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Groups of exponent eight

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Dedicated to B.H. Neumann

This paper is a survey of the current state of knowledge on groups of exponent 8. It contains a report on a first stage of an attempt to answer the Burnside questions for these groups.

One of the driving forces behind the study of the structure of groups is a number of questions posed by Burnside [2] in 1902. In particular he asked whether every m-generator group of exponent n (every element has order dividing n) is finite. It is known that the answer is 'yes' for exponents $2, 3, 4, 6$, and all (finite) numbers of generators (see, for example, Sections 5.12-5.13 of Magnus, Karrass, Solitar [10]) and 'no' for all *odd* exponents greater than or equal to 665 (see Adian [1]).

These results focus particular attention on the exponents which are powers of two. This paper is primarily a report on what might be described as the beginnings of an investigation of groups of exponent 8 with a view to answering the Burnside questions for them. Though some of the results have been obtained by subsets of us, the report is presented jointly because we believe that even the foothills on which we stand at present have been reached only as a consequence of our various interactions over the last few years. It is intended that the appropriate subsets will publish more detailed reports on their results.

Since there are comparatively few published results on groups of

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exponent 8, we preface our report with a synopsis of these results. The first published result specifically on groups of exponent 8 is that of Sanov [15] in 1947 which states that there is a 2-generator group of exponent 8 with order 2^{136}. This is proved by making use of the Schreier formula for the rank of a subgroup of finite index in a free group of finite rank. It is part of the folk-lore that the same argument applied to the subgroup F^4 generated by fourth powers of elements in a free group F of rank 2 shows that $F/\left(F^4\right)^2$ is a 2-generator group of exponent 8 and order 2^{4109} (because F/F^4 has order 2^{12} – see §6.8 of Coxeter and Moser [4]). A further measure of the complexity of $F/\left(F^4\right)^2$ is that its nilpotency class is 39 (Shield [17]). In 1951 Sanov [16] began another line of investigation by proving that all groups of exponent 8 satisfy the 23rd Engel congruence (see p. 48 of Robinson [14]). The best result in this direction is due to Krause [7] in 1964; he showed they satisfy the 14th Engel congruence. On the other hand they need not satisfy the 11th Engel congruence. This can be seen by considering the following group, which is also used to justify some of our later remarks. The group, K say, is generated by two elements h, k which satisfy

$$k^8 = h^8 = 1 , \quad k^{h^4} = k^{-1} , \quad kk^h = k^hk , \quad kk^{h^2} = k^{h^2}k , \quad kk^{h^3} = k^{h^3}k ,$$

as a defining set of relations. It is a split extension of the direct product of four cyclic subgroups of order 8 (generated by k, k^h, k^{h^2}, k^{h^3}) by the cyclic subgroup generated by h and has class 12. It is straight-forward to check that K has exponent 8 and that $[k, {}_{11}h] = \left(k\, k^hk^{h^2}k^{h^3}\right)^4 \neq 1$. Let B denote the free group of rank 2 and exponent 8, and B_c the cth term of its lower central series. Macdonald [9, p. 437], using a computer implementation of an algorithm for calculating nilpotent quotient (or factor) groups, showed that B/B_7 has order 2^{35} and that $\left(B_6\right)^2 \leq B_7$. Using the Canberra implementation of a related algorithm called the nilpotent quotient algorithm (see Newman [11]), these results have been extended to show that

$$|B/B_8| = 2^{53}\ ,\quad |B/B_9| = 2^{83}\ ,\quad |B/B_{10}| = 2^{139}\ ,\quad |B/B_{11}| = 2^{238}\ ,$$

$$|B/B_{12}| = 2^{415}\ ,\quad |B/B_{13}| = 2^{722}\ ;$$

$$(B_6)^2 \le B_{10}\ ,\quad (B_4)^4 \le B_{10}\ ,\quad (B_5)^4 \le B_{12}\ .$$

(The claims that $(B_6)^2 \le B_{10}$ and $(B_4)^4 \le B_{10}$ were first made by Skopin [18, Theorem B]. Unfortunately there seems to be an error in his calculations - for instance they lead to the subsequent claim by Lobyč and Skopin [8] that every 2-generator metabelian group of exponent 8 has class at most 10 - this is incorrect as the group K above shows.) Hermanns [6] has proved that the free 2-generator metabelian group of exponent 8, namely B/B'', has order 2^{63} and class 12 (this has been confirmed by using the nilpotent quotient algorithm). Hermanns also shows that $(B_5)^4 \le B''$ and that $(B_9)^2 \le B''$.

Finally, hot off the press, Razmyslov [12] has shown that there is a locally finite insoluble group of exponent 8.

