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ON POWERFUL AND p-CENTRAL RESTRICTED LIE ALGEBRAS

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In this note we analyse the analogy between m-potent and p-central restricted Lie algebras and p-groups. For restricted Lie algebras the notion of m-potency has stronger implications than for p-groups (Theorem A). Every finite-dimensional restricted Lie algebra $\mathfrak{L}$ is isomorphic to $\widetilde{\mathfrak{L}}/\widetilde{\mathfrak{L}}_{[p]}$ for some finite-dimensional p-central restricted Lie algebra $\widetilde{\mathfrak{L}}$ (Proposition B). In particular, for restricted Lie algebras there does not hold an analogue of J.Buckley's theorem. For p odd one can characterise powerful restricted Lie algebras in terms of the cup product map in the same way as for finite p-groups (Theorem C). Moreover, the p-centrality of the finite-dimensional restricted Lie algebra $\mathfrak{L}$ has a strong implication on the structure of the cohomology ring $H^\bullet(\mathfrak{L},\mathbb{F})$ (Theorem D).

1. Introduction

The structure theory of powerful p-groups had a strong impact on the study of finite and infinite pro-p groups (see [**15, 16**]). Moreover, the mod p cohomology of p-central groups has been studied quite intensively, since for these groups the cohomology ring $H^\bullet(G,\mathbb{F}_p)$ is easiest to analyse (see [**6, 28**]). In this note we shall analyse these concepts for restricted Lie algebras.

One would expect that powerful restricted Lie algebras play a similar role in the category of finite-dimensional p-nilpotent restricted Lie algebras as powerful p-groups play in the category of finite p-groups. However, this is not the case. Let $\mathbb{F}$ be a field of characteristic $p > 0$, and let $\mathfrak{F}_p$ denote the class of finite-dimensional p-nilpotent restricted $\mathbb{F}$-Lie algebras. For $p \neq 2$, the restricted Lie algebra $\mathfrak{L} \in \mathfrak{F}_p$ is called *m-potent*, $m < p-1$, if

$$\gamma_{m+1}(\mathfrak{L}) \leqslant \mathfrak{L}^{[p]}, \tag{1.1}$$

where $\gamma_k(\mathfrak{L})$ denotes the k^{th}-term of the descending central series of $\mathfrak{L}$, and $\mathfrak{L}^{[p]^i}$ denotes the $\mathbb{F}$-vector space spanned by the elements $x^{[p]^i}$, $x \in \mathfrak{L}$. So 1-potent restricted Lie algebras are just powerful restricted Lie algebras as introduced by Riley and Semple in

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[19]. For $m = p - 2$, our definition is in analogy with the definition used by González–Sánchez and Jaikin–Zapirain for p-groups (see [9]). For $p = 2$, $\mathfrak{L}$ is called *1-potent* - or *powerful* - if

$$[\mathfrak{L}, \mathfrak{L}] \leqslant (\mathfrak{L}^{[2]})^{[2]}. \tag{1.2}$$

Obviously, there exist powerful p-groups of arbitrary high nilpotency class. However, for restricted Lie algebras one has the following (see Theorem 2.6, Proposition 2.7).

THEOREM A. (a) *Let $\mathfrak{L} \in \mathfrak{F}_p$ be an m-potent restricted Lie algebra, $m < p - 1$ for p odd, or $m = 1$ for $p = 2$. Then $\mathfrak{L}$ is nilpotent of class $\mathrm{cl}(\mathfrak{L}) \leqslant m + 1$. Moreover, $\mathfrak{L}^{[p]^j}$ is a restricted Lie ideal for all $j \geqslant 0$, and one has $(\mathfrak{L}^{[p^i]})^{[p]^j} = \mathfrak{L}^{[p]^{i+j}}$. In particular, if $\mathbb{F}$ is perfect, then for every $x \in \mathfrak{L}^{[p]^j}$ there exists $y \in \mathfrak{L}$ such that $x = y^{[p]^j}$.*

(b) *Let $p \neq 2$, let $\mathfrak{L}$ be a finite-dimensional p-nilpotent restricted Lie algebra, and let $d: = d(\mathfrak{L})$ denote the minimal number of generators of $\mathfrak{L}$ as restricted Lie algebra. Then $\mathfrak{L}$ is powerful, if and only if $\mathfrak{L}$ is a sum of d cyclic restricted Lie algebras.*

For $m = 1$, the first part of Theorem A(a) has been proved already in [19, Section 5]. In section 3 we shall apply Theorem A in order to analyse properties of the restricted universal enveloping algebras of these algebras.

While m-potency has much stronger implications for restricted Lie algebras than for p-groups, the implications for p-centrality are sometimes stronger and sometimes weaker. A restricted Lie algebra $\mathfrak{L}$ is called *p-central*, if

$$\mathfrak{L}_{[p]}: = \{\, x \in \mathfrak{L} \mid x^{[p]} = 0 \,\} \leqslant \mathrm{Z}(\mathfrak{L}), \tag{1.3}$$

where $\mathrm{Z}(\mathfrak{L})$ denotes the centre of the restricted Lie algebra $\mathfrak{L}$. Hence, for a p-central restricted Lie algebra $\mathfrak{L}$, the subset $\mathfrak{L}_{[p]}$ is a restricted Lie ideal in $\mathfrak{L}$. Finite-dimensional restricted Lie algebras have the following property (see Proposition 2.10).

PROPOSITION B. *Let $\mathfrak{L}$ be a restricted Lie algebra of dimension $n < \infty$. Then there exists a p-central restricted Lie algebra $\widetilde{\mathfrak{L}}$ of dimension $2n$ such that $\mathfrak{L} \simeq \widetilde{\mathfrak{L}}/\widetilde{\mathfrak{L}}_{[p]}$.*

This property of restricted Lie algebras is in contrast to the situation one has for finite groups. Indeed, for p odd, Buckley's theorem states that for a finite p-central group G the group $G/\Omega_1(G)$ is p-central as well (see [7]). This phenomenon is also reflected by the fact that the characterisation of p-centrality given by Bianchi, Gillo Berta Mauri and Verardi (see [5]) for p-groups does not hold for restricted Lie algebras (see Proposition 2.11).

In the last section of the paper we consider cohomological properties of powerful and p-central restricted Lie algebras. For p odd, one can characterise powerful restricted Lie algebras in the class $\mathfrak{F}_p$ (see Theorem 4.1) in the same way as one can characterise finitely generated powerful pro-p groups in the class of all finitely generated pro-p groups (see [27, Theorem 5.1.6]).

THEOREM C. *Let p be odd and let $\mathfrak{L} \in \mathfrak{F}_p$. Then the following are equivalent:*

(i) *$\mathfrak{L}$ is powerful.*

(ii) *Cup product induces an injective map*

$$\_\cup\_ : H^1(\mathfrak{L},\mathbb{F}) \wedge H^1(\mathfrak{L},\mathbb{F}) \longrightarrow H^2(\mathfrak{L},\mathbb{F}), \tag{1.4}$$

where $\mathbb{F}$ denotes the trivial left $\mathfrak{u}(\mathfrak{L})$-module.

In [**6**], Broto and Henn showed that for an arbitrary p-central finite group G the cohomology ring $H^\bullet(G,\mathbb{F}_p)$ is a Cohen–Macaulay $\mathbb{F}_p$-algebra. Let p be odd. A finite group G satisfies the *Ω-extension property*, if there exists a finite p-central group $\widetilde{G}$ such that G is isomorphic to $\widetilde{G}/\Omega_1(\widetilde{G})$. In [**28**, Theorem A] it was shown that a finite p-group G satisfies the Ω-extension property, if and only if

$$H^\bullet(G,\mathbb{F}_p) \simeq C^\bullet \otimes_{\mathbb{F}_p} S^\bullet, \tag{1.5}$$

where $C^\bullet$ is a finite-dimensional graded commutative $\mathbb{F}_p$-algebra, and $S^\bullet$ is a polynomial algebra generated in degree 2. Another interpretation of Proposition B is that for restricted Lie algebras the Ω-extension property is always satisfied. For restricted Lie algebras, we shall prove the following theorem (see Theorem 4.3, Corollary 4.5) which can be seen as an analogue of [**28**, Theorem A].

THEOREM D. *Let p be odd and let $\mathfrak{L}$ be a finite-dimensional restricted p-central restricted Lie algebra. Then*

$$H^\bullet(\mathfrak{L},\mathbb{F}) \simeq C^\bullet \otimes_{\mathbb{F}} S^\bullet(\mathfrak{L}^*_{[p]}) \tag{1.6}$$

where $S^\bullet(\mathfrak{L}^*_{[p]})$ is the polynomial $\mathbb{F}$-algebra generated by $\mathfrak{L}^*_{[p]} := \mathrm{Hom}_{\mathbb{F}}(\mathfrak{L}_{[p]},\mathbb{F})$ in degree 2, and $C^\bullet$ is a finite-dimensional $\mathbb{F}$-algebra satisfying Poincaré duality in dimension $n := \dim_{\mathbb{F}_p}(\mathfrak{L}_{[p]})$. In particular, $H^\bullet(\mathfrak{L},\mathbb{F})$ is a Cohen–Macaulay $\mathbb{F}$-algebra.

If p is odd, one can characterise finite p-groups with the Ω-extension property by the structure of their cohomology ring (see [**28**]). Therefore, one would like to know whether the following problem has an affirmative answer.

PROBLEM 1. Let p be odd and let $\mathfrak{L} \in \mathfrak{F}_p$. Assume that $H^\bullet(\mathfrak{L},\mathbb{F}) \simeq C^\bullet \otimes S^\bullet$, where $C^\bullet$ is a finite-dimensional $\mathbb{F}$-algebra, and $S^\bullet$ is a polynomial $\mathbb{F}$-algebra generated in degree 2. Is it true that $\mathfrak{L}$ is p-central?

