 and K_2 be two 2-dimensional convex bodies with $K_1 \sim jK_2$. For $\boldsymbol{u} \in R^2$, let the total lengths of all straight segments parallel to $\boldsymbol{u}$ on the bounda-*

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ries of K_1 and K_2 be h_1 and h_2 respectively. Then $h_1 = jh_2$. In particular, if both tac planes of K_1 parallel to $\boldsymbol{u}$ are regular then so are the corresponding tac planes of K_2.

The result is well-known and follows from Lemma 1 and a result in [2], p. 31.

3. A new characterisation of the ellipsoid

We first state an important characterisation of the ellipsoid due to Blaschke, in Lemma 3. A proof is given of the n-dimensional generalization in [3], p. 93.

LEMMA 3. *Let K be a 3-dimensional convex body. Let $T(\boldsymbol{u})$ be the union set of all lines parallel to $\boldsymbol{u}$ which intersect K but not its interior. If the set*

$$C(\boldsymbol{u}) = T(\boldsymbol{u}) \cap K$$

lies in a plane for each $\boldsymbol{u} \in R^3$ then K is an ellipsoid.

We can now state the major result of this paper. Further results on ellipsoids are given in Corollaries 2 and 3 in the next section.

THEOREM 1. *Let K be a 3-dimensional convex body in R^3 with $\boldsymbol{O}$ in its interior and with all its tac planes regular. Let h be a constant with $0 < h < 1$. Suppose that the 2-dimensional convex bodies $W_1 \cap K$ and $W_2 \cap K$ are equivalent for all pairs of parallel planes W_1 and W_2 both on the same side of $\boldsymbol{O}$, provided W_1 and W_2 intersect the interior of K but not the interior of hK. Then K is an ellipsoid.*

The condition that $\boldsymbol{O}$ is in the interior of K is not really necessary, but it is convenient for the statement and proof of the theorem. The condition 'W_1 and W_2 do not intersect the interior of hK' is only used to show that we need only consider such planes which are 'close to the boundary' of K. We might have said instead that 'there is an $\varepsilon > 0$ so that W_1 and W_2 are within a perpendicular distance ε of a tac plane of K which is parallel to W_1 and W_2.

The following simple Corollary can be seen immediately and it incorporates some previous results in [2], p. 142.

COROLLARY 1. *Let K be a 3-dimensional convex body. If $W_1 \cap K$ and $W_2 \cap K$ are equivalent whenever W_1 and W_2 are two parallel planes intersecting the interior of K, then K is an ellipsoid.*

The proof of Theorem 1 is long and it is given in the series of Lemmas 4 – 10 to each of which the hypothesis of the Theorem applies. The aim of the proof is to show that the set $C(\boldsymbol{u})$, defined in Lemmas 3, lies in a plane, so that Lemma 3 may be applied to show K is an ellipsoid.

LEMMA 4. *Let $C(\boldsymbol{u})$ and $T(\boldsymbol{u})$ be defined as in Lemma 3.*

(a) *Any plane W parallel to the vector $\boldsymbol{u}$ which intersects the interior of K will intersect $C(\boldsymbol{u})$ in two distinct points.*

(b) *$C(\boldsymbol{u})$ is a simple closed curve in R^3.*

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PROOF. (a). $T(\boldsymbol{u})$ is a convex cylinder, and W clearly intersects $T(\boldsymbol{u})$ in two distinct generators of $T(\boldsymbol{u})$. These two generators must lie in distinct tac planes of K, and because of the regularity of the tac planes they each have a single point in common with K and so with $C(\boldsymbol{u})$.