Our study of the Burnside question for groups of exponent 8 has its origins in the hope that the nature of certain subgroups of the free group, $B(m, 8)$, of rank m with exponent 8 might give an indication of whether to expect that $B(m, 8)$ is finite or not. This hope is based on a result of Adian (see Theorem VII.1.8 of [1]) that for n odd and sufficiently large and for m at least 2 the infinite group $B(m, n)$ has all its finite subgroups cyclic. This suggests that a pointer to the finiteness of $B(m, 8)$ would be obtained if it were possible to exhibit comparatively large finite subgroups in it. The situation is not quite so simple however. Every infinite 2-group has an infinite abelian subgroup (see Theorem 3.41 in Robinson [13]) and therefore abelian subgroups of every 2-power order. In groups of exponent 8, these large finite abelian subgroups must of course have large generating sets. These considerations suggest the study of 2-generator subgroups. A subgroup generated by two (distinct) elements of order 2 is a homomorphic image of the infinite dihedral group (or free product of two cyclic groups of order 2) which has exponent dividing 8; it is therefore dihedral of order 4, 8, or 16, and so certainly finite.

From now on we concentrate entirely on the case $m = 2$ and take $\{a, b\}$ as a free generating set for $B = B(2, 8)$. Perhaps the first 2-generator subgroup of B to consider is that generated by $\{a^4, b^4\}$. This subgroup has order 16 (for the image of it under the mapping of B to K defined by $a \mapsto hk$, $b \mapsto h$ has order 16). As soon as one considers 2-generator subgroups one of whose generators does not (necessarily) have order 2, a non-trivial finiteness problem arises. The most natural subgroup to consider next is that generated by $\{a^4, b^2\}$: call it D, say. The question whether D is finite seems difficult and an answer to it can probably not be expected for some time.

The subgroup D can be regarded as a homomorphic image of the group Δ defined on the generating set $\{x, y\}$ by the relations $x^2 = y^4 = 1$ and the condition that it have exponent 8. One of our main results is that the restricted problem for Δ has a positive answer; that is: *the largest residually finite quotient*, $\bar{\Delta}$, *of* Δ *is finite.* This was established with the nilpotent quotient algorithm by a substantial calculation, involving tens of hours of computer time. Moreover the nilpotent quotient algorithm yields that the order of $\bar{\Delta}$ divides 2^{205} and that the class of $\bar{\Delta}$ is at most 26. Some initial progress, which is described below, has been made on the question of the finiteness of Δ itself. The basic idea is to obtain information about sections (quotient groups of subgroups) of Δ by studying centralizers of fourth powers in Δ and in certain groups which arise from Δ by taking sections and preimages (still of exponent 8) repeatedly. To date these lengthy calculations (over 100 pages) have been done by hand, although more recently use has been made of structural information about finite quotients provided by the nilpotent quotient algorithm.

Observe that D can be regarded as a product of its finite subgroups $\langle a^4, b^4\rangle$ and $\langle b^2\rangle$; these meet in $\langle b^4\rangle$ (in other words D is generated by an amalgam of finite groups). In $\langle a^4, b^4\rangle$ there are two proper subgroups $\langle (a^4b^4)^4, b^4\rangle$ and $\langle (a^4b^4)^2, b^4\rangle$ containing b^4. This suggests two intermediate finiteness problems. Is $L = \langle (a^4b^4)^4, b^2\rangle$ finite? Is $S = \langle (a^4b^4)^2, b^2\rangle$ finite? Both these questions can be pulled

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back into Δ . Is $\Lambda = \langle (xy^2)^4, y \rangle$ finite? Is $\Sigma = \langle (xy^2)^2, y \rangle$ finite? The first of these is easy to answer, for y^2 is central in Λ and $\Lambda/\langle y^2 \rangle$ is generated by two elements of order 2 , so Λ is finite - of order at most 32 . The second question is harder; our other main result implies that Σ is finite - with order dividing 2^{31} and divisible by 2^{14} . This was proved by taking further preimages.

Since $(xy^2)^8 = y^4 = (y^2(xy^2)^2)^2 = 1$, the group Σ is a homomorphic image of the group Γ^* defined on the generating set $\{w, y\}$ by the relations $w^4 = y^4 = (y^2w)^2 = 1$ and the condition that it have exponent 8 . The nilpotent quotient algorithm applied to Γ^* shows that the restricted problem for Γ^* has a positive solution and yields a consistent power-commutator presentation for the largest residually finite quotient $\overline{\Gamma^*}$ of Γ^* . This shows that $\overline{\Gamma^*}$ has order 2^{31} and class 19 , and gives other structural information about $\overline{\Gamma^*}$. With this as a guide, it was possible to prove our second main result: *the group* Γ^* *is finite*.