The main purpose of this paper is the study of m-potent restricted Lie algebras and p-central restricted Lie algebras in analogy to m-potent p-groups and p-central finite groups. However, there might be other contexts where these notions play an important role. We close the introduction with the following two open problems [The authors thank the referee for communicating these problems to them.] which might be the subject of further investigations.

PROBLEM 2. Investigate m-potent and p-central restricted Lie algebras represented as ring constructions defined in [**14**, Chapter 3].

PROBLEM 3. Describe m-potent and p-central restricted colour Lie superalgebras represented as blocked matrices of directed graphs (see [**13**]).

2. POTENT AND p-CENTRAL RESTRICTED LIE ALGEBRAS

Let $\mathfrak{L}$ be a restricted Lie algebra over the field $\mathbb{F}$ of characteristic $p > 0$. For a subset S of $\mathfrak{L}$, we denote by $\langle S\rangle_p$ the restricted subalgebra generated by S. If I is an ideal of $\mathfrak{L}$ then $I_p\colon = \langle I\rangle_p$ is a restricted ideal of $\mathfrak{L}$. By $S^{[p]^k}$, $k > 0$, we denote the $\mathbb{F}$-vector subspace of $\mathfrak{L}$ spanned by the elements $x^{[p]^k}$, $x \in S$. The restricted Lie algebra $\mathfrak{L}$ is *cyclic*, if there exists $x \in \mathfrak{L}$ such that $\mathfrak{L} = \langle x\rangle_p$.

For a positive integer i we denote by $\gamma_i(\mathfrak{L})$ the i^{th} term of the lower central series of $\mathfrak{L}$. For a restricted Lie algebra $\mathfrak{L} \in \mathfrak{F}_p$, we denote by $\mathrm{cl}(\mathfrak{L})$ the nilpotency class of $\mathfrak{L}$, and by $e(\mathfrak{L})$ its exponent, that is, the minimum number $m \in \mathbb{N}_0$ such that $\mathfrak{L}^{[p]^m} = 0$. The element $x \in \mathfrak{L}$ is called *of exponent* k, $k \in \mathbb{N}_0$, if and only if $\langle x\rangle_p$ is of exponent k. For an ideal I of the Lie algebra $\mathfrak{L}$ we put $[I,{}_n\mathfrak{L}]\colon = \Big[\dots[[I,\mathfrak{L}],\mathfrak{L}],\dots,\mathfrak{L}\Big]$, where $\mathfrak{L}$ appears in the latter expression n times.

2.1. THE FRATTINI IDEAL $\Phi(\mathfrak{L})$. Let $\mathfrak{L} \in \mathfrak{F}_p$. The restricted Lie ideal

$$\Phi(\mathfrak{L})\colon = \mathfrak{L}^{[p]} + [\mathfrak{L},\mathfrak{L}] \tag{2.1}$$

will be called the *Frattini ideal of* $\mathfrak{L}$. For the convenience of the reader we state its well-known properties in the following proposition (see [**21**]).

PROPOSITION 2.1. *Let* $\mathfrak{L} \in \mathfrak{F}_p$.

(a) $\Phi(\mathfrak{L})$ *is the intersection of all restricted Lie ideals* I *of* $\mathfrak{L}$ *of codimension* 1.
(b) *If* S *is a subset of* $\mathfrak{L}$ *whose image in* $\mathfrak{L}/\Phi(\mathfrak{L})$ *spans* $\mathfrak{L}/\Phi(\mathfrak{L})$, *then* $\langle S\rangle_p = \mathfrak{L}$.
(c) *Let* $d(\mathfrak{L})$ *denote the minimal number of generators of* $\mathfrak{L}$ *as restricted Lie algebra. Then* $d(\mathfrak{L}) = \dim_{\mathbb{F}}\big(\mathfrak{L}/\Phi(\mathfrak{L})\big)$.
(d) *Let* J *be a restricted ideal of* $\mathfrak{L}$ *being contained in* $\Phi(\mathfrak{L})$. *Then* $\Phi(\mathfrak{L}/J) = \Phi(\mathfrak{L})/J$.
(e) *Let* J *be a* 1*-dimensional restricted Lie ideal of* $\mathfrak{L}$ *such that the short exact sequence* $0 \to J \to \mathfrak{L} \to \mathfrak{L}/J \to 0$ *is non-split. Then* J *is contained in* $\Phi(\mathfrak{L})$.

2.2. POTENTLY EMBEDDED IDEALS. Let p be odd and $m < p-1$. A restricted ideal I of $\mathfrak{L} \in \mathfrak{F}_p$ is called *m-potently embedded* in $\mathfrak{L}$, if $[I,{}_m\mathfrak{L}]$ is contained in $I^{[p]}$. If $p = 2$, then I is called 1*-potently embedded* in $\mathfrak{L}$, if $[I,\mathfrak{L}]$ is contained in $(I^{[2]})^{[2]}$. A 1-potently embedded ideal will also be called a *powerfully embedded ideal.* Obviously, if I is m-potently embedded in $\mathfrak{L}$, then $I^{[p]}$ is a restricted ideal of L. One has the following:

LEMMA 2.2. *Let* $\mathfrak{L} \in \mathfrak{F}_p$ *and let* I *be a restricted ideal of* $\mathfrak{L}$.

(a) *Let p be odd and $m < p-1$. Then I is m-potently embedded in $\mathfrak{L}$, if and only if $I/[I,{}_{p-1}\mathfrak{L}]_p$ is m-potently embedded in $\mathfrak{L}/[I,{}_{p-1}\mathfrak{L}]_p$. In this case one has $[I,{}_{p-1}\mathfrak{L}]_p = 0$.*

(b) *Let $p=2$. Then I is 1-potently embedded in $\mathfrak{L}$, if and only if $I/[I,{}_3\mathfrak{L}]_2$ is 1-potently embedded in $\mathfrak{L}/[I,{}_3\mathfrak{L}]_2$. In this case one has $[I,{}_3\mathfrak{L}]_2 = 0$.*

PROOF: (a) Assume that $I/[I,{}_{p-1}\mathfrak{L}]_p$ is m-potently embedded in $\mathfrak{L}/[I,{}_{p-1}\mathfrak{L}]_p$. It suffices to show that $[I,{}_{p-1}\mathfrak{L}] = 0$. Suppose $[I,{}_{p-1}\mathfrak{L}] \neq 0$. By hypothesis,

$$[I,{}_m\mathfrak{L}]_p = \left([I,{}_m\mathfrak{L}]_p \cap I^{[p]}\right) + [I,{}_{p-1}\mathfrak{L}]_p. \tag{2.2}$$

Put $J := \left([I,{}_m\mathfrak{L}]_p \cap I^{[p]}\right) + [I,{}_p\mathfrak{L}]_p$. Then J is a restricted ideal of $\mathfrak{L}$, and by definition, $\left([I,{}_m\mathfrak{L}]_p \cap I^{[p]}\right) \subseteq J \subseteq [I,{}_m\mathfrak{L}]_p$. As $\mathfrak{L}$ is nilpotent and $[I,{}_{p-1}\mathfrak{L}] \neq 0$, one has $[J,\mathfrak{L}] \subseteq [I,{}_p\mathfrak{L}] \subsetneq \left[[I,{}_m\mathfrak{L}],\mathfrak{L}\right] = \left[[I,{}_m\mathfrak{L}]_p,\mathfrak{L}\right]$. In particular, $J \neq [I,{}_m\mathfrak{L}]_p$. Since $\mathfrak{L}$ is finite-dimensional and p-nilpotent, there exists a restricted ideal K of $\mathfrak{L}$ such that $J \subseteq K \subsetneq [I,{}_m\mathfrak{L}]_p$, and K has codimension 1 in $[I,{}_m\mathfrak{L}]_p$. Put $[I,{}_m\mathfrak{L}]_p = K + \mathbb{F}.x$ for a suitable $x \in [I,{}_m\mathfrak{L}]_p$. Since every 1-dimensional left $\mathfrak{L}$-module is trivial, one concludes that $\left[[I,{}_m\mathfrak{L}],\mathfrak{L}\right]_p \subseteq K$. By (2.2) and as $m < p-1$, it follows that $[I,{}_m\mathfrak{L}]_p \subseteq K$, a contradiction, and this yields the claim.

(b) The proof for $p=2$ can be obtained in a similar way by replacing the role of $I^{[p]}$ by $(I^{[2]})^{[2]}$ and $[I,{}_{p-1}\mathfrak{L}]$ by $[I,{}_3\mathfrak{L}]$. ∎

For the reminder of this section we assume that m is a positive integer satisfying $m < p-1$ for p odd or $m = 1$ in case $p = 2$.

PROPOSITION 2.3. *Let $\mathfrak{L} \in \mathfrak{F}_p$ and let I and J be two restricted ideals of $\mathfrak{L}$. If I and J are m-potently embedded in $\mathfrak{L}$, so are $[I,\mathfrak{L}]_p$, $I^{[p]}$, $[I,J]_p$ and $I+J$.*

PROOF: Let p be odd. First we show that $[I,\mathfrak{L}]$ is m-potently embedded. Without loss of generality we may assume that $\left[[I,\mathfrak{L}],{}_{p-1}\mathfrak{L}\right] = 0$ (see Lemma 2.2(a)). Hence, for any $x \in I$ and $a \in \mathfrak{L}$, one has $(\operatorname{ad} x)^p(a) = 0$, and thus $I^{[p]} \subseteq Z(\mathfrak{L})$. Since I is m-potently embedded in $\mathfrak{L}$, this yields $\left[[I,\mathfrak{L}]_p,{}_m\mathfrak{L}\right] \subseteq [I^{[p]},\mathfrak{L}] = 0$ and the claim follows.

