ON A SUBLATTICE OF THE LATTICE OF CONGRUENCES ON A SIMPLE REGULAR ω-SEMIGROUP

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The set E of idempotents of a semigroup S can be partially ordered by defining $e \leqq f$ if and only if $ef = fe = e$ $(e, f \in E)$. If $E = \{e_i : i = i = 0, 1, \cdots\}$ and under this ordering

$$e_0 > e_1 > e_2 \cdots,$$

then we call S an ω-semigroup. Munn [10] has given a complete classification of simple regular ω-semigroups in terms of groups and group homomorphisms. Let $\Lambda_0(S)$ denote the set of congruences on a simple regular ω-semigroup S consisting of those congruences which either are idempotent-separating or are group congruences on S. It is evident that $\Lambda_0(S)$ is a sublattice of the lattice of all congruences on S.

In this paper we determine a necessary and sufficient condition for the sublattice $\Lambda_0(S)$ to be modular.

If we further restrict S and insist that it be bisimple then $\Lambda_0(S)$ becomes the full lattice of congruences on S (Munn [9]). For this case Munn [9] has determined a necessary and sufficient condition for $\Lambda_0(S)$ to be modular. Our work generalizes Munn's theorem from bisimple ω-semigroups to simple regular ω-semigroups. Many of the results in this paper are straightforward generalizations of results given by Munn for the bisimple case. Whenever possible Munn's results are used in obtaining our generalizations.

For notation and definitions not given in this paper the reader is referred to Clifford and Preston [1] and [2].

1. Preliminary results

Following Munn [10], let d be a positive integer and let $\{G_i : i = 0, 1, \cdots, d-1\}$ be a set of pairwise disjoint groups. Let γ_{d-1} be a homomorphism of G_{d-1} into G_0 and let γ_i be a homomorphism of G_i into G_{i+1} $(i = 0, 1, \cdots, d-2)$. Thus

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we have a sequence

$$G_0 \xrightarrow{\gamma_0} G_1 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_{d-2}} G_{d-1} \xrightarrow{\gamma_{d-1}} G_0 .$$

Denote by N the set of non-negative integers. For $n \in N$ denote by $n \pmod d$ the integer equivalent to n modulo d, belonging to N, and less than d. Define $\gamma_n = \gamma_{n(\mathrm{mod}\, d)}$ for $n \in N$. For $(m, n) \in N \times N$ and $m < n$ write

$$\alpha_{m,n} = \gamma_m \gamma_{m+1} \cdots \gamma_{n-1},$$

and for all $n \in N$ let $\alpha_{n,n}$ denote the identity automorphism of $G_{n(\mathrm{mod}\, d)}$. Let S be the set of all ordered triples (m, a_i, n), where $m, n \in N$, $0 \leqq i \leqq d-1$ and $a_i \in G_i$. Define a multiplication in S as follows:

$$(m, a_i, n) \cdot (p, b_j, q) = (m+p-p \wedge n, (a_i \alpha_{u,w})(b_j \alpha_{v,w}), q+n-p \wedge n),$$

where $p \wedge n = \min \{p, n\}$, $u = nd+i$, $v = pd+j$ and $w = \max \{u, v\}$. Denote the groupoid so formed by $S(d; G_0, \cdots, G_{d-1}; \gamma_0, \cdots, \gamma_{d-1})$ or, more compactly, by $S(d; G_i; \gamma_i)$. Then, as was shown in [10], $S(d; G_i; \gamma_i)$ is a simple regular ω-semigroup and any simple regular ω-semigroup is isomorphic to a semigroup $S(d; G_i; \gamma_i)$.

For $0 \leqq i \leqq d-1$ put $S_i = \{(m, a_i, n) : m, n \in N, a_i \in G_i\}$. S_i is a bisimple subsemigroup of S; further

$$S = \bigcup_{0 \leqq i \leqq d-1} S_i .$$

It is evident that $\alpha_{i, i+d}$ is an endomorphism of G_i. In the terminology of Reilly [11], $S_i = S_i(G_i, \alpha_{i, i+d})$.

For $n \in N$ and $i = 0, 1, \cdots, d-1$ write $e_i^n = (n, e_i, n)$, where e_i is the identity of the group G_i. The elements e_i^n are the idempotents of $S(d; G_i; \gamma_i)$ and we have

$$e_0^0 > e_1^0 > \cdots > e_{d-1}^0 > e_0^1 > e_1^1 > \cdots > e_{d-1}^1 > e_0^2 > \cdots .$$

The semigroup $S(d; G_i; \gamma_i)$ is in fact an inverse semigroup with identity e_0^0. Further, $(m, a_i, n)^{-1} = (n, a_i^{-1}, m)$.

