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COMMENTS ON SOME INEQUALITIES OF PEARCE AND PEČARIĆ

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We prove inequalities for convex functions, L^p norms, and sums of powers. Our results sharpen recently published inequalities of C. E. M. Pearce and J. E. Pečarić.

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1. Introduction

The study of singular measures led G. Brown [**1**] in 1988 to several interesting new inequalities involving polynomials and fractional powers. These inequalities were re-examined by A. W. Kemp [**4**] in 1992. In order to shorten the proof of Brown's main theorem, Kemp applied a specific inequality for convex functions which was generalized by C. E. M. Pearce and J. E. Pečarić. Their result says:

Theorem A. (Pearce and Pečarić, [**5**]). *Let a, b, s_i and t_i $(i = 0, 1, 2)$ be positive real numbers with $a/s_i + b/t_i = 1$ $(i = 0, 1, 2)$ and $s_1 < s_0 < s_2$. If $f, g : (0, \infty) \to \mathbb{R}$ are convex functions, then*

$$\frac{f(s_0)}{s_0} + \frac{g(t_0)}{t_0} \le \max_{i=1,2}\left(\frac{f(s_i)}{s_i} + \frac{g(t_i)}{t_i}\right). \tag{1.1}$$

In 1989, G. Brown and L. Shepp [**2**] presented two "key lemmas" [**6**, p. 60] which are useful in calculations with fractal sets. They proved two inequalities for L^p norms and sums of powers which are of the same type of inequalities as given in Theorem A. Recently, C. E. M. Pearce and J. E. Pečarić have published the following extensions.

Theorem B. (Pearce and Pečarić, [**6**]). *Let the positive real numbers a, b, s_i and t_i $(i = 0, 1, 2)$ satisfy $a/s_i + b/t_i = 1$ $(i = 0, 1, 2)$ and $s_1 < s_0 < s_2$.*

(i) *If $\|\cdot\|_p$ denotes the L^p norm of a real-valued function, then*

$$\|f\|_{s_0}\|g\|_{t_0} \leq \max_{i=1,2}\left(\|f\|_{s_i}\|g\|_{t_i}\right), \tag{1.2}$$

provided that all quantities exist.

(ii) *Let* $x = (x_i)$, $u = (u_i)$ $(i = 1, \ldots, n)$ *and* $y = (y_j)$, $v = (v_j)$ $(j = 1, \ldots, m)$ *be sequences of positive real numbers. If*

$$S_n^{[t]}(x, u) = \left(\sum_{i=1}^{n} u_i x_i^t\right)^{1/t},$$

then

$$S_n^{[s_0]}(x, u) S_m^{[t_0]}(y, v) \leq \max_{i=1,2}\left(S_n^{[s_i]}(x, u) S_m^{[t_i]}(y, v)\right). \tag{1.3}$$

If $\sum_{i=1}^{n} u_i = 1$, then $S_n^{[t]}(x, u)$ is the weighted power mean of order t. And, if $u_i = 1$ $(i = 1, \ldots, n)$, then we have the sum of order t. Many remarkable properties of these power sums can be found in the monograph [3].

It is the main purpose of this paper to present refinements of Theorem A and Theorem B. A central role in our proofs plays an inequality for arithmetic means which might be of independent interest.

2. An inequality for arithmetic means

The following inequality sharpens $A_n \leq \max_{1 \leq i \leq n} a_i$, where A_n denotes the weighted arithmetic mean of the real numbers $a_1, \ldots, a_n$.

Theorem 1. *Let* p_i $(i = 1, \ldots, n; n \geq 2)$ *be positive real numbers with* $\sum_{i=1}^{n} p_i = 1$. *Then we have for all real numbers* $a_1, \ldots, a_n$:

$$\frac{p}{n-1} \sum_{1 \leq i < j \leq n} \left(\sqrt{|a_i|} - \sqrt{|a_j|}\right)^2 \leq \max_{1 \leq i \leq n} a_i - \sum_{i=1}^{n} p_i a_i, \tag{2.1}$$

where

$$p = \min_{1 \leq i \leq n} p_i.$$

Proof. Let

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$$S(a_1, \ldots, a_n) = \max_{1\le i\le n} a_i - \sum_{i=1}^{n} p_i a_i - \frac{p}{n-1} \sum_{1\le i<j\le n} \left(\sqrt{|a_i|} - \sqrt{|a_j|}\right)^2.$$

Without loss of generality we may assume that $a_n \le a_{n-1} \le \ldots \le a_1$. We consider two cases.