Concerning $I^{[p]}$ we have already observed that $I^{[p]}$ is a restricted ideal of $\mathfrak{L}$. By Lemma 2.2(a), we may assume that $[I^{[p]},{}_{p-1}\mathfrak{L}] = 0$. As I is m-potently embedded in $\mathfrak{L}$, it follows that $[I,{}_{m+p-1}\mathfrak{L}] = 0$. Hence $[I^{[p]},{}_m\mathfrak{L}] = 0$, and $I^{[p]}$ is m-potently embedded in $\mathfrak{L}$.

Next consider $[I,J]_p$. As above we may assume that $\left[[I,J]_p,{}_{p-1}\mathfrak{L}\right] = 0$. This forces $[I^{[p]},J] = [I,J^{[p]}] = 0$. Since I and J are m-potently embedded in $\mathfrak{L}$, this implies that $\left[[I,{}_m\mathfrak{L}],J\right] \subseteq [I^{[p]},J] = 0$ and $\left[[J,{}_m\mathfrak{L}],I\right] \subseteq [I,J^{[p]}] = 0$. By Jacobi's identity, one has $0 = \left[[I,J],{}_m\mathfrak{L}\right] = \left[[I,J]_p,{}_m\mathfrak{L}\right]$, and thus $[I,J]_p$ is m-potently embedded in $\mathfrak{L}$.

Finally, for $I+J$ one has

$$[I+J,{}_m\mathfrak{L}] = [I,{}_m\mathfrak{L}] + [J,{}_m\mathfrak{L}] \subseteq I^{[p]} + J^{[p]} \subseteq (I+J)^{[p]}, \tag{2.3}$$

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therefore $I + J$ is m-potently embedded in $\mathfrak{L}$.

For $p = 2$, the proof is analogous to the case p odd using Lemma 2.2(b) and suitable modifications. ∎

As a consequence of Proposition 2.3 one obtains the following corollary.

COROLLARY 2.4. *Any restricted Lie algebra $\mathfrak{L} \in \mathfrak{F}_p$ contains a unique maximal m-potently embedded restricted ideal.*

A restricted m-potently embedded ideal I of $\mathfrak{L}$ is obviously m-potent. If $\mathfrak{H}$ is a restricted subalgebra of $\mathfrak{L}$ and $\mathfrak{H}/I$ is cyclic, then $\mathfrak{H}$ is m-potent. Indeed, in this case there is $x \in \mathfrak{H}$ such that every element of $\mathfrak{H}/I$ is a linear combination of the elements $x^{[p]^i} + I$, $i \in \mathbb{N}_0$. Consequently, $[\mathfrak{H}, \mathfrak{H}] = [I, \mathfrak{H}]$. As I is m-potently embedded in $\mathfrak{L}$, one has

$$(2.4) \qquad \begin{aligned} \gamma_{m+1}(\mathfrak{H}) &\subseteq I^{[p]} &&\subseteq \mathfrak{H}^{[p]} && \text{for } p \text{ odd,} \\ \gamma_{m+1}(\mathfrak{H}) &\subseteq (I^{[2]})^{[2]} &&\subseteq (\mathfrak{H}^{[2]})^{[2]} && \text{for } p = 2. \end{aligned}$$

The m-potency of a restricted Lie algebras is preserved by extension of the ground field. Furthermore, quotient Lie algebras and direct sums of m-potent restricted Lie algebras are m-potent as well. The following example shows that a restricted ideal of a m-potent restricted Lie algebra need not be m-potent.

EXAMPLE 2.5. Let $\mathfrak{L}$ be the Lie algebra over a field $\mathbb{F}$ of odd characteristic with $\mathbb{F}$-basis $\{x, y, z, v\}$ and with relations $[x, y] = z$ and $z, v \in Z(\mathfrak{L})$. The p-map of $\mathfrak{L}$ is given by

$$(2.5) \qquad x^{[p]} = y^{[p]} = z^{[p]} = 0, \qquad v^{[p]} = z.$$

One has $[\mathfrak{L}, \mathfrak{L}] = \mathfrak{L}^{[p]} = \mathbb{F}.z$, and thus $\mathfrak{L}$ is powerful. For the restricted ideal $I\colon = \mathbb{F}.x + \mathbb{F}.y + \mathbb{F}.z$ one has $[I, I] = \mathbb{F}.z$, while $I^{[p]} = 0$. Therefore, I is not powerful.

THEOREM 2.6. *Let $\mathfrak{L} \in \mathfrak{F}_p$ be an m-potent restricted Lie algebra.*

(a) *$\mathfrak{L}$ is nilpotent of class $\mathrm{cl}(\mathfrak{L}) \leqslant m + 1$.*

(b) *For $i \geqslant 0$ the $\mathbb{F}$-vector space $\mathfrak{L}^{[p]^i}$ is a restricted ideal of $\mathfrak{L}$. Moreover, $(\mathfrak{L}^{[p]^j})^{[p]^i} = \mathfrak{L}^{[p]^{i+j}}$.*

(c) *Let $\{b_1, \dots, b_r\}$ be an $\mathbb{F}$-basis of $\mathfrak{L}$. Then $\mathfrak{L}^{[p]^i} = \sum_{1 \leqslant k \leqslant r} \mathbb{F}.b_k^{[p]^i}$.*

(d) *If $\mathbb{F}$ is perfect, for every element x of $\mathfrak{L}^{[p]^i}$ there exists $y \in \mathfrak{L}$ such that $y^{[p]^i} = x$.*

PROOF: (a) Let p be odd. By Lemma 2.2(a), $\mathrm{cl}(\mathfrak{L}) \leqslant p - 1$. For every $x, y \in \mathfrak{L}$ one has $\mathrm{ad}\, x^{[p]}(y) = 0$ and thus $\mathfrak{L}^{[p]} \subseteq Z(\mathfrak{L})$. Moreover, as $\mathfrak{L}$ is m-potent, $\gamma_{m+1}(\mathfrak{L}/\mathfrak{L}^{[p]}) = 0$ and thus $\mathrm{cl}(\mathfrak{L}) \leqslant m + 1$.

Let $p = 2$. By Lemma 2.2(b), $\mathrm{cl}(\mathfrak{L}) \leqslant 3$. One concludes that $(\mathrm{ad}\, x^{[2]})^2(y) = 0$ for every $x, y \in \mathfrak{L}$. Hence $(\mathfrak{L}^{[2]})^{[2]} \subseteq Z(\mathfrak{L})$. Since $\mathfrak{L}$ is 1-potent, $\gamma_2\big(\mathfrak{L}/(\mathfrak{L}^{[2]})^{[2]}\big) = 0$ and therefore $\mathrm{cl}(\mathfrak{L}) \leqslant 2$.

(b) By (a), the $\mathbb{F}$-vector subspace $\mathfrak{L}^{[p]^i}$ of $\mathfrak{L}$ is contained in $\mathrm{Z}(\mathfrak{L})$ for every $i > 0$. This yields (b). Part (c) and (d) follow from the fact that $\_^{[p]^i}: \mathfrak{L} \to \mathrm{Z}(\mathfrak{L})$ is a p-semilinear map. ∎

One has the following characterisation of powerful restricted Lie algebras.

Proposition 2.7. *Let $\mathfrak{L} \in \mathfrak{F}_p$ be a restricted Lie algebra with $d{:} = d(\mathfrak{L})$.*

(a) *If $\mathfrak{L}$ is powerful, then $\mathfrak{L}$ is a sum of d cyclic restricted Lie algebras.*

(b) *If $p \neq 2$, then $\mathfrak{L}$ is powerful, if and only if $\mathfrak{L}$ is the sum of d cyclic restricted Lie algebras.*

Proof: (a) Since $\mathfrak{L}$ is powerful, one has $\Phi(\mathfrak{L}) = \mathfrak{L}^{[p]}$. Let $\pi_\Phi : \mathfrak{L} \to \mathfrak{L}/\Phi(\mathfrak{L})$ denote the canonical projection, and let $S = \{x_1, x_2, \dots, x_d\}$ be a subset of $\mathfrak{L}$ such that $\pi_\Phi(S)$ is a basis of the $\mathbb{F}$-vector space $\mathfrak{L}/\mathfrak{L}^{[p]}$. Denote by $H{:} = \sum_{i=1}^{d} \langle x_i \rangle_p$ the sum of the cyclic restricted Lie algebras $\langle x_i \rangle_p$. By construction, one has $\pi_\Phi(H) = \pi_\Phi(\mathfrak{L})$. Hence $H + \mathfrak{L}^{[p]} = \mathfrak{L}$. As $\_^{[p]}: \mathfrak{L} \to \mathfrak{L}^{[p]}$ is p-semilinear and $\mathfrak{L}^{[p]} \leqslant \mathrm{Z}(\mathfrak{L})$, this implies $\mathfrak{L}^{[p]} = H^{[p]} + \mathfrak{L}^{[p]^2}$. Thus, by induction, $\mathfrak{L}^{[p]} = H^{[p]}$ and this yields the claim.