(b) If Q is the line through O parallel to the vector $\boldsymbol{u}$ then from the proof of (a) it follows that any half-plane bounded by Q has a single point in common with $C(\boldsymbol{u})$. Choose any one such half-plane and designate it $V(O)$ and let $V(s)$ designate the half-plane making an angle s with $V(O)$, for $0 \leqq s \leqq 2\pi$. Let $\boldsymbol{a}(s)$ be the unique point of $C(\boldsymbol{u})$ in $V(s)$. Now

$$C(\boldsymbol{u}) = \{\boldsymbol{a}(s) | 0 \leqq s < 2\pi\},$$

while $\boldsymbol{a}(O) = \boldsymbol{a}(2\pi)$ and $\boldsymbol{a}(s) \neq \boldsymbol{a}(t)$ if $0 \leqq s < t < 2\pi$. We must show $\boldsymbol{a}(s)$ is a continuous function of s. $C(\boldsymbol{u})$ is a closed set since it is the intersection of the closed sets K and $T(\boldsymbol{u})$. If $\{s_i\}$ is a sequence with $s_i \to s$ then $\{\boldsymbol{a}(s_i)\}$ must have at least one limit point which can only be in $V(s)$. Since $C(\boldsymbol{u})$ is closed this limit point must be $\boldsymbol{a}(s)$. This completes the proof.

LEMMA 5. *Let W_1 and W_2 be any two planes as in the hypothesis of Theorem 1 so that $W_1 \cap K$ is equivalent to $W_2 \cap K$. If W_1 intersects $C(\boldsymbol{u})$ in $\boldsymbol{a}_1$ and $\boldsymbol{b}_1$, and W_2 intersects $C(\boldsymbol{u})$ in $\boldsymbol{a}_2$ and $\boldsymbol{b}_2$, then $\boldsymbol{a}_1\boldsymbol{b}_1$ is parallel to $\boldsymbol{a}_2\boldsymbol{b}_2$.*

PROOF. There is a line U_1 in W_1 parallel to $\boldsymbol{u}$ which intersects K in $\boldsymbol{a}_1$, and similarly lines V_1 in W_1 and U_2 and V_2 in W_2 intersecting K in $\boldsymbol{b}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{b}_2$. Clearly U_1 and V_1 are parallel tac planes of the 2-dimensional convex body $W_1 \cap K$ at $\boldsymbol{a}_1$ and $\boldsymbol{b}_1$, and U_2 and V_2 similarly for $W_2 \cap K$ at $\boldsymbol{a}_2$ and $\boldsymbol{b}_2$. Hence by Lemma 1 the result follows immediately.

In order to simplify the following proofs we now construct a function $f: [0, 2\pi] \to R$. By an orthogonal transformation of R^n we can make $\boldsymbol{u} = (0, 0, 1)$. Then each point of $C(\boldsymbol{u})$ is the point of contact with K of a tac plane with an outer normal $(\cos r, \sin r, 0)$ for some $r \in [0, 2\pi]$, and we write this point $f(r)$. We easily see the following: f is continuous, f is a surjection, f is injective except at most countably many points of $C(\boldsymbol{u})$ where K has more than one tac plane; f induces the ordering of $[0, 2\pi]$ on $C(\boldsymbol{u})$ (excluding $f(0)$). We loose no generality in assuming f is injective at $r = 0$. Notice that the tac plane of K at $f(r)$ makes an angle r with the tac plane of K at $f(O)$.

Notice also that a plane perpendicular to $(\cos r, \sin r, 0)$ cuts $C(\boldsymbol{u})$ not at all, or in one of $f(r)$, $f(r \pm \pi)$, or in two points $f(s)$ and $f(s')$. In the last case we call $f(s)f(s')$ an *r-chord*. Clearly there is a unique r-chord through each point of $C(\boldsymbol{u})$ other than $f(r)$ or $f(r \pm \pi)$.

We now prove some continuity properties of r-chords.

LEMMA 6. (a) *We may define r as a continuous function of s and s' $(s < s')$ by $f(s)f(s')$ being an r-chord with $s < r < s'$, in the cases when this r-chord is defined. If s' is fixed then r is a monotonic increasing function of s.*

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(b) *If $f(s')$ is defined as a function of r and s by $f(s)f(s')$ being an r-chord then the function is continuous.*

PROOF. (a) Let $f(s)f(s')$ be a q-chord with say $q < \pi$, then the plane W parallel to $\boldsymbol{u}$ and containing $f(s)f(s')$ separates the tac planes T_1 and T_2 of K at $f(q)$ and $f(q+\pi)$. In addition W separates $C(\boldsymbol{u})$ into two parts, one of which is $\{f(x)|s < x < s'\}$ and so we define r to be whichever of q or $q+\pi$ lies in this part. Since f is continuous, the plane W containing $f(s)$ and $f(s')$ must vary continuously with s and s', and therefore also the planes T and T'. Yet T and T' make angles q and $q+\pi$ with a fixed plane parallel to $\boldsymbol{u}$, so it follows that r also varies continuously with s and s'.