Case 1: $0 \le a_n$ or $a_1 \le 0$.
Then we have

$$\begin{aligned}\sum_{1\le i<j\le n} \left(\sqrt{|a_i|} - \sqrt{|a_j|}\right)^2 \\ &= (n-1)\sum_{i=1}^{n} |a_i| - 2\sum_{1\le i<j\le n} \sqrt{|a_i a_j|} \\ &\le (n-1)\sum_{i=1}^{n} |a_i| - 2\sum_{1\le i<j\le n} \min(|a_i|, |a_j|) \\ &= \sum_{i=1}^{n} (n+1-2i)a_i.\end{aligned} \tag{2.2}$$

We set $q_i = p_i + [(n+1-2i)/(n-1)]p$ $(i = 1, \ldots, n)$; then we have $q_i \ge 0$ $(i = 1, \ldots, n)$ and $\sum_{i=1}^{n} q_i = 1$. Hence, we conclude from (2.2):

$$S(a_1, \ldots, a_n) \ge a_1 - \sum_{i=1}^{n} q_i a_i \ge 0.$$

Case 2: $a_n \le \cdots \le a_{k+1} \le 0 \le a_k \le \cdots \le a_1$, where $k \in \{1, \ldots, n-1\}$.
Then

$$S(a_1, \ldots, a_n) = a_1 - \sum_{i=1}^{n} p_i a_i - p\sum_{i=1}^{n} |a_i| + \frac{2p}{n-1} \sum_{1\le i<j\le n} \sqrt{|a_i a_j|}.$$

We define for $r = 1, \ldots, k$ and $a > 0$:

$$S_r(a) = S(a, \ldots, a, a_{r+1}, \ldots, a_n).$$

Differentiation yields

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$$S_r'(a) = 1 - \sum_{i=1}^{r} p_i - pr + \left(\frac{(r-1)r}{2} + \frac{r}{2\sqrt{a}} \sum_{i=r+1}^{n} \sqrt{|a_i|}\right) \frac{2p}{n-1}$$
$$\geq 1 - \sum_{i=1}^{r} p_i - pr + \frac{(r-1)rp}{n-1}$$
$$\geq \left(n - 2r + \frac{(r-1)r}{n-1}\right) p$$
$$\geq 0,$$

where the last inequality follows from $1 \leq r \leq n-1$. Hence, S_r is increasing on $[0, \infty)$. Using the monotonicity of S_r and $S_{m-1}(a_m) = S_m(a_m)$ $(m = 2, \ldots, k)$, we get

$$S(a_1, \ldots, a_n) = S_1(a_1) \geq S_1(a_2) = S_2(a_2) \geq S_2(a_3)$$
$$\geq \cdots \geq S_k(a_k) \geq S_k(0) \geq \sum_{i=k+1}^{n} (p - p_i) a_i \geq 0.$$

This completes the proof of Theorem 1.

Remarks. (1) Let $p_k = \min_{1 \leq i \leq n} p_i$; if we set $a_i = 1$ $(i = 1, \ldots, n; i \neq k)$ and $a_k = 0$, then we have

$$\left(\max_{1 \leq i \leq n} a_i - \sum_{i=1}^{n} p_i a_i\right) \Big/ \sum_{1 \leq i < j \leq n} \left(\sqrt{|a_i|} - \sqrt{|a_j|}\right)^2 = \frac{p_k}{n-1}.$$

Hence, in (2.1) the constant factor $[1/(n-1)] \min_{1 \leq i \leq n} p_i$ cannot be replaced by a greater number.

(2) If we set $0 < a_1 = \cdots = a_{n-1} < a_n = 1$, then we obtain

$$\lim_{a_1 \to 1} \left(\max_{1 \leq i \leq n} a_i - \sum_{i=1}^{n} p_i a_i\right) \Big/ \sum_{1 \leq i < j \leq n} \left(\sqrt{|a_i|} - \sqrt{|a_j|}\right)^2$$
$$= \lim_{a_1 \to 1} \frac{1 - p_n}{n-1} \frac{1 + \sqrt{a_1}}{1 - \sqrt{a_1}} = \infty.$$

This implies that there does not exist a converse of inequality (2.1), that is, there does not exist a constant $c(p_1, \ldots, p_n)$ such that the inequality

$$\max_{1 \leq i \leq n} a_i - \sum_{i=1}^{n} p_i a_i \leq c(p_1, \ldots, p_n) \sum_{1 \leq i < j \leq n} \left(\sqrt{|a_i|} - \sqrt{|a_j|}\right)^2$$

holds for all real numbers $a_1, \ldots, a_n$.