(b) Let $\mathfrak{L} = \sum_{i=1}^{d} \langle x_i \rangle_p$. The $\mathbb{F}$-subspace $\sum_{i=1}^{d} \langle x_i^{[p]} \rangle_p$ has codimension $d = d(\mathfrak{L})$ and is contained in $\ker(\pi_\Phi)$. Hence $\ker(\pi_\Phi) = \sum_{i=1}^{d} \langle x_i^{[p]} \rangle_p$. This implies $[\mathfrak{L}, \mathfrak{L}] \leqslant \Phi(\mathfrak{L}) \leqslant \mathfrak{L}^{[p]}$ and $\mathfrak{L}$ is powerful. ∎

The following example shows that Proposition 2.7(b) does not hold in even characteristic:

Example 2.8. Let $\mathfrak{H}$ be the 3-dimensional Heisenberg algebra over a field $\mathbb{F}$ of characteristic 2. Then H has a basis $\{x, y, z\}$ with

$$[x, y] = z, \qquad [x, z] = [y, z] = 0. \tag{2.6}$$

Consider the p-map on $\mathfrak{H}$ given by

$$x^{[2]} = y^{[2]} = z, \qquad z^{[2]} = 0. \tag{2.7}$$

Then $d(\mathfrak{H}) = 2$ and $\mathfrak{H} = \langle x \rangle_p + \langle y \rangle_p$, but $\mathfrak{H}$ is not powerful.

The following property is useful for the characterisation of powerful restricted Lie algebras in terms of cohomological properties.

Proposition 2.9. *Let p be odd, and let $\mathfrak{L} \in \mathfrak{F}_p$ be a non-powerful restricted Lie algebra. Then there exists a restricted Lie ideal J of $\mathfrak{L}$, such that*

(i) *$\mathfrak{L}^{[p]}$ is contained in J.*

(ii) *J is contained in $\Phi(\mathfrak{L})$ and has codimension 1.*

PROOF: The restricted Lie algebra $\mathfrak{L}$ is powerful, if and only if $\mathfrak{L}/\gamma_p(\mathfrak{L})_p$ is powerful (see Lemma 2.2(a)). Since $\gamma_p(\mathfrak{L})_p$ is contained in $\Phi(\mathfrak{L})$, we may therefore assume that $\gamma_p(\mathfrak{L})_p = 0$ (see Proposition 2.1(d)). In particular, $\mathfrak{L}^{[p]}$ is a restricted Lie ideal contained in $Z(\mathfrak{L})$. Since $\mathfrak{L}$ is non-powerful, $\mathfrak{L}^{[p]}$ is properly contained in $\Phi(\mathfrak{L})$. Let J be a maximal ideal being properly contained in $\Phi(\mathfrak{L})$ containing $\mathfrak{L}^{[p]}$. Then J has the desired properties. ∎

2.3. p-CENTRAL RESTRICTED LIE ALGEBRAS. For a restricted Lie algebra $\mathfrak{L}$ with p-map $\_^{[p]}\colon \mathfrak{L} \to \mathfrak{L}^{[p]}$ we denote by $\mathfrak{L}_{[p]}$ the set of all zeros of $[p]$. Thus, $\mathfrak{L}$ is p-central, if and only if $\mathfrak{L}_{[p]} \subseteq Z(\mathfrak{L})$. If $\mathfrak{L}$ is a p-central restricted Lie algebra, $\mathfrak{L}_{[p]}$ is a restricted ideal. The property of p-centrality will be inherited on restricted subalgebras and is preserved by direct sums and extensions of the ground field. However, homomorphic images of p-central restricted Lie algebras need not be p-central. More precisely, any restricted Lie algebras is the homomorphic image of a p-central restricted Lie algebra.

PROPOSITION 2.10. *Let $\mathfrak{L}$ be a restricted Lie algebra of dimension n over a field $\mathbb{F}$ of characteristic $p > 0$. Then there exists a p-central restricted Lie algebra $\widetilde{\mathfrak{L}}$ such that $\dim_{\mathbb{F}}(\widetilde{\mathfrak{L}}) = 2n$ and $\mathfrak{L}$ is isomorphic to $\widetilde{\mathfrak{L}}/\widetilde{\mathfrak{L}}_{[p]}$ as a restricted Lie algebra.*

PROOF: Let $\{x_1, \dots, x_n\}$ be an $\mathbb{F}$-basis for $\mathfrak{L}$ and let $\mathfrak{B}$ be an Abelian n-dimensional Lie algebra over $\mathbb{F}$ with basis $\{y_1, \dots, y_n\}$. Let $\widetilde{\mathfrak{L}}$ denote the Lie algebra $\mathfrak{L} \oplus \mathfrak{B}$ with p-map $[p']$ given by

$$x_1^{[p']} = x_1^{[p]} + y_1; \quad \dots \quad x_n^{[p']} = x_n^{[p]} + y_n; \quad y_1^{[p']} = \dots = y_n^{[p']} = 0. \tag{2.8}$$

Clearly, for $z = x + y \in \widetilde{\mathfrak{L}}$ with $x = \sum_{i=1}^{n} \lambda_i x_i \in \mathfrak{L}$ and $y \in \mathfrak{B}$ one has

$$z^{[p']} = \sum_{i=1}^{n} \lambda_i^p . x_i^{[p]} + \sum_{i=1}^{n} \lambda_i^p . y_i. \tag{2.9}$$

The linear independence of the elements $x_1, \dots, x_n, y_1, \dots, y_n$ forces $\widetilde{\mathfrak{L}}_{[p]} = \mathfrak{B}$, and this yields the claim. ∎

The following property which has been studied for finite groups in [5] yields a criterion for p-centrality in case that the nilpotency class is less than p.

PROPOSITION 2.11. *Let $\mathfrak{L}$ be a nilpotent restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 0$ with $\mathrm{cl}(\mathfrak{L}) < p$. Then $\mathfrak{L}$ is p-central, if and only if one has $[x, y] = 0$ for every $x, y \in \mathfrak{L}$ satisfying $x^{[p]} = y^{[p]}$.*

PROOF: Assume that $\mathfrak{L}$ is p-central. Since $\mathrm{cl}(\mathfrak{L}) < p$, the p-map is p-semilinear. Hence $x^{[p]} = y^{[p]}$ forces $(x - y)^{[p]} = 0$. This yields $x - y \in Z(\mathfrak{L})$, and thus $[x, y] = 0$.

Conversely, suppose that for $x, y \in \mathfrak{L}$, $x^{[p]} = y^{[p]}$ implies $[x, y] = 0$. Since $\mathrm{cl}(\mathfrak{L}) < p$, for every $x \in \mathfrak{L}$ and $z \in \mathfrak{L}_{[p]}$, one has $(x + z)^{[p]} = x^{[p]} + z^{[p]} = x^{[p]}$. So, by hypothesis, $z \in Z(\mathfrak{L})$ and this yields the claim. ∎

The following examples show that in contrast to the situation for finite groups (see [5]), one cannot drop the hypothesis on the nilpotency class.

EXAMPLE 2.12. Let $\mathbb{F}$ be a field of characteristic $p > 0$. Let $\mathfrak{L}$ be the restricted $\mathbb{F}$-Lie algebra with basis $x, y, z, a_1, \dots, a_{p-1}, b_1, \dots, b_{p-1}$ subject to the following relations: $b_i, z \in Z(\mathfrak{L})$, $1 \leqslant i \leqslant p-1$ and $[x,y] = a_1$, $[x, a_i] = [a_i, a_j] = 0$ for every $i, j < p$, $[a_i, y] = a_{i+1}$ for $i < p-1$ and $[a_{p-1}, y] = 0$. In particular, $\mathrm{cl}(\mathfrak{L}) = p$. The p-map is given by $x^{[p]} = 0$, $y^{[p]} = z$, $z^{[p]} = 0$, $a_i^{[p]} = b_i$, $b_i^{[p]} = 0$, $1 \leqslant i \leqslant p-1$. A straightforward verification shows that any two elements of $\mathfrak{L}$ having the same image under the p-map commute. However, $x^{[p]} = 0$ while $x \notin Z(\mathfrak{L})$. Therefore, $\mathfrak{L}$ is not p-central.

EXAMPLE 2.13. Let $\mathfrak{M}$ be the restricted Lie algebra which coincides with $\mathfrak{L}$ of Example 2.12 as $\mathbb{F}$-Lie algebra, but which p-map is given by $x^{[p]} = y^{[p]} = z$, $z^{[p]} = 0$, $a_i^{[p]} = b_i$ and $b_i^{[p]} = 0$ for $1 \leqslant i \leqslant p-1$. It is an easy exercise to verify that $\mathfrak{L}$ is p-central. However, $x^{[p]} = y^{[p]}$, but $[x,y] \neq 0$.

3. THE RESTRICTED ENVELOPING ALGEBRA OF POWERFUL RESTRICTED LIE ALGEBRAS

Let $\mathfrak{L}$ be a restricted Lie algebra over a field of characteristic $p > 0$. By $\mathrm{u}(\mathfrak{L})$ we shall denote the *restricted universal enveloping algebra* of $\mathfrak{L}$, and by $\omega(\mathfrak{L})$ we shall denote the *augmentation ideal* of $\mathrm{u}(\mathfrak{L})$, that is, $\omega(\mathfrak{L})$ is the kernel of the counit $\varepsilon\colon \mathrm{u}(\mathfrak{L}) \to \mathbb{F}$ of the $\mathbb{F}$-Hopf algebra $\mathrm{u}(\mathfrak{L})$. In particular, $\omega(\mathfrak{L})$ is the associative ideal generated by $\mathfrak{L}$ in $\mathrm{u}(\mathfrak{L})$.