Put $B = \{(m, e_i, n) : m, n \in N, 0 \leqq i \leqq d-1\}$. B is a subsemigroup of $S(d; G_i, \gamma_i)$. We note that B is uniquely determined by the number d. When $d = 1$, B becomes the bicyclic semigroup.

A congruence ρ on a semigroup S is called idempotent-separating if each congruence class contains at most one idempotent of S. Lallement [5] has proved that a congruence on a regular semigroup is idempotent-separating if and only if it is contained in Green's equivalence $\mathscr{H}$. From the definition of multiplication in $S(d; G_i; \gamma_i)$ it is easy to show that the equivalence $\mathscr{H}$ on a simple regular ω-semigroup is given by:

$$((m, a_i, n), (p, b_j, q)) \in \mathscr{H} \text{ if and only if } i = j,\ m = p \text{ and } n = q.$$

This result will be used frequently. In fact $\mathscr{H}$ is a congruence, as we now prove.

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LEMMA 1.1. (cf. Munn [10], Theorem 2.1). *Let* $S = S(d; G_i; \gamma_i)$. *Then* $\mathscr{H}$ *is a congruence on* S *and* $S/\mathscr{H} \cong B$.

PROOF. The mapping θ of S onto B defined by $(m, a_i, n)\theta = (m, e_i, n)$ is a homomorphism. Further $((m, a_i, n), (p, b_j, q)) \in \mathscr{H}$ if and only if $(m, e_i, n) = (p, e_j, q)$; hence $\theta \circ \theta^{-1} = \mathscr{H}$ and the result follows.

A congruence ρ on a semigroup S is called a group congruence if S/ρ is a group. The following lemma provides a characterization of the minimum group congruence σ on an inverse semigroup.

LEMMA 1.2. (Munn [6], Theorem 1). *Let* S *be an inverse semigroup and let a relation* σ *be defined on* S *by the rule that* $(x, y) \in \sigma$ *if and only if* $ex = ey$ *for some idempotent* e *in* S (*or, equivalently, if and only if* $xf = yf$ *for some idempotent* f *in* S). *Then* σ *is a group congruence on* S. *Furthermore, a congruence* ρ *on* S *is a group congruence if and only if* $\sigma \subseteq \rho$ *and so* S/ρ *is isomorphic with some quotient group of* S/σ.

Let $\Lambda_0(S)$ denote the set of congruences on a simple regular ω-semigroup S consisting of those congruences which either are idempotent-separating or are group congruences on S.

Evidently $\Lambda_0(S)$ is a sublattice of the lattice $\Lambda(S)$ of all congruences on S. For $\alpha, \beta \in \Lambda(S)$ and $\alpha \subseteq \beta$, define

$$[\alpha, \beta] = \{\lambda \in \Lambda(S) : \alpha \subseteq \lambda \subseteq \beta\};$$

$[\alpha, \beta]$ is a sublattice of $\Lambda(S)$. Then $[i, \mathscr{H}]$ is the set of all idempotent-separating congruences on S and $[\sigma, S \times S]$ is the set of all group congruences on S. Thus

$$\Lambda_0(S) = [i, \mathscr{H}] \cup [\sigma, S \times S].$$

A lattice Π is modular if and only if for any elements $\alpha, \beta, \gamma \in \Pi$,

$$\alpha \leqq \beta \text{ implies } \alpha \vee (\beta \wedge \gamma) = \beta \wedge (\alpha \vee \gamma).$$

It is well known that the lattice of normal subgroups of a group is modular (see for instance Hall [3]); hence $[\sigma, S \times S]$ is modular since it is isomorphic to the lattice of congruences on the group S/σ. Munn [7] has proved that for regular semigroups the lattice of all congruences contained in Green's equivalence $\mathscr{H}$ is modular. Hence $[i, \mathscr{H}]$ is modular,

2. Idempotent-separating congruences

Let $S = S(d; G_i; \gamma_i)$ be a simple regular ω-semigroup. Put $G = G_0 \times G_1 \times \cdots \times G_{d-1}$, the cartesian product of G_i, $i = 0, 1, \cdots, d-1$. A subset A of G will be called γ-*admissible* if it satisfies the following three conditions:

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(i) $A = A_0 \times A_1 \times \cdots \times A_{d-1}$, for some $A_i \subseteq G_i$, $i = 0, 1, \cdots, d-1$,

(ii) $A_i \trianglelefteq G_i$, for $i = 0, 1, \cdots, d-1$, and

(iii) $A_{d-1}\gamma_{d-1} \subseteq A_0$ and $A_i\gamma_i \subseteq A_{i+1}$, for $i = 0, 1, \cdots, d-2$.