3. An inequality for convex functions

The following proposition provides a sharpening of inequality (1.1).

Theorem 2. *Let a, b, s_i and t_i $(i=0, 1, 2)$ be positive real numbers with $a/s_i + b/t_i = 1$ $(i = 0, 1, 2)$ and $s_1 < s_0 < s_2$, and let $f, g : (0, \infty) \to \mathbb{R}$ be convex functions. Then*

$$h_0 \le \max(h_1, h_2) - \alpha\left[\sqrt{|h_1|} - \sqrt{|h_2|}\right]^2,$$

where

$$h_i = \frac{f(s_i)}{s_i} + \frac{g(t_i)}{t_i} \quad (i = 0, 1, 2)$$

and

$$\alpha = \min\left(\frac{(s_2 - s_0)s_1}{(s_2 - s_1)s_0}, \frac{(s_0 - s_1)s_2}{(s_2 - s_1)s_0}\right) \in (0, 1).$$

Proof. We follow the method of proof given in **[5]**. There exists a real number $\delta \in (0, 1)$ such that

$$s_0 = \delta s_1 + (1 - \delta)s_2. \tag{3.1}$$

We set $\eta = \delta(s_1/s_0)(t_0/t_1)$; then we obtain

$$\begin{aligned}
\eta + (1 - \delta)\frac{s_2}{s_0}\frac{t_0}{t_2} &= \frac{t_0}{s_0}\left(\delta\frac{s_1}{t_1} + (1 - \delta)\frac{s_2}{t_2}\right) \\
&= \frac{b}{s_0 - a}\left(\delta\frac{s_1 - a}{b} + (1 - \delta)\frac{s_2 - a}{b}\right) \\
&= \frac{\delta s_1 + (1 - \delta)s_2 - a}{s_0 - a} \\
&= 1,
\end{aligned}$$

which leads to

$$t_0 = \eta t_1 + (1 - \eta)t_2, \quad \eta \in (0, 1). \tag{3.2}$$

Applying Jensen's inequality we get from (3.1) and (3.2):

$$f(s_0) \le \delta f(s_1) + (1 - \delta)f(s_2) \tag{3.3}$$

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and

$$g(t_0) \le \eta g(t_1) + (1-\eta) g(t_2). \tag{3.4}$$

Since $\eta(t_1/t_0) = \delta(s_1/s_0)$ and $(1-\eta)(t_2/t_0) = (1-\delta)(s_2/s_0)$, we conclude from (3.3), (3.4) and (2.1) (with $n = 2$):

$$\begin{aligned} h_0 &= \frac{f(s_0)}{s_0} + \frac{g(t_0)}{t_0} \\ &\le \delta \frac{s_1}{s_0}\left(\frac{f(s_1)}{s_1} + \frac{g(t_1)}{t_1}\right) + (1-\delta)\frac{s_2}{s_0}\left(\frac{f(s_2)}{s_2} + \frac{g(t_2)}{t_2}\right) \\ &= \delta \frac{s_1}{s_0} h_1 + (1-\delta)\frac{s_2}{s_0} h_2 \\ &\le \max(h_1, h_2) - \min\left(\delta\frac{s_1}{s_0}, (1-\delta)\frac{s_2}{s_0}\right)\left[\sqrt{|h_1|} - \sqrt{|h_2|}\right]^2 \\ &= \max(h_1, h_2) - \alpha\left[\sqrt{|h_1|} - \sqrt{|h_2|}\right]^2. \end{aligned}$$

4. Inequalities for L^p norms and power sums

We establish the following refinements of inequalities (1.2) and (1.3).

Theorem 3. *Let a, b, s_i and t_i $(i = 0, 1, 2)$ be positive real numbers such that $a/s_i + b/t_i = 1$ $(i = 0, 1, 2)$ and $s_1 < s_0 < s_2$. Furthermore, let*

$$\alpha = \min\left(\frac{(s_2 - s_0)s_1}{(s_2 - s_1)s_0}, \frac{(s_0 - s_1)s_2}{(s_2 - s_1)s_0}\right).$$