3.1. THE NILPOTENCY INDEX OF THE AUGMENTATION IDEAL. It is well known (see [20]) that $\omega(\mathfrak{L})$ is nilpotent, if and only if $\mathfrak{L} \in \mathfrak{F}_p$. The *nilpotency index* $t(\mathrm{u}(\mathfrak{L}))$ of $\omega(\mathfrak{L})$ is defined to be the smallest positive integer k such that $\omega(\mathfrak{L})^k = 0$. Relations beetwen the nilpotency index $t(\mathrm{u}(\mathfrak{L}))$ of $\omega(\mathfrak{L})$ and the exponent $e(\mathfrak{L})$ of $\mathfrak{L}$ were studied in [**21**]: for example, it was shown that $p^{e(\mathfrak{L})} \leqslant t(\mathrm{u}(\mathfrak{L}))$ for all $\mathfrak{L} \in \mathfrak{F}_p$. For powerful restricted Lie algebras one has also the following.

PROPOSITION 3.1. *Let $\mathfrak{L}$ be a powerful restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 0$. Then one has*

$$t(\mathrm{u}(\mathfrak{L})) \leqslant 1 + d(\mathfrak{L}) \cdot (p^{e(\mathfrak{L})} - 1). \tag{3.1}$$

Moreover, equality holds in (3.1), *if and only if every element $x \in \mathfrak{L} \setminus \mathfrak{L}^{[p]}$ is of exponent $e(\mathfrak{L})$.*

PROOF: Put

$$\mathfrak{D}_1(\mathfrak{L}) := \mathfrak{L}, \qquad \mathfrak{D}_m(\mathfrak{L}) := \left\langle \mathfrak{D}_{\lceil m/p \rceil}(\mathfrak{L})^{[p]} \right\rangle_p + [\mathfrak{L}, \mathfrak{D}_{m-1}(\mathfrak{L})] \quad \text{for } m > 1. \tag{3.2}$$

By [**21**], one has

$$t(\mathrm{u}(\mathfrak{L})) = 1 + (p-1) \cdot \sum_{n \geqslant 1} n \cdot d_n, \tag{3.3}$$

where $d_n := \dim_{\mathbb{F}}(\mathfrak{D}_n(\mathfrak{L})/\mathfrak{D}_{n+1}(\mathfrak{L}))$. By Theorem 2.6 and induction, one concludes easily that

$$\mathfrak{D}_n(\mathfrak{L}) = \mathfrak{L}^{[p]^{k(n)}} \tag{3.4}$$

where $k(n) := \lceil \log_p n \rceil$. This yields

$$d_n = \begin{cases} \dim_{\mathbb{F}}(\mathfrak{L}^{[p]^i}/\mathfrak{L}^{[p]^{i+1}}) & \text{if } n = p^i \text{ with } 0 \leqslant i < e(\mathfrak{L}), \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Formula (3.3) implies that

$$t(\mathrm{u}(\mathfrak{L})) = 1 + (p-1) \cdot \sum_{i=0}^{e(\mathfrak{L})-1} p^i \cdot \dim_{\mathbb{F}}(\mathfrak{L}^{[p]^i}/\mathfrak{L}^{[p]^{i+1}}). \tag{3.6}$$

Moreover, by Theorem 2.6, $\dim_{\mathbb{F}}(\mathfrak{L}^{[p]^i}/\mathfrak{L}^{[p]^{i+1}}) \leqslant d(\mathfrak{L})$ which yields (3.1). One has equality in (3.1), if and only if $\dim_{\mathbb{F}}(\mathfrak{L}^{[p]^i}/\mathfrak{L}^{[p]^{i+1}}) = d(\mathfrak{L})$ for all $i = 0, \ldots, e(\mathfrak{L}) - 1$. By Theorem 2.6, this is equivalent to the property that every element $x \in \mathfrak{L} \setminus \mathfrak{L}^{[p]}$ is of exponent $e(\mathfrak{L})$. □

3.2. The Lie derived length. Let $\mathfrak{A}$ be any associative $\mathbb{F}$-algebra with unit. The associative $\mathbb{F}$-algebra $\mathfrak{A}$ can be regarded as an $\mathbb{F}$-Lie algebra via the Lie commutator $[x, y] = xy - yx$, $x, y \in \mathfrak{A}$. The *Lie derived series* $\delta^{[n]}(\mathfrak{A})$ and the *strong Lie derived series* $\delta^{(n)}(\mathfrak{A})$ of $\mathfrak{A}$ are given by

$$\begin{aligned} \delta^{[0]}(\mathfrak{A}) &:= \delta^{(0)}(\mathfrak{A}) = \mathfrak{A}, \\ \delta^{[n]}(\mathfrak{A}) &:= [\delta^{[n-1]}(\mathfrak{A}), \delta^{[n-1]}(\mathfrak{A})], \\ \delta^{(n)}(\mathfrak{A}) &:= [\delta^{(n-1)}(\mathfrak{A}), \delta^{(n-1)}(\mathfrak{A})]\mathfrak{A}. \end{aligned} \tag{3.7}$$

The associative $\mathbb{F}$-algebra $\mathfrak{A}$ is called *Lie solvable* (respectively *strongly Lie solvable*), if $\delta^{[n]}(\mathfrak{A}) = 0$ (respectively $\delta^{(n)}(\mathfrak{A}) = 0$) for some $n > 0$. The smallest such number n is called the *Lie derived length* (respectively *strong Lie derived length*) and will be denoted by $\mathrm{dl}_{\mathrm{Lie}}(\mathfrak{A})$ (respectively $\mathrm{dl}^{\mathrm{Lie}}(\mathfrak{A})$). Obviously, if $\mathfrak{A}$ is strongly Lie solvable, then $\mathfrak{A}$ is Lie solvable and $\mathrm{dl}_{\mathrm{Lie}}(\mathfrak{A}) \leqslant \mathrm{dl}^{\mathrm{Lie}}(\mathfrak{A})$.

Let $\mathfrak{L}$ be a finite-dimensional restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 0$. Under the assumption that $\mathbb{F}$ is of odd characteristic, Riley and Shalev proved in [**20**] that $\mathrm{u}(\mathfrak{L})$ is Lie solvable, if and only if $\mathfrak{L}'_p := [\mathfrak{L}, \mathfrak{L}]_p$ is p-nilpotent. In [**23**] it was shown that - without any restriction on the ground field - $\mathrm{u}(\mathfrak{L})$ is strongly Lie solvable, if and only if $\mathfrak{L}'_p$ is p-nilpotent. However, for such a restricted Lie algebra it can happen that $\mathrm{dl}_{\mathrm{Lie}}(\mathrm{u}(\mathfrak{L})) \neq \mathrm{dl}^{\mathrm{Lie}}(\mathrm{u}(\mathfrak{L}))$. Apart from the results in [**22, 23, 25**], very little is known about the Lie derived lengths of the $\mathbb{F}$-algebra $\mathrm{u}(\mathfrak{L})$. For powerful restricted Lie algebras one has the following property.

Proposition 3.2. *Let $\mathfrak{L}$ be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 0$. If $\mathfrak{L}$ is powerful, then*

$$\begin{aligned}(3.8)\qquad \min\left(\lceil \log_2(p^{e(\mathfrak{L}'_p)}+1)\rceil, p-1\right) &\leqslant \mathrm{dl}_{\mathrm{Lie}}(\mathrm{u}(\mathfrak{L})) \\ &\leqslant \mathrm{dl}^{\mathrm{Lie}}(\mathrm{u}(\mathfrak{L})) \leqslant \left\lceil \log_2(2 + d(\mathfrak{L}'_p)\cdot(p^{e(\mathfrak{L}'_p)}-1))\right\rceil.\end{aligned}$$

Proof: By [**23**, Lem.2] and Proposition 3.1, one has

$$\mathrm{dl}^{\mathrm{Lie}}(\mathrm{u}(\mathfrak{L})) \leqslant \lceil \log_2(2 + d(\mathfrak{L}'_p)\cdot(p^{e(\mathfrak{L}'_p)}-1))\rceil. \tag{3.9}$$

It remains to show, that if $\mathrm{dl}_{\mathrm{Lie}}(u(\mathfrak{L})) < p$, then $\mathrm{dl}_{\mathrm{Lie}}(u(\mathfrak{L})) \geqslant \lceil \log_2(p^{e(\mathfrak{L}'_p)}+1)\rceil$. If $\mathfrak{L}$ is Abelian, the claim is trivial. Assume that $\mathfrak{L}$ is non-Abelian, that is, $\mathrm{cl}(\mathfrak{L}) = 2$. By Theorem A, $\mathfrak{L}'_p$ is Abelian. Consequently, there exist two non-commuting elements $a, b \in \mathfrak{L}$, such that $z := [a,b]$ is of exponent $e(\mathfrak{L}'_p)$. We claim that $a^h z^{2^m-1}, b^k z^{2^m-1} \in \delta^{[m]}(\mathrm{u}(\mathfrak{L}))$ for every non-negative integer m and for every $0 \leqslant h, k \leqslant p-m-1$. We proceed by induction on m. For $m=0$, the claim is trivial. Assume that by induction, one has $a^{h+1}z^{2^{m-1}-1} \in \delta^{[m-1]}(\mathrm{u}(\mathfrak{L}))$ and $bz^{2^{m-1}-1} \in \delta^{[m-1]}(\mathrm{u}(\mathfrak{L}))$. As z centralises a and b, the Leibnitz rule implies that

$$[a^{h+1}, b] = \sum_{i=1}^{h+1} a^{i-1}[a,b]a^{h-i+1} = \sum_{i=1}^{h+1} a^h z = (h+1)a^h z. \tag{3.10}$$

In particular,

$$[a^{h+1}z^{2^{m-1}-1}, bz^{2^{m-1}-1}] = [a^{h+1}, b]z^{2^m-2} = (h+1)a^h z^{2^m-1}. \tag{3.11}$$