A subset A of G will be called *normal* if it satisfies (i) and (ii) above.

We have already noted that $\alpha_{i,\,i+d}$ is an endomorphism of G_i, for $i = 0, 1, \cdots, d-1$. Munn [9] defines a subgroup A_i of G_i to be $\alpha_{i,\,i+d}$*-admissible* if (i) $A_i \trianglelefteq G_i$ and (ii) $A_i\alpha_{i,\,i+d} \subseteq A_i$. Suppose that $A = A_0 \times A_1 \times \cdots \times A_{d-1}$ is a γ-admissible subset of G. Then it is easily verified that A_i is $\alpha_{i,\,i+d}$-admissible for $i = 0, 1, \cdots, d-1$.

If $A = A_0 \times A_1 \times \cdots \times A_{d-1}$ and $B = B_0 \times B_1 \times \cdots \times B_{d-1}$ are normal subsets of G, we define

$$A \vee B = A_0B_0 \times A_1B_1 \times \cdots \times A_{d-1}B_{d-1} \quad \text{and}$$
$$A \wedge B = A_0 \cap B_0 \times A_1 \cap B_1 \times \cdots \times A_{d-1} \cap B_{d-1}.$$

We shall often write $AB = A \vee B$. It is clear that AB and $A \wedge B$ are again normal subsets of G. Let Γ denote the set of γ-admissible subsets of G. It is easily checked that the property (iii) above is preserved under the operations of union ($\vee$) and intersection ($\wedge$). Thus Γ is a sublattice of the lattice of normal subsets of G.

Since the lattice of normal subgroups of a group is modular it follows that the lattice of normal subsets of G is modular, since the direct product of modular lattices is modular. Further, a sublattice of a modular lattice is modular and hence the lattice Γ of γ-admissible subsets of G is modular.

For any congruence λ on S we define *a subset A^λ of G as follows*:

$$A^\lambda = A_0^\lambda \times A_1^\lambda \times \cdots \times A_{d-1}^\lambda, \quad \text{where}$$
$$A_i^\lambda = \{a_i \in G_i : (0, a_i, 0) \in e_i^0\lambda\}, \; i = 0, 1, \cdots, d-1.$$

Evidently $A^\lambda = A^{\lambda \wedge \mathscr{H}}$, since the $\mathscr{H}$-class containing e_i^0 is $\{(0, a_i, 0) \in S : a_i \in G_i\}$.

LEMMA 2.1. *For any congruence λ on $S(d; G_i; \lambda_i)$ A^λ is a γ-admissible subset of G.*

PROOF. Put $H_i = \{(0, a_i, 0) \in S : a_i \in G_i\}$; H_i is a subgroup of S. Set $\lambda_i = \lambda \cap (H_i \times H_i)$. Then λ_i is a congruence on H_i and so $e_i^0\lambda$ is a normal subgroup of H_i. Now G_i is isomorphic to H_i under the mapping $a_i \to (0, a_i, 0)$ and it follows that A_i^λ is a normal subgroup of G_i. Thus A_i^λ is a normal subgroup of G_i ($i = 0, 1, \cdots, d-1$), since our argument is independent of i.

Suppose $a_i \in A_i^\lambda$ and assume $i \neq d-1$. Put $x = (0, a_i, 0)$. Then $(x, e_i^0) \in \lambda$ and so

$$(xe_{i+1}^0, e_i^0e_{i+1}^0) \in \lambda,$$

that is,

$$((0, a_i\gamma_i, 0), e_{i+1}^0) \in \lambda;$$

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hence $a_i\gamma_i \in A_{i+1}^{\lambda}$. On the other hand, if $i = d-1$, put $x = (0, a_{d-1}, 0)$ and $y = (0, e_0, 1)$. Then $(x, e_{d-1}^0) \in \lambda$ and so

$$(yxy^{-1}, ye_{d-1}^0 y^{-1}) \in \lambda.$$

But

$$yxy^{-1} = (0, a_{d-1}\gamma_{d-1}, 0)$$

and $ye_{d-1}^0 y^{-1} = e_0^0$. Hence $a_{d-1}\gamma_{d-1} \in A_0^{\lambda}$. Thus A^{λ} is γ-admissible.