To prove the finiteness of Γ^* it suffices to prove the finiteness of some finitely presented preimage Γ of Γ^* . Such a Γ was not prescribed in advance, but rather eighth power relations were added successively to the three relations $w^4 = y^4 = (y^2w)^2 = 1$ as required. The proof utilizes various subgroups of Γ , ultimately showing that a certain subgroup of finite index in Γ is finite. For example, the first stage of the proof shows that the subgroup Γ_1 of Γ , generated by $\{w^2, (wy^{-1})^2, (ywy^{-1})^2\}$ is of finite index. This uses just the three relations above, without any additional eighth power relations.

An important step in the proof that Γ_1 is finite is to show that the subgroup Π generated by $\{w^2(yw^{-1})^2, ywyw^{-1}\}$ is finite. This was, as usual, done by finding an adequate set of non-trivial relations between these generators and then considering the corresponding preimage G^* of Π : G^* is defined on the generating set $\{p, q\}$ by the relations

$$p^4 = q^4 = (pq)^4 = (pq^{-1})^4 = (p^2q^2)^4 = (pqp^{-1}q^{-1})^4 = (p^2qpq^2pq)^2 = 1 ,$$

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and the condition that it have exponent 8 ; the map to Π is $p \to w^2(yw^{-1})^2$, $q \to ywyw^{-1}$. Again, a finitely presented preimage G of G^* was actually shown to be finite. The presentation for G , which includes the seven relations given explicitly above for G^* , provides an indication as to how some of the eighth power relations for Γ were selected. For example the relation $q^4 = 1$ is included in the presentation for G , so the relation $(wy^{-1})^8 = 1$ is included in the presentation for Γ , because q^4 maps to

$$(ywyw^{-1})^4 = y(wyw^{-1}y)^4y^{-1} = y(wy^{-1}wy^{-1})^4y^{-1} .$$

There are now two proofs for the finiteness of G^* . Both in their final form rely on detailed structural information about the largest residually finite quotient $\overline{G^*}$ of G^* obtained from a power-commutator presentation for $\overline{G^*}$ calculated by the nilpotent quotient algorithm (the group has order 2^{21} and class 9). The original proof, which was the subject of the dissertation by Grunewald [5], involves detailed study of fourth powers and their centralizers. The idea is to exploit the affirmative answer of the Burnside question for groups of exponent 4 , which guarantees that a finitely generated group with enough fourth power relations is finite. For this to yield a finiteness proof one must show that enough fourth powers lie in a finite normal subgroup of the group in question. The keys to this are the two, seemingly trite, observations that

(a) it suffices to work in subgroups of finite index (depending on the fourth power), and

(b) fourth powers have order two.

(The rest of the proof of the finiteness of Γ^* is similar in spirit to this, though the details are more complex.)

The other proof uses coset enumeration (see Cannon, Dimino, Havas, and Watson [3]) machine implemented with the following *normalizing coset* facility. Since complete enumeration of the cosets of an obviously finite subgroup of G^* is impracticable, because of very high indexes, a subgroup building technique is used. After a convenient number of cosets of a finite subgroup have been enumerated, a check is made as to whether the information in the incomplete coset table implies that some coset is

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stabilized by the subgroup. If so, then the coset lies in the normalizer of the subgroup, and hence a larger, finite subgroup is obtained by adjoining a coset representative of the stabilized coset to the subgroup generating set. Repeated applications of this process, when successful, lead to the construction of an ascending chain of finite subgroups. To prove the finiteness of G^*, a finite subgroup large enough to obtain finite index by coset enumeration is reached this way, even though the index of the original subgroup was too high. Further, it is possible this way to show that only two eighth power relations, $(p^2q)^8 = (pq^2)^8 = 1$, need be added to the above seven relations to give a set of defining relations for G^*. The nilpotent quotient algorithm shows that the resulting presentation is minimal.

Another result has been proved with these methods; namely that the subgroup N of Δ which is the normal closure of $(y^2xyx)^4$ has index 2^{26} in Δ. Moreover the largest residually finite quotient of N is elementary abelian of order dividing 2^{179}.

The subgroup Π of Γ^* can be shown to have order 2^{14}. The kernel of the mapping from G^* to Π is generated as a normal subgroup of G^* by

$$\{[[q, p], (q\ p)^4], [q, p^2]^2[q^2, p]^2, (qpq^{-1}p)^4\} .$$

The corresponding relations in y, w can be regarded as non-obvious consequences of the exponent 8 condition. These can be translated via the other mappings to non-obvious consequences of the exponent 8 condition in B. They are rather unpleasant to write down. Some examples of consequences of the exponent 8 condition derived this way are

$$[(a^4b^4a^4b^2)^2, b^4(a^4b^4a^4b^2)^4b^4]^4 = 1 ,$$

$$((a^4b^4a^4b^2)^3b^4(a^4b^4a^4b^2)^3b^2(a^4b^4a^4b^2)^2b^2)^4 = 1 .$$

Perhaps the methods described here can be further refined to lead, eventually, to a finiteness proof for Δ and, consequently, for D - not to mention B !

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