As $0 < h+1 < p$, one concludes that $a^h z^{2^m-1} \in \delta^{[m]}(\mathrm{u}(\mathfrak{L}))$, and a similar argument shows that $b^k z^{2^m-1} \in \delta^{[m]}(\mathrm{u}(\mathfrak{L}))$. This yields the claim. The Poincaré–Birkhoff–Witt theorem for restricted universal enveloping algebras (see [**26**, Chapter 2, Theorem 5.1]) implies that for $2^m - 1 < p^{e(\mathfrak{L}'_p)}$, the element z^{2^m-1} is non-trivial. The claim has shown that for $0 \leqslant m \leqslant p-1$ the element z^{2^m-1} is contained in $\delta^{[m]}(\mathrm{u}(\mathfrak{L}))$, completing the proof of the proposition. ∎

3.3. The Lie nilpotency class and the nilpotency class of the group of units. Let $\mathbb{F}$ be a field, and let $\mathfrak{A}$ be an associative $\mathbb{F}$-algebra with unit. One calls $\mathfrak{A}$ *Lie nilpotent*, if $\mathfrak{A}$ is nilpotent as $\mathbb{F}$-Lie algebra. In this case we denote by $\mathrm{cl}_{\mathrm{Lie}}(\mathfrak{A})$ the Lie nilpotency class of $\mathfrak{A}$. Put $\mathfrak{A}^{(1)} := \mathfrak{A}$ and $\mathfrak{A}^{(n+1)} = [\mathfrak{A}^{(n)}, \mathfrak{A}^{(n)}]\mathfrak{A}$, $n \geqslant 2$. One says that $\mathfrak{A}$ is *strongly Lie nilpotent*, if $\mathfrak{A}^{(n)} = 0$ for some n. In this case one calls the minimal non-negative integer $\mathrm{cl}^{\mathrm{Lie}}(\mathfrak{A}) := m$ satisfying $\mathfrak{A}^{(m+1)} = 0$ the *strong Lie nilpotency class* of $\mathfrak{A}$.

In [**20**], Riley and Shalev proved that if $\mathfrak{L}$ is a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 0$, then $\mathrm{u}(\mathfrak{L})$ is Lie nilpotent, if and only it is strongly Lie nilpotent. Moreover, this happens precisely when $\mathfrak{L}$ is nilpotent and $\mathfrak{L}'_p \in \mathfrak{F}_p$. They also showed that $\mathrm{cl}_{\mathrm{Lie}}(\mathrm{u}(\mathfrak{L})) = \mathrm{cl}^{\mathrm{Lie}}(\mathrm{u}(\mathfrak{L}))$ provided $p > 3$, while it is unknown whether this equality holds in the exceptional cases $p = 2, 3$ as well. In [**24**] it was shown, that if $\mathfrak{L} \in \mathfrak{F}_p$ and $\mathfrak{L}'_p$ is cyclic, then $\mathrm{cl}_{\mathrm{Lie}}(\mathrm{u}(\mathfrak{L})) = \mathrm{cl}^{\mathrm{Lie}}(\mathrm{u}(\mathfrak{L})) = p^{\dim_{\mathbb{F}} \mathfrak{L}'_p}$. Here we prove the following result:

PROPOSITION 3.3. *Let $\mathfrak{L} \in \mathfrak{F}_p$. If $\mathfrak{L}'_p$ is powerfully embedded in $\mathfrak{L}$, then*

$$p^{e(\mathfrak{L}'_p)} \leqslant \mathrm{cl}_{\mathrm{Lie}}(\mathrm{u}(\mathfrak{L})) \leqslant \mathrm{cl}^{\mathrm{Lie}}(\mathrm{u}(\mathfrak{L})) \leqslant 1 + d(\mathfrak{L}'_p) \cdot (p^{e(\mathfrak{L}'_p)} - 1). \tag{3.12}$$

PROOF: From [**24**, Theorem 1] and Theorem 2.6 it follows that $\mathrm{cl}_{\mathrm{Lie}}(\mathrm{u}(\mathfrak{L})) \geqslant p^{e(\mathfrak{L}'_p)}$. Consider the chain of restricted ideals of $\mathfrak{L}$ defined recursively by

$$\begin{aligned} &\mathfrak{D}_1(\mathfrak{L}) := \mathfrak{L}, \qquad \mathfrak{D}_2(\mathfrak{L}) := \mathfrak{L}'_p, \\ &\mathfrak{D}_{m+1}(\mathfrak{L}) := \langle \mathfrak{D}_{(\lceil (m+p)/p \rceil)}(\mathfrak{L})^{[p]} \rangle_p + [\mathfrak{D}_{(m)}(\mathfrak{L}), \mathfrak{L}], \qquad m \geqslant 2. \end{aligned} \tag{3.13}$$

According to [**21**], one has

$$\mathrm{cl}^{\mathrm{Lie}}(\mathrm{u}(\mathfrak{L})) = 1 + (p-1) \cdot \sum_{m \geqslant 1} m \cdot d_{(m+1)}, \tag{3.14}$$

where $d_{(m)} := \dim_{\mathbb{F}}(\mathfrak{D}_{(m)}(\mathfrak{L})/\mathfrak{D}_{(m+1)}(\mathfrak{L}))$. As $\mathfrak{L}'_p$ is powerfully embedded in $\mathfrak{L}$, Proposition 2.3 and Theorem 2.6 imply that for $n > 1$ one has

$$\mathfrak{D}_{(n)}(\mathfrak{L}) = (\mathfrak{L}'_p)^{[p]^{h(n)}}, \tag{3.15}$$

where $h(n) := \lceil \log_p(n-1) \rceil$. From this identity one concludes that for $n \geqslant 2$

$$d_{(n)} = \begin{cases} \dim_{\mathbb{F}}((\mathfrak{L}'_p)^{[p]^i}/(\mathfrak{L}'_p)^{[p]^{i+1}}) & \text{if } n = p^i + 1 \text{ with } 0 \leqslant i < e(\mathfrak{L}'_p), \\ 0 & \text{otherwise.} \end{cases} \tag{3.16}$$

From formula (3.14) one deduces that

$$\mathrm{cl}^{\mathrm{Lie}}(\mathrm{u}(\mathfrak{L})) = 1 + (p-1) \cdot \sum_{n=0}^{e(\mathfrak{L}'_p)-1} p^n \cdot \dim_{\mathbb{F}}((\mathfrak{L}'_p)^{[p]^n}/(\mathfrak{L}'_p)^{[p]^{n+1}}). \tag{3.17}$$

As in Proposition 3.1, this yields $\mathrm{cl}^{\mathrm{Lie}}(\mathrm{u}(\mathfrak{L})) \leqslant 1 + d(\mathfrak{L}'_p) \cdot (p^{e(\mathfrak{L}'_p)} - 1)$. ∎

For an associative $\mathbb{F}$-algebra $\mathfrak{A}$ with unit, we denote by $\mathfrak{A}^*$ the group of units of $\mathfrak{A}$. Let $\mathrm{cl}(G)$ denote the nilpotency class of the nilpotent group G. If $\mathfrak{L} \in \mathfrak{F}_p$, then $\omega(\mathfrak{L})$ is nilpotent and $\mathrm{u}(\mathfrak{L})^* = \mathbb{F}^* \times (1 + \omega(\mathfrak{L}))$. Hence, $\mathrm{u}(\mathfrak{L})^*$ is nilpotent and $\mathrm{cl}(\mathrm{u}(\mathfrak{L})^*) = \mathrm{cl}(1 + \omega(\mathfrak{L}))$. According to a result of Du (see [**8**]), if an associative $\mathbb{F}$-algebra $\mathfrak{T}$ is radical, that is, $\mathfrak{T}$ coincides with its Jacobson radical, and Lie nilpotent, then $\mathrm{cl}_{\mathrm{Lie}}(\mathfrak{T})$ coincides with the nilpotency class of the adjoint group $\mathfrak{T}^\circ = 1 + \mathfrak{T}$. As a consequence one obtains the following:

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COROLLARY 3.4. *Let $\mathfrak{L} \in \mathfrak{F}_p$. If $\mathfrak{L}'_p$ is powerfully embedded in $\mathfrak{L}$, then*

$$p^{e(\mathfrak{L}'_p)} \leqslant \mathrm{cl}(\mathrm{u}(\mathfrak{L})^*) \leqslant 1 + d(\mathfrak{L}'_p) \cdot (p^{e(\mathfrak{L}'_p)} - 1). \tag{3.18}$$