LEMMA 2.2. (i) *Let λ be an idempotent-separating congruence on $S(d; G_i; \lambda_i)$. Then*

$$((m, a_i, n), (p, b_j, q)) \in \lambda \text{ if and only if } i = j, m = p, n = q \text{ and } a_i b_j^{-1} \in A_i^{\lambda}.$$

(ii) *Let $A = A_0 \times A_1 \times \cdots \times A_{d-1}$ be a γ-admissible subset of G. Then the relation*

$$\lambda = \{((m, a_i, n), (p, b_j, q)) \in S \times S : i = j, m = p, n = q \text{ and } a_i b_j^{-1} \in A_i\}$$

is an idempotent-separating congruence on $S(d; G_i; \gamma_i)$. Furthermore, $A^{\lambda} = A$.

The proof of lemma 2.2 is a straightforward computation and is omitted. It follows that there exists a one-to-one correspondence between the idempotent-separating congruences of S and the γ-admissible subsets of G. Furthermore, $\lambda \subseteq \lambda'$ if and only if $A^{\lambda} \subseteq A^{\lambda'}$.

By the above remark the mapping $\phi : [i, \mathscr{H}] \to \Gamma$ given by $\lambda\phi = A^{\lambda}$, which is onto by lemma 2.2 (ii), is a lattice isomorphism. This affords us a direct proof of the fact that $[i, \mathscr{H}]$ is modular, since we have noted that Γ is a modular lattice.

Let A be a γ-admissible subset of G. We define a mapping $\gamma_{d-1}|A$ of G_{d-1}/A_{d-1} into G_0/A_0 as follows:

$$(a_{d-1}A_{d-1})(\gamma_{d-1}|A) = (a_{d-1}\gamma_{d-1})A_0, \text{ for all } a_{d-1}A_{d-1} \in G_{d-1}/A_{d-1}.$$

Further, for $i = 0, 1, \cdots, d-2$, we define a mapping $\gamma_i|A$ of G_i/A_i into G_{i+1}/A_{i+1} as follows:

$$(a_i A_i)(\gamma_i|A) = (a_i\gamma_i)A_{i+1}, \text{ for all } a_i A_i \in G_i/A_i.$$

That these mappings are well defined is a consequence of the γ-admissibility of A. It is immediate that $\gamma_i|A$ is a homomorphism for $i = 0, 1, \cdots, d-1$. For $n \in N$ define $\gamma_n|A = \gamma_{n(\text{mod } d)}|A$. For $(m, n) \in N \times N$ and $m < n$ write

$$\alpha_{m,n}|A = (\gamma_m|A)(\gamma_{m+1}|A) \cdots (\gamma_{n-1}|A),$$

and for all $n \in N$ let $\alpha_{n,n}|A$ denote the identity automorphism of $G_{n(\text{mod } d)}/A_{n(\text{mod } d)}$. Further, if we assume that $m = pd+i$ and $n = qd+j$, then

$$(a_i A_i)(\alpha_{m,n}|A) = (a_i\alpha_{m,n})A_j.$$

THEOREM 2.1. *Let λ be an idempotent-separating congruence on $S(d; G_i; \gamma_i)$. Then*

$$S/\lambda \cong S(d; G_i/A_i^{\lambda}; \gamma_i|A^{\lambda}).$$

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PROOF. Consider the mapping θ of S onto $S(d; G_i/A_i^\lambda; \gamma_i|A^\lambda)$ defined by

$$(m, a_i, n)\theta = (m, a_i A_i^\lambda, n).$$

Then θ is a homomorphism. Further $(m, a_i, n)\theta = (p, b_j, q)\theta$ if and only if $i = j$, $m = p$, $n = q$ and $a_i A_i^\lambda = b_j A_j^\lambda$. By lemma 2.2 (i) these equations hold if and only if $((m, a_i, n), (p, b_j, q)) \in \lambda$. Hence $\theta \circ \theta^{-1} = \lambda$ which gives the result.

Let $\ker \alpha_{i,\, i+d}^n$ denote the kernel of the endomorphism $\alpha_{i,\, i+d}^n$ of A_i, for $i = 0, 1, \cdots, d-1$ and $n = 1, 2, \cdots$. Put $K_i^n = \ker \alpha_{i,\, i+d}^n$, $K_i = \bigcup_{n=1}^\infty K_i^n$ and $K = K_0 \times K_1 \times \cdots \times K_{d-1}$.