(i) *Assuming that all quantities exist, we have*

$$P_0 \le \max(P_1, P_2) - \alpha\left[\sqrt{P_1} - \sqrt{P_2}\right]^2, \tag{4.1}$$

where

$$P_i = \|f\|_{s_i} \|g\|_{t_i} \quad (i = 0, 1, 2).$$

(ii) *If $x = (x_i)$, $u = (u_i)$ $(i = 1, \ldots, n)$ and $y = (y_j)$, $v = (v_j)$ $(j = 1, \ldots, m)$ are sequences of positive real numbers, then*

$$Q_0 \le \max(Q_1, Q_2) - \alpha\left[\sqrt{Q_1} - \sqrt{Q_2}\right]^2, \tag{4.2}$$

where

$$Q_i = S_n^{[s_i]}(x,u)S_m^{[t_i]}(y,v) \quad (i = 0, 1, 2) \quad \textit{and}$$
$$S_n^{[t]}(x,u) = \left(\sum_{i=1}^{n} u_i x_i^t\right)^{1/t}.$$

Proof. (i) Let

$$\delta = \frac{s_2 - s_0}{s_2 - s_1} \qquad \text{and} \qquad \eta = \delta \frac{s_1}{s_0}\frac{t_0}{t_1}. \tag{4.3}$$

Then, as shown in the proof of Theorem 2, we have $\delta, \eta \in (0, 1)$ and

$$s_0 = \delta s_1 + (1-\delta)s_2 \qquad \text{and} \qquad t_0 = \eta t_1 + (1-\eta)t_2. \tag{4.4}$$

Applying Hölder's inequality for integrals we obtain

$$\|f\|_{s_0}^{s_0} \le \|f\|_{s_1}^{\delta s_1}\ \|f\|_{s_2}^{(1-\delta)s_2}$$

and

$$\|g\|_{t_0}^{t_0} \le \|g\|_{t_1}^{\eta t_1}\ \|g\|_{t_2}^{(1-\eta)t_2}$$

which implies, since $\eta(t_1/t_0) = \delta(s_1/s_0)$ and $(1-\eta)(t_2/t_0) = (1-\delta)(s_2/s_0)$,

$$\begin{aligned} P_0 &= \|f\|_{s_0}\ \|g\|_{t_0} \\ &\le \left(\|f\|_{s_1}\ \|g\|_{t_1}\right)^{\delta s_1/s_0}\left(\|f\|_{s_2}\ \|g\|_{t_2}\right)^{(1-\delta)s_2/s_0} \\ &= P_1^{\delta s_1/s_0} P_2^{(1-\delta)s_2/s_0}. \end{aligned} \tag{4.5}$$

From (4.5), the arithmetic mean-geometric mean inequality $a^p b^q \le pa + qb$ $(a, b \ge 0;$ $p, q > 0, p+q = 1)$ and (2.1) (with $n = 2$) we get

$$\begin{aligned} P_0 &\le \delta\frac{s_1}{s_0}P_1 + (1-\delta)\frac{s_2}{s_0}P_2 \\ &\le \max(P_1, P_2) - \min\left(\delta\frac{s_1}{s_0}, (1-\delta)\frac{s_2}{s_0}\right)\left[\sqrt{P_1} - \sqrt{P_2}\right]^2 \\ &= \max(P_1, P_2) - \alpha\left[\sqrt{P_1} - \sqrt{P_2}\right]^2. \end{aligned}$$

(ii) If δ and η are given by (4.3), then we conclude from (4.4) and Hölder's inequality for sums that

$$\left(S_n^{[s_0]}(x,u)\right)^{s_0} \le \left(S_n^{[s_1]}(x,u)\right)^{\delta s_1} \left(S_n^{[s_2]}(x,u)\right)^{(1-\delta)s_2}$$

and

$$\left(S_m^{[t_0]}(y,v)\right)^{t_0} \le \left(S_m^{[t_1]}(y,v)\right)^{\eta t_1} \left(S_m^{[t_2]}(y,v)\right)^{(1-\eta)t_2}.$$

This leads to

$$\begin{aligned} Q_0 &= S_n^{[s_0]}(x,u) S_m^{[t_0]}(y,v) \le \left(S_n^{[s_1]}(x,u) S_m^{[t_1]}(y,v)\right)^{\delta s_1/s_0} \left(S_n^{[s_2]}(x,u) S_m^{[t_2]}(y,v)\right)^{(1-\delta)s_2/s_0} \\ &= Q_1^{\delta s_1/s_0} Q_2^{(1-\delta)s_2/s_0}, \end{aligned}$$

which implies, as shown in (i), that

$$Q_0 \le \max(Q_1, Q_2) - \alpha\left[\sqrt{Q_1} - \sqrt{Q_2}\right]^2.$$

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