If s' is fixed and s decreases monotonically then W and so T_1 and T_2 must rotate monotonically, and they must rotate in the sense of decreasing r (for otherwise r would not stay in the interval (s, s'). This completes part (a).

(b) The plane W varies continuously with r and s, and therefore so does its second point of intersection with $C(\boldsymbol{u})$, namely $f(s')$. This completes the proof.

We next describe an important property of r-chords.

LEMMA 7. *For each $r \in (0, 2\pi)$ there is a neighborhood $(p, p') \subset (0, 2\pi)$, designated $N(r)$, satisfying:*

(a) *$f(r)$ is in the interior of $\{f(x)|p < x < p'\}$ on $C(\boldsymbol{u})$;*

(b) *$f(p)$ and $f(p')$ are continuous functions of r;*

(c) *if $s, t \in N(r)$ and $f(s)f(s')$ and $f(t)f(t')$ are r-chords then they are parallel.*

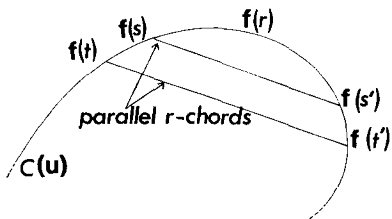

Diagram 2

PROOF. Let W be a plane perpendicular to $(\cos r, \sin r, 0)$ intersecting $C(\boldsymbol{u})$ in $f(q)$ and $f(q')$ with $q < q'$, (so $f(q)f(q')$ is an r-chord). Then W divides $C(\boldsymbol{u})$ into two parts one of which is $\{f(x)|q < x < q'\}$, and then the other part contains $f(0)$. However $f(r) \neq f(0)$ since $r \neq 0$ and f is injective at $r = 0$, thus we can choose W so that it separates $f(0)$ and $f(r)$ and then it follows that $q < r < q'$. We choose p and p' so that $f(p)f(p')$ is an r-chord and p is the minimum value of q with this property $q < r < q'$, subject also to the condition that W does not intersect the interior of hK (h as in the hypothesis of Theorem 1). This defines p and p' uniquely and $f(p) \neq f(r)$ (since $h < 1$) so that (a) is satisfied. Property (c) follows from Lemma 5 and the hypothesis of the theorem with $N(r) = (p, p')$. We still have to establish property (b).

Let W now be the plane perpendicular to $(\cos r, \sin r, 0)$ containing $f(p)$ and $f(p')$. It is clear from the construction of $f(p)$ and $f(p')$ that either W contains $f(0)$ (in which case either $p = 0$ or $p' = 2\pi$) or if not, then W is a tac plane of hK. In the first case the continuity follows from Lemma 6 (b)(since one end point of $f(p)f(p')$ is fixed). In the second case the tac plane W varies continuously with r and therefore the points of intersection of W with $C(\boldsymbol{u})$, namely $f(p)$ and $f(p')$, vary continuously with r. This completes the proof.

We next show that $C(\boldsymbol{u})$ is a plane curve and thus complete the proof of Theorem 1 by applying Lemma 3.

LEMMA 8. *$C(\boldsymbol{u})$ is a plane curve.*

PROOF. Let us take a closed subinterval of $(0, 2\pi)$, say $[a, b]$. Define the function $D : [a, b] \to R$ by

$$D : r \mapsto \min. (|f(r)-f(p)|, |f(r)-f(p')|), \tag{1}$$

where $N(r) = (p, p')$. Then by Lemma 7 (a) and (b), D is strictly positive and continuous, so there exists a constant $m > 0$ so that $D(r) > m$ for $r \in [a, b]$.