LEMMA 2.3. *Let σ be the minimal group congruence on $S = S(d; G_i; \gamma_i)$. Then $A^{\sigma \wedge \mathscr{H}} = A^\sigma = K$.*

PROOF. We have noted earlier that $A^{\sigma \wedge \mathscr{H}} = A^\sigma$. Let $a_i \in A_i^\sigma$. Then $((0, a_i, 0), e_i^0) \in \sigma$ and so, by Lemma 1.2, there exists e_j^m such that $e_j^m(0, a_i, 0) = e_j^m e_i^0$. We may suppose without loss of generality that $m > 0$. Hence $(m, a_i \alpha_{i,\, md+j}, m) = e_j^m$ and so $a_i \alpha_{i,\, md+j} = e_j$; it follows that $(a_i \alpha_{i,\, md+j})\alpha_{j,\, d+i} = e_i$. Thus $a_i \in K_i$.

Conversely, let $a_i \in K_i$. Then $a_i \alpha_{i,\, i+d}^m = e_i$ for some m, and so $e_i^m(0, a_i, 0) = e_i^m e_i^0$. Hence, by lemma 1.2, $((0, a_i, 0), e_i^0) \in \sigma$; that is, $a_i \in A_i^\sigma$. We conclude that $A_i^\sigma = K_i$ $(i = 0, 1, \cdots, d-1)$, and the result follows.

COROLLARY 2.1. *Let $S = S(d; G_i; \gamma_i)$. Then*

$$S/\sigma \wedge \mathscr{H} \cong S(d; G_i/K_i : \gamma_i|K).$$

Corollary 2.1 follows immediately from theorem 2.1.

3. Group congruences

We begin this section with a general result about inverse semigroups with identity.

LEMMA 3.1. (Munn [9], Lemma 4 (ii)). *Let ρ be a group congruence on an inverse semigroup S with identity e. Then $(x, y) \in \rho$ if and only if $xy^{-1} \in e\rho$.*

Let $A = A_0 \times A_1 \times \cdots \times A_{d-1}$ be a γ-admissible subset of G. Then, as was noted earlier, A_i is a $\alpha_{i,\, i+d}$-admissible subgroup of G_i, for $i = 0, 1, \cdots, d-1$. Following Munn [9], we define $\operatorname{rad} A_i$, the *radical of* A_i relative to the endomorphism $\alpha_{i,\, i+d}$, as follows:

$$\operatorname{rad} A_i = \{a_i \in G_i : a_i \alpha_{i,\, i+d}^n \in A_i \text{ for some } n\}.$$

Using these radicals we define *the radical of* A, Rad A, as follows:

$$\operatorname{Rad} A = \operatorname{rad} A_0 \times \operatorname{rad} A_1 \times \cdots \times \operatorname{rad} A_{d-1}.$$

Rad A is a normal subset of G, since each $\operatorname{rad} A_i \trianglelefteq G_i$ (Munn [9]). In fact Rad A is a γ-admissible subset of G, as we now show. Let $a_i \in \operatorname{rad} A_i$, that is, let

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$$a_i \alpha^n_{i,i+d} \in A_i \text{ for some } n,$$

and so

$$(a_i \alpha^n_{i,i+d})\gamma_i \in A_{i+1 \,(\mathrm{mod}\, d)}.$$

But

$$(a_i \alpha^n_{i,i+d})\gamma_i = (a_i \gamma_i)\alpha^n_{i+1,i+1+d}.$$

Thus Rad $A \in \Gamma$.

Again, directly generalizing Munn's procedure in [9], we denote the γ-admissible set $\{(e_0, e_1, \cdots, e_{d-1})\}$ by 1. Then Rad $1 = K$. Hence it follows that $A^\sigma = \text{Rad } 1$. Put $\Gamma^* = \{A \in \Gamma : \text{Rad } A = A\}$.

That the following properties hold for Rad follows immediately from Munn's lemma 2 in [9], where it is shown that the analogous properties hold for rad in each component G_i of G.

(i) $A \subseteq \text{Rad } A$,

(ii) $A \subseteq A'$ implies that Rad $A \subseteq$ Rad A',

(iii) Rad (Rad A) = Rad A,

(iv) A Rad $1 \subseteq$ Rad A,

(v) Rad (A Rad 1) = Rad A.

Properties (i), (ii) and (iii) imply that Rad is a closure operator on the set of γ-admissible subsets of G.

The next lemma follows from and generalizes Munn's lemma 4 (i) in [9].