4. COHOMOLOGY FOR RESTRICTED LIE ALGEBRAS

Let $\mathfrak{L}$ be a restricted Lie algebra and let $\mathrm{u}(\mathfrak{L})$ denote its restricted universal enveloping $\mathbb{F}$-algebra. The k^{th}*-cohomology group* with coefficients in the left $\mathfrak{L}$-module M is given by

$$H^k(\mathfrak{L}, M) \colon = \mathrm{Ext}^k_{\mathrm{u}(\mathfrak{L})}(\mathbb{F}, M). \tag{4.1}$$

where $\mathbb{F}$ denotes the trivial left $\mathfrak{L}$-module. Cup-product

$$\_\cup\_ \colon H^\bullet(\mathfrak{L}, \mathbb{F}) \times H^\bullet(\mathfrak{L}, \mathbb{F}) \longrightarrow H^\bullet(\mathfrak{L}, \mathbb{F}), \tag{4.2}$$

which coincides with the Yoneda composition of Ext-groups, gives $H^\bullet(\mathfrak{L}, \mathbb{F})$ naturally the structure of a graded commutative $\mathbb{F}$-algebra. Moreover, every homomorphism $\phi \colon \mathfrak{L} \to \mathfrak{M}$ induces a homomorphism of graded commutative $\mathbb{F}$-algebras $\phi^\bullet \colon H^\bullet(\mathfrak{M}, \mathbb{F}) \to H^\bullet(\mathfrak{L}, \mathbb{F})$. The *reduced cohomology* $\mathbb{F}$*-algebra* of the restricted Lie algebra $\mathfrak{L}$ is given by

$$H^\bullet(\mathfrak{L}, \mathbb{F})_{\mathrm{red}} \colon = H^\bullet(\mathfrak{L}, \mathbb{F}) / \operatorname{nil}\big(H^\bullet(\mathfrak{L}, \mathbb{F})\big), \tag{4.3}$$

where $\operatorname{nil}\big(H^\bullet(\mathfrak{L}, \mathbb{F})\big)$ denotes the graded ideal of nilpotent elements of the graded $\mathbb{F}$-algebra $H^\bullet(\mathfrak{L}, \mathbb{F})$. Certainly, one of the most striking result on the cohomology of finite-dimensional restricted Lie algebras is the theorem of Jantzen (see [**11**]). It states that if $\mathbb{F}$ is an algebraically closed field of characteristic p, p odd, then $H^\bullet(\mathfrak{L}, \mathbb{F})_{\mathrm{red}}$ can be identified with the rational functions on the algebraic set $\mathfrak{L}_{[p]} = \{\, x \in \mathfrak{L} \mid x^{[p]} = 0 \,\}$ generated as $\mathbb{F}$-algebra in degree 2. One can think of this theorem as the analogue of Quillen's theorem which describes $H^\bullet(G, \mathbb{F}_p)$ of a finite group G up to F-isomorphism (see [**18**]).

4.1. POWERFUL RESTRICTED LIE ALGEBRAS. If p is odd, one can characterise powerful restricted Lie algebras in the class $\mathfrak{F}_p$ in the same way as powerful pro-p groups (see [**27**, Theorem 5.1.6]).

THEOREM 4.1. *Let p be odd and let $\mathfrak{L} \in \mathfrak{F}_p$. Then the following are equivalent:*

(i) *$\mathfrak{L}$ is powerful.*

(ii) *The mapping $\beta_{\mathfrak{L}} \colon H^1(\mathfrak{L}, \mathbb{F}) \wedge H^1(\mathfrak{L}, \mathbb{F}) \to H^2(\mathfrak{L}, F)$ induced by cup-product is injective.*

The proof of Theorem 4.1 makes use of the following simple fact.

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FACT 4.2. Let p be odd, and let $\mathfrak{A}$ be a finite-dimensional Abelian restricted Lie algebra with trivial p-map. Let $\eta \in H^2(\mathfrak{A}, F)$ and let

$$\mathsf{s}_\eta\colon\quad 0 \longrightarrow \mathbb{F} \longrightarrow \mathfrak{A}_\eta \xrightarrow{\tau_\eta} \mathfrak{A} \longrightarrow 0 \tag{4.4}$$

denote the corresponding short exact sequence of restricted Lie algebras (see [**10**]). Then one has $\mathfrak{A}_\eta^{[p]} = 0$, if and only if $\eta \in \operatorname{im}(\beta_{\mathfrak{A}})$.

PROOF: As p is odd, the p-map on $\mathfrak{A}_\eta$ induces a p-semilinear map $\psi(\eta) \in \operatorname{Hom}^p_{\mathbb{F}}(\mathfrak{A}, \mathbb{F})$ of degree p. This yields a short exact sequence

$$0 \longrightarrow H^1(\mathfrak{A}, \mathbb{F}) \wedge H^1(\mathfrak{A}, \mathbb{F}) \xrightarrow{\beta_{\mathfrak{A}}} H^2(\mathfrak{A}, \mathbb{F}) \xrightarrow{\psi} \operatorname{Hom}^p_{\mathbb{F}}(\mathfrak{A}, \mathbb{F}) \longrightarrow 0, \tag{4.5}$$

which implies the claim. ∎

PROOF: [Proof of Theorem 4.1] Let $\pi\colon \mathfrak{L} \to \mathfrak{A}$, $\mathfrak{A}\colon = \mathfrak{L}/\Phi(\mathfrak{L})$, denote the canonical projection. One has a commutative diagram

$$\begin{array}{ccc} H^1(\mathfrak{A}, \mathbb{F}) \wedge H^1(\mathfrak{A}, \mathbb{F}) & \xrightarrow{\beta_{\mathfrak{A}}} & H^2(\mathfrak{A}, \mathbb{F}) \\ \downarrow{\scriptstyle \pi^1 \wedge \pi^1} & & \downarrow{\scriptstyle \pi^2} \\ H^1(\mathfrak{L}, \mathbb{F}) \wedge H^1(\mathfrak{L}, \mathbb{F}) & \xrightarrow{\beta_{\mathfrak{L}}} & H^2(\mathfrak{L}, \mathbb{F}). \end{array} \tag{4.6}$$

Moreover, $\pi^1 \wedge \pi^1$ is an isomorphism, and $\beta_{\mathfrak{A}}$ is injective. For $\eta \in H^2(\mathfrak{A}, \mathbb{F})$, let $\mathfrak{L}_\eta$ denote the pull back of the mappings $\mathfrak{L} \to \mathfrak{A} \leftarrow \mathfrak{A}_\eta$, that is, one has a commutative diagram

$$\begin{array}{lccccccccc} \mathsf{s}'_\eta\colon & 0 & \longrightarrow & \mathbb{F} & \longrightarrow & \mathfrak{L}_\eta & \longrightarrow & \mathfrak{L} & \longrightarrow & 0 \\ & & & \| & & \downarrow & \swarrow{\scriptstyle \alpha} & \downarrow{\scriptstyle \pi} & & \\ \mathsf{s}_\eta\colon & 0 & \longrightarrow & \mathbb{F} & \longrightarrow & \mathfrak{A}_\eta & \xrightarrow{\tau_\eta} & \mathfrak{A} & \longrightarrow & 0 \end{array} \tag{4.7}$$

If s'_η is split, there exists a mapping α making the diagram (4.7) commute. On the other hand, $\mathfrak{L}_\eta$ is the pull back of the mappings π and τ_n. Hence the existence of the mapping α in (4.7) implies that s'_η is split. If $\eta \neq 0$, s_η is a Frattini extension (see Proposition 2.1(e)), and therefore, a mapping α making (4.7) commute must be surjective (see Proposition 2.1(b)).

Let $\mathfrak{L}$ be powerful. Let $\xi' \in H^1(\mathfrak{L}, \mathbb{F}) \wedge H^1(\mathfrak{L}, \mathbb{F})$, $\xi' \neq 0$, and assume that $\beta_{\mathfrak{L}}(\xi') = 0$. Let $\xi \in H^1(\mathfrak{A}, \mathbb{F}) \wedge H^1(\mathfrak{A}, \mathbb{F})$ such that $(\pi^1 \wedge \pi^1)(\xi) = \xi'$. Hence $\beta_{\mathfrak{A}}(\xi) \neq 0$ and $\mathsf{s}_{\beta_{\mathfrak{A}}(\xi)}$ is a Frattini extension. The commutativity of the diagram (4.6) and the previously mentioned remark imply that there exists a surjective map $\alpha\colon \mathfrak{L} \to \mathfrak{A}_{\beta_{\mathfrak{A}}(\xi)}$ making the diagram (4.7) commute for $\eta\colon = \beta_{\mathfrak{A}}(\xi)$. However, by Fact 4.2, one has $\mathfrak{A}^{[p]}_{\beta_{\mathfrak{A}}(\xi)} = 0$. Hence $\mathfrak{A}_{\beta_{\mathfrak{A}}(\xi)}$ is not powerful. On the other hand, as a homomorphic image of $\mathfrak{L}$ the restricted Lie algebra $\mathfrak{A}_{\beta_{\mathfrak{A}}(\xi)}$ must be powerful, a contradiction. This yields the implication (i)⇒(ii).

Let $\beta_{\mathfrak{L}}$ be injective, and assume that $\mathfrak{L}$ is not powerful. Hence there exists a restricted ideal J of $\mathfrak{L}$ such that $\mathfrak{L}^{[p]} \subseteq J \subsetneq \Phi(\mathfrak{L})$ and J has codimension 1 in $\Phi(\mathfrak{L})$ (see Proposition

2.9). Let $\mathfrak{H}\colon = \mathfrak{L}/J$ and let

$$\text{(4.8)} \qquad \mathsf{s}\colon \quad 0 \longrightarrow \mathbb{F} \longrightarrow \mathfrak{H} \xrightarrow{\sigma} \mathfrak{A} \longrightarrow 0$$

denote the canonical short exact sequence. By construction, s is non-split and $\mathfrak{H}^{[p]} = 0$. Hence by Fact 4.2, there exists an element $\xi \in H^1(\mathfrak{A},\mathbb{F}) \wedge H^1(\mathfrak{A},\mathbb{F})$, $\xi \neq 0$, such that $\mathsf{s} = \mathsf{s}_{\beta_{\mathfrak{A}}(\xi)}$. From the commutative diagram

$$\text{(4.9)} \qquad \begin{array}{lccccccccc} \mathsf{s}'_{\beta_{\mathfrak{A}}(\xi)}\colon & 0 & \longrightarrow & \mathbb{F} & \longrightarrow & \mathfrak{L}_{\beta_{\mathfrak{A}}(\xi)} & \longrightarrow & \mathfrak{L} & \longrightarrow & 0 \\ & & & \| & & \downarrow & \overset{\alpha}{\swarrow} & \downarrow \pi & & \\ \mathsf{s}_{\beta_{\mathfrak{A}}(\xi)}\colon & 0 & \longrightarrow & \mathbb{F} & \longrightarrow & \mathfrak{H} & \xrightarrow{\sigma} & \mathfrak{A} & \longrightarrow & 0 \end{array}$$

one concludes that $\beta_{\mathfrak{L}}\big((\pi^1 \wedge \pi^1)(\xi)\big) = 0$. Hence $\beta_{\mathfrak{L}}$ is not injective, a contradiction, and this completes the proof of the theorem. □