Now we choose an open subinterval of $[a, b]$, I say, satisfying $|f(x)-f(y)| \leqq m$ whenever $x, y \in I$, and also so that the set $f(I)$ is more then one point. We will show that the set $f(I)$ lies in a plane. Notice the following property of I which follows from the definition (1) and definitions of m and I,

$$N(r) \supset I \text{ for } r \in I. \tag{2}$$

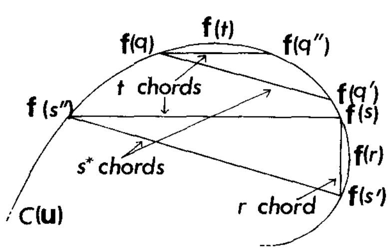

Diagram 3

Let r and $t \in I$ and let $f(s)f(s')$ be an r-chord with $s \in I$ and $f(s)f(s'')$ a t-chord as in Diagram 3. As $s \to r$ we have $s' \to r$ by Lemma 6(b), and therefore, since I is open, there is an $s_0 \in I$ so that $s' \in I$ whenever $s_0 < s < s_0'$. Let s^* be so that $f(s')f(s'')$ is an s^*-chord. By Lemmas 6(a) and (b), s^* is a continuous function of s and so $s^* \to t$ as $s \to r$. Hence there is an s_1 with $s_1 \geqq s_0$ so that $s^* \in I$ whenever $s_1 < s < s_1'$ (and we can also easily ensure $f(s_1) \neq f(r)$).

All of the r-chords $f(s)f(s')$ are parallel for $s_1 < s < s_1'$ by Lemma 7 since $s \in I$ and $N(r) \supset I$ by (2). Similarly the t-chords $f(s)f(s'')$ are parallel for $s_1 < s < s_1'$. Hence the s^*-chords $f(s')f(s'')$, for $s_1 < s < s_1'$, are parallel to the

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plane U through $\boldsymbol{O}$ spanned by $\boldsymbol{f}(s)-\boldsymbol{f}(s')$ and $\boldsymbol{f}(s)-\boldsymbol{f}(s'')$ for any such s.

Let $q \in I$ and $\boldsymbol{f}(q)\boldsymbol{f}(q')$ be the s^*-chord through $\boldsymbol{f}(q)$. Then $\boldsymbol{f}(q)\boldsymbol{f}(q')$ is parallel to $\boldsymbol{f}(s')\boldsymbol{f}(s'')$, for $s_1 < s < s_1'$, by Lemma 7, since in these cases $s' \in I$ and $N(s^*) \supset I$. Hence all of these points $\boldsymbol{f}(q')$ lie in the plane V through $\boldsymbol{f}(q)$ parallel to U. By Lemma 6(b) these points $\boldsymbol{f}(q')$ comprise a proper interval on $C(\boldsymbol{u})$ containing $\boldsymbol{f}(q'')$ in its interior, where $\boldsymbol{f}(q)\boldsymbol{f}(q'')$ is a t-chord. Hence by an appropriate choice of q and t we can show that any point of I, is in the interior of an interval of points of $C(\boldsymbol{u})$ lying in a plane. Therefore all of the points $\boldsymbol{f}(x)$, for $x \in I$, lie in a plane.

It easily follows from the arbitrary choice of the subinterval I of $[a, b]$ and of the interval $[a, b]$ itself that all of the points $\boldsymbol{f}(x)$, for $x \in (0, 2\pi)$, lie in a plane and so $C(\boldsymbol{u})$ is a plane curve. This completes Lemma 8.

Theorem 1 now follows from Lemmas 3 and 8.

4. Further results on ellipsoids

The result of section 3, namely Theorem 1, is now generalized in several ways. In Corollary 2 we extend the result to R^n without difficulty.