LEMMA 3.2. *Let* $\rho \in [\sigma, S \times S]$. *Then* $A^\rho \in \Gamma^*$.

PROOF. $\rho | S_i$ is a group congruence on S_i. Hence rad $A_i^\rho = A_i^\rho$ by Munn's lemma 4 (i) in [9]. Thus Rad $A^\rho = A^\rho$ and $A^\rho \in \Gamma^*$ as required.

We now fix our attention on the sublattice $[\sigma, \sigma \vee \mathscr{H}]$, and begin by determining the congruence $\sigma \vee \mathscr{H}$. Note that the restriction to each S_i gives Munn and Reilly's determination of $\sigma \vee \mathscr{H}$ on a bisimple ω-semigroup [8].

LEMMA 3.3. *Let* $S = S(d; G_i; \gamma_i)$. *Then*

$$((m, a_i, n), (p, b_j, q)) \in \sigma \vee \mathscr{H} \text{ if and only if } m-n = p-q.$$

PROOF. Let $x = (m, a_i, n)$ and $y = (p, b_j, q)$. Suppose that $(x, y) \in \sigma \vee \mathscr{H}$. Then since $\sigma \vee \mathscr{H} = \sigma \circ \mathscr{H} \circ \sigma$ (Howie [4], Theorem 3.9) there exist u, v in S such that $(x, u) \in \sigma$, $(u, v) \in \mathscr{H}$, and $(v, y) \in \sigma$. Let $u = (m', g_k, n')$ and $v = (p', h_t, q')$. Since $(x, u) \in \sigma$ there exists, by lemma 1.2, an idempotent e_s^r such that $e_s^r x = e_s^r u$, and we can assume without loss of generality that $r \geqq m, m'$. Hence we have that $r+n-m = r+n'-m'$ and so $m-n = m'-n'$. Similarly, since $(v, y) \in \sigma$ we have $p-q = p'-q'$. But $m'-p' = n'-q'$ since $(u, v) \in \mathscr{H}$. Hence $m-n = p-q$.

Conversely, let x and y be such that $m-n = p-q$. We may assume that $m \leqq p$. Suppose $m < p$ or $m = p$ and $j \geqq i$. Then

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$$\begin{aligned} e_j^p x &= (p, e_j, p)(m, a_i, n) \\ &= (p, a_i \alpha_{md+i, pd+j}, p+n-m) \\ &= (p, a_i \alpha_{md+i, pd+j}, q) \end{aligned}$$

and so $(e_j^p x, y) \in \mathscr{H}$. But $(x, e_j^p x) \in \sigma$ since e_j^p is idempotent. Hence $(x, y) \in \sigma \circ \mathscr{H} \subseteq \sigma \vee \mathscr{H}$.

Suppose now that $m = p$ and $j < i$. Then

$$\begin{aligned} e_i^p y &= (p, e_i, p)(p, b_j, q) \\ &= (p, b\alpha_{pd+j, pd+i}, q) \end{aligned}$$

and so $(e_i^p y, x) \in \mathscr{H}$. But $(y, e_i^p y) \in \sigma$ since e_i^p is idempotent. Hence $(x, y) \in \sigma \circ \mathscr{H} \subseteq \sigma \vee \mathscr{H}$.

COROLLARY 3.1. *$S/\sigma \vee \mathscr{H} \cong Z$, where Z denotes the integers.*

PROOF. Consider the mapping $\theta : S \to Z$ defined by $(m, a_i, n)\theta = m-n$. The mapping θ is a homomorphism and it follows that $\theta \circ \theta^{-1} = \sigma \vee \mathscr{H}$.

LEMMA 3.4. (i) *Let $\rho \in [\sigma, \sigma \vee \mathscr{H}]$. Then*

$$e_0^0 \rho = \{(m, a_i, m) \in S : m \in N, a_i \in A_i, 0 \leqq i \leqq d-1\}.$$

(ii) *Let $\rho, \rho' \in [\sigma, \sigma \vee \mathscr{H}]$. Then $\rho \subseteq \rho'$ if and only if $A^\rho \subseteq A^{\rho'}$.*

PROOF. (i) Since ρ is a group congruence, $(e_0^0, e_i^0) \in \rho$ for $0 \leqq i \leqq d-1$. Hence

$$e_0^0 \rho = \bigcup_{0 \leqq i \leqq d-1} e_i^0(\rho | S_i)$$

and our result follows from Munn [9] lemma 5(i).