4.2. COHOMOLOGY FOR p-CENTRAL RESTRICTED LIE ALGEBRAS. Let $\mathfrak{L}$ be a finite-dimensional p-central restricted Lie algebra. For such a restricted Lie algebra one has a surjective homomorphism

$$\text{(4.10)} \qquad \rho\colon \mathfrak{L}_{[p]} \oplus \mathfrak{L} \longrightarrow \mathfrak{L}, \quad \rho(z,x)\colon = z + x.$$

Applying Künneth' theorem one obtains a mapping

$$\text{(4.11)} \qquad \Delta_{\mathfrak{L}}\colon = (\mathrm{red} \otimes \mathrm{id}) \circ \rho^\bullet\colon H^\bullet(\mathfrak{L},\mathbb{F}) \longrightarrow H^\bullet(\mathfrak{L}_{[p]},\mathbb{F})_{\mathrm{red}} \otimes H^\bullet(\mathfrak{L},\mathbb{F}),$$

which gives $H^\bullet(\mathfrak{L},\mathbb{F})$ the structure of a left $H^\bullet(\mathfrak{L}_{[p]},\mathbb{F})_{\mathrm{red}}$-comodule algebra. The Hopf algebra structure on $H^\bullet(\mathfrak{L}_{[p]},\mathbb{F})_{\mathrm{red}}$ is induced by the mapping $\Delta_{\mathfrak{L}_{[p]}}$ (see [**17**]). Using this additional structure one deduces the following.

THEOREM 4.3. *Let p be odd and let $\mathfrak{L}$ be a finite-dimensional restricted Lie algebra. Then one has an isomorphism of graded commutative $\mathbb{F}$-algebras*

$$\text{(4.12)} \qquad H^\bullet(\mathfrak{L},\mathbb{F}) \simeq C^\bullet \otimes S^\bullet(\mathfrak{L}^*_{[p]})$$

where $S^\bullet(\mathfrak{L}^*_{[p]})$ is generated in degree 2 and $C^\bullet$ is a finite-dimensional graded commutative $\mathbb{F}$-algebra. In particular, $H^\bullet(\mathfrak{L},\mathbb{F})$ is a graded commutative Cohen–Macaulay $\mathbb{F}$-algebra.

PROOF: Let $\iota\colon \mathfrak{L}_{[p]} \to \mathfrak{L}$ denote the canonical map. The theorem of Jantzen implies that the reduced restriction map

$$\text{(4.13)} \qquad j^\bullet\colon = \mathrm{red} \circ \iota^\bullet\colon H^\bullet(\mathfrak{L},\mathbb{F}) \longrightarrow H^\bullet(\mathfrak{L}_{[p]},\mathbb{F})_{\mathrm{red}}$$

is surjective. Thus [**28**, Theorem 3.1] implies that one has an isomorphism of $\mathbb{F}$-algebras

$$\text{(4.14)} \qquad H^\bullet(\mathfrak{L},\mathbb{F}) \simeq C^\bullet \otimes H^\bullet(\mathfrak{L}_{[p]},\mathbb{F})_{\mathrm{red}},$$

where $C^\bullet := \mathbb{F}\Box_{H^\bullet(\mathfrak{L}_{[p]},\mathbb{F})_{\mathrm{red}}} H^\bullet(\mathfrak{L},\mathbb{F})$ and $\Box$ denotes the cotensor product (see [**17**]). Moreover, since $H^\bullet(\mathfrak{L},\mathbb{F})$ is a finitely generated graded commutative $\mathbb{F}$-algebra (see [**12**, Section 1.11, Proposition]), $C^\bullet$ is also finitely generated. By Jantzen's theorem,

$$\iota^\bullet_{\mathrm{red}}\colon H^\bullet(\mathfrak{L},\mathbb{F})_{\mathrm{red}} \longrightarrow H^\bullet(\mathfrak{L}_{[p]},\mathbb{F})_{\mathrm{red}} \tag{4.15}$$

is an isomorphism. This implies that the augmentation ideal $\omega(C^\bullet)$ of $C^\bullet$ consists entirely of nilpotent elements. In particular, $C^\bullet$ is finite-dimensional, and $H^\bullet(\mathfrak{L},\mathbb{F})$ is a graded commutative Cohen–Macaulay $\mathbb{F}$-algebra. ∎

Let $B^\bullet$ be a graded commutative $\mathbb{F}$-algebra. Then $B^\bullet$ is said to *satisfy Poincaré duality in dimension* n, if $\dim_{\mathbb{F}}(B^n) = 1$, $B^{n+j} = 0$ for all $j > 0$, and if for all $k \in \{0,\dots,n\}$ multiplication induces a non-degenerate pairing $B^k \otimes B^{n-k} \to B^n$. In [3], Benson and Carlson developed a method for studying the cohomology ring $H^\bullet(G,\mathbb{F}_p)$ for a finite group G provided one knows that $H^\bullet(G,\mathbb{F}_p)$ is a Cohen–Macaulay $\mathbb{F}$-algebra and p is odd. Their main result can be summarised as follows:

Theorem 4.4. ([**3**, Theorem 6.3]) *Let $\mathbb{F}$ be a field of characteristic $p \neq 2$, and let $\mathbf{A}$ be a finite-dimensional cocommutative $\mathbb{F}$-Hopf algebra such that*

(i) *$\mathbf{A}$ is a Frobenius algebra.*

(ii) *$H^\bullet(\mathbf{A},\mathbb{F})$ is a finitely generated Cohen–Macaulay $\mathbb{F}$-algebra.*

Let $\xi_1,\dots,\xi_n$ be a homogeneous system of parameters of degree $s_1,\dots,s_n$, $s_i \geqslant 2$. Then $C^\bullet := H^\bullet(\mathbf{A},\mathbb{F})/\langle\xi_1,\dots,\xi_n\rangle$ satisfies Poincaré duality in dimension $s := \sum_{i=1}^{n}(s_i-1)$.

Proof: The cocommutativity of the Hopf algebra $\mathbf{A}$ ensures that for left $\mathbf{A}$-modules M and N, the tensor product $M \otimes_{\mathbb{F}} N$ is a projective left $\mathbf{A}$-module whenever one of the factors is projective. The property of being a Frobenius algebra implies that the left regular $\mathbf{A}$-module $\mathbf{A}$ is also injective (see [**1**, Proposition 1.6.2]). Therefore one can transfer the proof of [**3**, Theorem 6.3] ad verbatim. ∎

It is well-known that for a finite-dimensional restricted Lie algebra $\mathfrak{L}$, the restricted universal enveloping algebra $\mathfrak{u}(\mathfrak{L})$ is a Frobenius algebra (see [**4**]). Moreover, if p is odd and $\mathfrak{L}$ is a p-central restricted Lie algebra, Theorem 4.3 has shown that the cohomology ring $H^\bullet(\mathfrak{L},\mathbb{F})$ is a Cohen–Macaulay algebra with a homogeneous system of parameters $\xi_1,\dots,\xi_n$ all of degree 2, where $n := \dim_{\mathbb{F}}(\mathfrak{L}_{[p]})$. Hence from Theorem 4.4 one obtains:

Corollary 4.5. *The finite-dimensional $\mathbb{F}$-algebra $C^\bullet$ of Theorem 4.3 satisfies Poincaré duality in dimension $n := \dim_{\mathbb{F}}(\mathfrak{L}_{[p]})$.*

Remark 4.6. Let $\mathfrak{L}$ be a finite-dimensional p-central restricted Lie algebra, and let

$$h_{\mathfrak{L}}(t) := \sum_{k\in\mathbb{N}_0} \dim_{\mathbb{F}}\big(H^k(\mathfrak{L},\mathbb{F})\big)\cdot t^k \tag{4.16}$$

denote the *Hilbert series* of its cohomology algebra $H^\bullet(\mathfrak{L},\mathbb{F})$. One has a multiplicative decomposition $h_{\mathfrak{L}}(t) = c(t)\cdot(1-t^2)^{-n}$, where $c(t)$ denotes the Hilbert series of $C^\bullet$ and

$n := \dim_{\mathbb{F}}(\mathfrak{L}_{[p]})$. The Poincaré duality of $C^\bullet$ implies that $c(t) = t^n \cdot c(1/t)$. Hence $h_{\mathfrak{L}}(t)$ satisfies the functional equation

$$h_{\mathfrak{L}}(1/t) = (-t)^{\dim_{\mathbb{F}}(\mathfrak{L}_{[p]})} \cdot h_{\mathfrak{L}}(t). \tag{4.17}$$

The analogous functional equation also holds for p-central groups. Let G be a finite p-central group, that is, $\Omega_1(G) := \{ g \in G \mid g^p = 1 \} \leqslant Z(G)$, and let

$$h_G(t) := \sum_{k \in \mathbb{N}_0} \dim_{\mathbb{F}_p}\big(H^k(G, \mathbb{F}_p)\big) \cdot t^k \tag{4.18}$$

denote the Hilbert series of the mod p cohomology ring of G. By [**6**], $H^\bullet(G, \mathbb{F}_p)$ is a Cohen–Macaulay $\mathbb{F}_p$-algebra, and thus by [**2**, Theorem 5.18.1]),

$$h_G(1/t) = (-t)^{\dim_{\mathbb{F}_p}(\Omega_1(G))} \cdot h_G(t). \tag{4.19}$$

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