COROLLARY 2. *Let K be an n-dimensional convex body in R^n for $n \geqq 3$ with $\boldsymbol{O}$ in its interior and with all of its tac planes regular. Let u be a constant with $0 < h < 1$. Suppose that the $(n-1)$-dimensional convex bodies $(W_1 \cap K)$ and $(W_2 \cap K)$ are equivalent for all parallel hyperplanes W_1 and W_2 both on the same side of $\boldsymbol{O}$, provided W_1 and W_2 intersect the interior of K but not the interior of hK. Then K is an ellipsoid.*

PROOF. In the case $n = 3$ this is Theorem 1. The proof uses induction on n. Let K be an n-dimensional convex body satisfying the hypothesis and assume the result is true for $(n-1)$-dimensional convex bodies. It is then easy to show that if U is any hyperplane through $\boldsymbol{O}$ then the $(n-1)$-dimensional convex body $(U \cap K)$ is an ellipsoid. It follows that K is an ellipsoid (see [3], p. 91). This completes the proof.

The condition that the tac planes be regular is not necessary and we now show that the hypothesis of Corollary 2 can be restated in terms of the regular tac planes of an arbitrary convex body K. The hypothesis of the next result is stated in a slightly different form from Corollary 2 for convenience in the proof.

COROLLARY 3. *Let K be an n-dimensional convex body in R^n with $\boldsymbol{O}$ in its interior and $n \geqq 3$. Let h be a constant with $0 < h < 1$. Suppose that the $(n-1)$-dimensional convex bodies $(W_1 \cap K)$ and $(W_2 \cap K)$ are equivalent whenever there are $\boldsymbol{u}$, h_1, and h_2 with $h \leqq h_1 < 1$, and $h \leqq h_2 < 1$, and W_1 and W_2 are regular tac planes of $h_1 K$ and $h_2 K$, respectively, both with outer normal $\boldsymbol{u}$. Then K is an ellipsoid.*

PROOF. We prove that every tac plane of K is regular, in which case the result follows from Corollary 1. We prove this regularity only when $n = 3$ since the general case follows by a simple induction argument.

As a first step we show that the equivalence condition of the hypothesis can be extended to include planes W_1 and W_2 which are not regular tac planes of $h_1 K$ and $h_2 K$. We can certainly find a sequence $\{\boldsymbol{u}_i\}$ with $\boldsymbol{u}_i \to \boldsymbol{u}$ as $i \to \infty$ so that the tac planes W_{1i} of $h_1 K$ and W_{2i} of $h_2 K$ with outer normals $\boldsymbol{u}_i$, for each i, are regular. This is because the normals of non-regular tac planes form a set of measure zero in R^3 (see [2], section 9). For each i the hypothesis shows that $(W_{1i} \cap K)$ is equivalent to $(W_{2i} \cap K)$. However $(W_1 \cap K)$ is the limit as $i \to \infty$ of the convex bodies $(W_{1i} \cap K)$ in the sense that the boundary points of $(W_{1i} \cap K)$ become uniformly arbitrarily close to those of $(W_1 \cap K)$ as $i \to \infty$ and vice-versa (an accurate description of this can be given in terms of a metric on convex bodies as given in [5] p. 235 for example). Using the definition of equivalence, it follows that $(W_1 \cap K)$ is equivalent to $(W_2 \cap K)$.

We now show that no tac plane of K can intersect K in just a segment. Suppose a tac plane with outer normal $\boldsymbol{u}$ intersects K in just a segment. Let W_1 be the tac plane of $h_1 K$ with outer normal $\boldsymbol{u}$. If we let $h_1 \to 1$ (from below) then we can easily show that the ratio of the minimum to maximum width of $(W_1 \cap K)$ must approach zero since the set $(W_1 \cap K)$ must have the segment as its limit. This is incompatible with the equivalence condition of the hypothesis. Hence any tac plane of K intersects K in a single point or a proper plane face.

Suppose that a tac plane T intersects K in a plane face, and T' is the other tac plane of K parallel to T. Let $D = T \cap K$, $D' = T' \cap K$ (considered as 2-dimensional convex bodies) where D' may be a single point, and choose any proper chord $\boldsymbol{a}_1 \boldsymbol{a}_2$ of D. We divide the argument into two cases: (i) every tac plane of K parallel to $\boldsymbol{a}_1 \boldsymbol{a}_2$ intersects D or D'; (ii) there is a tac plane W of K parallel to $\boldsymbol{a}_1 \boldsymbol{a}_2$ which does not intersect D or D'.