(ii) If $\rho \subseteq \rho'$ then clearly $A^\rho \subseteq A^{\rho'}$. Suppose conversely that $A^\rho \subseteq A^{\rho'}$. Then by (i), $e_0^0 \rho \subseteq e_0^0 \rho'$. Let $(x, y) \in \rho$. Then $xy^{-1} \in e_0^0 \rho$ by lemma 3.1. Hence $xy^{-1} \in e_0^0 \rho'$ and so $(x, y) \in \rho$, again by lemma 3.1. We conclude that $\rho \subseteq \rho'$.

We now prove a partial converse to lemma 3.2 which generalizes Munn's lemma 5 (iii) in [9].

LEMMA 3.5. *Let $A \in \Gamma^*$. Then there exists τ in $[\sigma, \sigma \vee \mathscr{H}]$ such that $A^\tau = A$.*

PROOF. For $x = (m, a_i, n)$ and $y = (p, b_j, q)$ in S write

$$||xy|| = a_i \alpha_{u, w} b_j^{-1} \alpha_{v, w},$$

where $u = nd+i, v = qd+j$ and $w = \max\{u, v\}$. Define

$$\tau = \{(x, y) \in S \times S : x = (m, a_i, n), y = (p, b_j, q), \\ m-n = p-q \text{ and } ||xy|| \in A_k \text{ for some } k\}.$$

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We shall show that τ is a congruence on S with the desired property. It is straightforward to show that τ is reflexive and symmetric.

Let now $x = (m, a_i, n)$, $y = (p, b_j, q)$ and $z = (r, c_k, s)$. Suppose $(x, y) \in \tau$ and $(y, z) \in \tau$. Then $m-n = p-q$ and $p-q = r-s$ and so $m-n = r-s$. To prove that τ is transitive it remains to show that $\|xz\| \in A_t$, for some t. We may assume without loss of generality that $n < s$ or $n = s$ and $i \leqq k$. We now proceed by cases.

Case (i): $q < n$ or $q = n$ and $j \leqq i$.

Then
$$\begin{aligned}
\|xz\| &= a_i \alpha_{nd+i,\, sd+k} c_k^{-1} \\
&= a_i \alpha_{nd+i,\, sd+k}(b_j^{-1} b_j)\alpha_{qd+j,\, sd+k} c_k^{-1} \\
&= (a_i \alpha_{nd+i,\, sd+k} b_j^{-1} \alpha_{qd+j,\, sd+k})(b_j \alpha_{qd+j,\, sd+k} c_k^{-1}) \\
&= (a_i \alpha_{nd+i,\, sd+k}(b_j^{-1} \alpha_{qd+j,\, nd+i})\alpha_{nd+i,\, sd+k})(b_j \alpha_{qd+j,\, sd+k} c_k^{-1}) \\
&= (a_i b_j^{-1} \alpha_{qd+j,\, nd+i})\alpha_{nd+i,\, sd+k}(b_j \alpha_{qd+j,\, sd+k} c_k^{-1}) \\
&= \|xy\| \alpha_{nd+i,\, sd+k} \|yz\| \\
&\in A_k, \text{ since } \|xy\| \in A_i,\ \|yz\| \in A_k \text{ and } A \text{ is } \gamma\text{-admissible.}
\end{aligned}$$

Hence $(x, z) \in \tau$.

Case (ii): $n < q < s$ or $n = q$ and $i \leqq j$ or $q = s$ and $j \leqq k$.

Then
$$\begin{aligned}
\|xz\| &= a_i \alpha_{nd+i,\, sd+k} c_k^{-1} \\
&= a_i \alpha_{nd+i,\, sd+k}(b_j b_j^{-1})\alpha_{qd+j,\, sd+k} c_k^{-1} \\
&= (a_i \alpha_{nd+i,\, qd+j})\alpha_{qd+j,\, sd+k} b_j \alpha_{qd+j,\, sd+k} b_j^{-1} \alpha_{qd+j,\, sd+k} c_k^{-1} \\
&= (a_i \alpha_{nd+i,\, qd+j} b_j^{-1})\alpha_{qd+j,\, sd+k} b_j \alpha_{qd+j,\, sd+k} c_k^{-1} \\
&= \|xy\| \alpha_{qd+j,\, sd+k} \|yz\| \\
&\in A_k, \text{ since } \|xy\| \in A_j,\ \|yz\| \in A_k \text{ and } A \text{ is } \gamma\text{-admissible.}
\end{aligned}$$

Hence $(x, z) \in \tau$.

Case (iii): $s < q$ or $s = q$ and $k \leqq j$.