In case (i) we can clearly find a tac plane W of K parallel to $\boldsymbol{a}_1 \boldsymbol{a}_2$ which intersects both D and D'. Then $W \cap K$ must contain a segment which does not lie in T or T'. It follows from the foregoing that this segment lies in a plane face of K parallel to $\boldsymbol{a}_1 \boldsymbol{a}_2$ and distinct from D and D'.

In case (ii) let W_1 be the tac plane of hK with the same outer normal as W. We can show by a simple continuity argument that W may be rechosen parallel to $\boldsymbol{a}_1 \boldsymbol{a}_2$ and still distinct from D and D', so that W_1 intersects the interior of D. In this case we choose W_2 to be another plane parallel to W lying strictly between W and W_1 which does not intersect D or D'. Clearly W_1 and W_2 satisfy the equivalence condition of the hypothesis so that $(W_1 \cap K)$ is equivalent to $(W_2 \cap K)$. However the frontier of $(W_1 \cap K)$ contains a proper segment parallel to $\boldsymbol{a}_1 \boldsymbol{a}_2$ (in its intersection with D), and so by Lemma 2 the frontier of $(W_2 \cap K)$ also contains a proper segment parallel to $\boldsymbol{a}_1 \boldsymbol{a}_2$. This segment of $(W_2 \cap K)$ does not lie in D or D' and so, as we showed previously, it lies in a plane face of K distinct from D and D'.

In both cases (i) and (ii) we deduce that there is a plane face F_1 of K parallel to $\boldsymbol{a}_1\boldsymbol{a}_2$ and distinct from D and D'. For another chord $\boldsymbol{b}_1\boldsymbol{b}_2$ of D which is not parallel to $\boldsymbol{a}_1\boldsymbol{a}_2$ we find another plane face F_2 parallel to $\boldsymbol{b}_1\boldsymbol{b}_2$, which is distinct from D and D'. However F_1 and F_2 are distinct since the only plane faces of K parallel to $\boldsymbol{a}_1\boldsymbol{a}_2$ and $\boldsymbol{b}_1\boldsymbol{b}_2$ are D and D'. It follows that there are uncountably many different plane faces of K since there are uncountably many mutually nonparallel chords of D. However K cannot have uncountably many plane faces, and so this is a contradiction of the supposition that it had one plane face. Hence all tac planes of K are regular, and this completes the proof of Corollary 3.

Another possible generalization of Theorem 1 would be to relax the condition that h be a constant and define h locally for each set of parallel planes. More exactly we would consider the following condition: *K is an n-dimensional convex body in R^n, for $n \geqq 3$, with $\boldsymbol{O}$ in its interior. For each $\boldsymbol{u}$ there is a $h(\boldsymbol{u})$ with $0 < h(\boldsymbol{u}) < 1$. $T(\boldsymbol{u})$ is the tac plane of K with outer normal $\boldsymbol{u}$. Then for each $\boldsymbol{u}$, $W_1 \cap K$ is equivalent to $W_2 \cap K$ whenever W_1 and W_2 are any two hyperplanes, both parallel to, and on the same side of $\boldsymbol{O}$, as $T(\boldsymbol{u})$, where W_1 and W_2 intersect the interior of K but not the interior of $h(\boldsymbol{u})K$.*

Rather surprisingly K need not be an ellipsoid now since the body K composed of a hemisphere and half-ellipsoid given by

$$\tfrac{1}{4}x_1^2+x_2^2+x_3^2 \leqq 1 \quad \text{for } x_1 \geqq 0$$
$$x_1^2+x_2^2+x_3^2 \leqq 1 \quad \text{for } 0 \geqq x_1$$

satisfies this condition. There is strong evidence that any K satisfying this condition must be composed of 'pieces' of second order surfaces.

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