Then
$$\begin{aligned}
\|xz\| \alpha_{sd+k,\, qd+j} &= (a_i \alpha_{nd+i,\, sd+k} c_k^{-1})\alpha_{sd+k,\, qd+j} \\
&= a_i \alpha_{nd+i,\, sd+k} \alpha_{sd+k,\, qd+j} c_k^{-1} \alpha_{sd+k,\, qd+j} \\
&= a_i \alpha_{nd+i,\, qd+j} b_j^{-1} b_j c_k^{-1} \alpha_{sd+k,\, qd+j} \\
&= \|xy\| \, \|yz\| \\
&\in A_j, \text{ since } \|xy\| \in A_j \text{ and } \|yz\| \in A_j.
\end{aligned}$$

Hence $\|xz\| \alpha_{sd+k,\, qd+j} \alpha_{j+d,\, k} \in A_k$, that is $\|xz\| \alpha_{k,\, k+d}^{q-s+1} \in A_k$. Now Rad $A = A$ and so $\|xz\| \in A_k$. Hence $(x, z) \in \tau$. We conclude that τ is an equivalence relation on S.

That τ is left and right compatible follows from computations similar to those above and they are left to the reader.

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It follows from lemma 3.3 and the definition of τ that $\tau \in [\sigma, \sigma \vee \mathscr{H}]$. It also follows from the definition of τ that $A^\tau = A$.

We note here that Munn's proof of lemma 5 (iii) in [9] can be generalized to give an alternative proof of our lemma 3.5 by making use of the closed self-conjugate subsemigroup of S.

$$M = \{(m, a_i, m) : m \in N, a_i \in A_i, 0 \leqq i \leqq d-1\}$$

Then $\tau = \{(x, y) \in S \times S : xy^{-1} \in M\}$ is a congruence on S and $A = A^\tau$.

Partially ordering Γ^* by inclusion we obtain from Lemma 3.5 and lemma 3.4 (ii) that

$$\Gamma^* \cong [\sigma, \sigma \vee \mathscr{H}].$$

Now $[\sigma, \sigma \vee \mathscr{H}]$ is a modular lattice since it is a sublattice of the modular lattice $[\sigma, S \times S]$. We conclude that Γ^* is a modular lattice.

4. The main result

We are now in a position to generalize Munn's argument to the present situation and prove an analogue of his main theorem in [9], viz.

THEOREM 4.1. *Let S be a simple regular ω-semigroup. Then the sublattice $\Lambda_0(S)$ is modular, if and only if* Rad $A = A$ Rad 1 *for all A in Γ.*

To establish this result we establish the analogues of lemmas 8, 9 and 10 of [9]. To do this we need two preliminary lemmas which generalize Munn's lemmas 6 and 7 of [9].

LEMMA 4.1. *Let $\lambda \in [i, \mathscr{H}]$ and $\rho \in [\sigma, \sigma \vee \mathscr{H}]$. Then*

(i) $\lambda \vee \rho \in [\sigma, \sigma \vee \mathscr{H}]$ *and* $A =$ Rad $(A^\lambda A^\rho)$, *and*

(ii) $\lambda \wedge \rho \in [i, \mathscr{H}]$ *and* $A^{\lambda \wedge \rho} = A^\lambda \wedge A^\rho$.

LEMMA 4.2. $A^{\lambda \vee \sigma} =$ Rad A^λ *for any* λ in $[i, \mathscr{H}]$.

To prove lemma 4.1 and 4.2 it suffices to note that Munn's argument in lemma 6 and lemma 7 can be applied in a co-ordinatewise fashion to the present situation.

The next three lemmas are the analogues of lemmas 8, 9 and 10 of [9].

LEMMA 4.3. *Let $\Lambda_0(S)$ be modular and let $A \in \Gamma$. Then* Rad $A = A$ Rad 1.

LEMMA 4.4. *Let* Rad $A = A$ Rad 1 *for all $A \in \Gamma$. Then $[i, \mathscr{H}] \cup [\sigma, \sigma \vee \mathscr{H}]$ is modular.*

LEMMA 4.5. *Let $[i, \mathscr{H}] \cup [\sigma, \sigma \vee \mathscr{H}]$ be modular. Then $\Lambda_0(S)$ is modular.*

To prove lemmas 4.3 and 4.4 we argue as Munn does in his lemmas 8 and 9 using the preceeding generalized lemmas. The proof of lemma 4.5 is identical to Munn's proof of lemma 10.

Theorem 4.1 now follows from lemmas 4.3, 4.4 and 4.5.

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