EXTENSION OF A BOUNDED VECTOR MEASURE WITH VALUES IN A REFLEXIVE BANACH SPACE

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Introduction. A vector measure (countable additive set function with values in a Banach space) on a field may be extended to a vector measure on the generated σ-field, under certain hypotheses. For example, the extension is established for the bounded variation case [2, 5, 8], and there are more general conditions under which the extension exists [1]. The above results have as hypotheses fairly strong boundedness conditions on the norm of the measure to be extended. In this paper we prove an extension theorem of the same type with a restriction on the range, supposing further that the measure is merely bounded. In fact a vector measure on a σ-field is bounded (III. 4.5 of [3]) but it is conceivable that a vector measure on a field could be unbounded.

The proof of the extension theorem of this paper will depend on the following theorem of B.J. Pettis [3]:

"A weakly countably additive vector valued set function on a σ-field is countably additive, that is, a vector measure".

To indicate that a union is disjoint we will write $E_1 + E_2 + \ldots$ instead of $E_1 \cup E_2 \cup \ldots$ and ΣE_n instead of $\cup E_n$.

Extension theorem. A bounded vector measure on a field, taking its values in a reflexive Banach space, extends uniquely to a vector measure on the generated σ-field.

Proof. Let μ be a bounded vector measure on a field Σ , taking its values in a reflexive Banach space X . Let Σ' denote the σ-field generated by Σ and let X^* denote the dual space of X . Denote by χ the natural isomorphism of X onto X^{**} (dual space of X^*):

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$$\chi(x)\, x^* = x^* x \qquad (x \in X,\ x^* \in X^*) .$$

For each $x^* \in X^*$ the scalar set function

$$x^*\mu : E \to x^*\mu(E) \qquad (E \in \Sigma)$$

is a scalar measure on Σ and so has a unique extension to a scalar measure, denoted $\overline{x^*\mu}$, on Σ' .

With each $E \in \Sigma'$ we associate[1] the scalar function f_E on X^* :

$$f_E(x^*) = \overline{x^*\mu}(E) \qquad (x^* \in X^*) .$$

It follows, from the uniqueness of the scalar measure extensions, that f_E is a linear functional on X^* . In fact, for x^*, $y^* \in X^*$ and scalars α , β ,

$$\overline{(\alpha x^* + \beta y^*)\mu} = \alpha\, \overline{x^*\mu} + \beta\, \overline{y^*\mu}$$

because each member is a scalar measure on Σ' , and they coincide on Σ . Therefore

$$\begin{aligned} f_E(\alpha x^* + \beta y^*) &= \overline{(\alpha x^* + \beta y^*)\mu}\ (E) \\ &= \alpha\, \overline{x^*\mu}\ (E) + \beta\, \overline{y^*\mu}\ (E) \\ &= \alpha f_E(x^*) + \beta f_E(x^*) . \end{aligned}$$

We next show that, for any $E \in \Sigma'$, the linear functional f_E is continuous. For arbitrary $x^* \in X^*$, $\lambda = \overline{x^*\mu}$ is a finite scalar measure. Let $\lambda_1, \ldots, \lambda_4$ be the positive finite measures such that $\lambda_1 - \lambda_2$, $\lambda_3 - \lambda_4$ are the Jordan decompositions of the real, imaginary parts, respectively of λ. Let $A \in \Sigma'$, $\varepsilon > 0$ be arbitrary. It is clear from the proof of 13, D[G] that there exists a $B \in \Sigma$ such that, simultaneously, $\lambda_i(A-B) + \lambda_i(B-A) = \lambda_i(A \triangle B) < \frac{\varepsilon}{8}$ $(i = 1, \ldots, 4)$. Then,

$$|\lambda(A-B)| \le \sum_1^4 \lambda_i(A-B) < \frac{\varepsilon}{2}, \quad \text{and} \quad |\lambda(B-A)| < \frac{\varepsilon}{2}, \quad \text{so that}$$

$$||\lambda(A)| - |\lambda(B)|| \le |\lambda(A) - \lambda(B)| \le |\lambda(A-B)| + |\lambda(B-A)| < \varepsilon .$$

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[1] as suggested to the author by Professor E. Granirer

Since ε is arbitrary, it follows that, for all $x^* \varepsilon X^*$,

$$\sup_{A\varepsilon\Sigma'} |\overline{x^*\mu}(A)| = \sup_{A\varepsilon\Sigma} |x^*\mu(A)| .$$

Hence, for fixed $E\varepsilon\Sigma'$,

$$|f_E(x^*)| = |\overline{x^*\mu}(E)| \leq \sup_{A\varepsilon\Sigma'} |\overline{x^*\mu}(A)|$$

$$= \sup_{A\varepsilon\Sigma} |\overline{x^*\mu}(A)| = \sup_{A\varepsilon\Sigma} |x^*\mu(A)|$$

$$\leq \|x^*\| \sup_{A\varepsilon\Sigma} \|\mu(A)\| .$$

The continuity established, for each $E\varepsilon\Sigma'$, f_E is an element of X^{**}. So we have the set function ν on Σ', with values in X^{**}:

$$\nu(E) = f_E \qquad (E\varepsilon\Sigma') .$$

For disjoint sets E, F belonging to Σ',

$$f_{(E+F)}(x^*) = \overline{x^*\mu}(E+F) = \overline{x^*\mu}(E) + \overline{x^*\mu}(F)$$
$$= f_E(x^*) + f_F(x^*) .$$

This holding for all $x^* \varepsilon X^*$, we have

$$\nu(E+F) = f_{(E+F)} = f_E + f_F = \nu(E) + \nu(F) .$$

Thus the set function ν is finitely additive.

Let $E = \sum_1^\infty E_n$ $(E_n \varepsilon \Sigma')$ and let x^* be an arbitrary element of X^*:

$$\nu(E)x^* = f_E(x^*) = \overline{x^*\mu}(E) = \sum_1^\infty \overline{x^*\mu}(E_n)$$

$$= \sum_1^\infty f_{E_n}(x^*) = \sum_1^\infty \nu(E_n) x^* .$$

Let $\overline{\mu}$ be the set function on Σ' with values in X, defined: $\overline{\mu}(E) = \chi^{-1} \nu(E)$ $(E \varepsilon \Sigma')$. Then the last equation becomes

$$x^* \bar{\mu}(E) = \sum_1^\infty x^* \bar{\mu}(E_n) .$$

Since this equation holds for all $x^* \varepsilon X^*$ whenever $E = \sum_1^\infty E_n$ $(E_n \varepsilon \Sigma')$, the theorem of Pettis asserts that $\bar{\mu}$ is a vector measure on Σ'.

Let $E \varepsilon \Sigma$; then for arbitrary $x^* \varepsilon X^*$,

$$x^* \bar{\mu}(E) = \nu(E)\, x^* = f_E(x^*) = \overline{x^*\mu}\,(E) = x^* \mu(E) .$$

Therefore $\bar{\mu}(E) = \mu(E)$ for all $E \varepsilon \Sigma$. The existence of the extension established, it remains to prove its uniqueness. Let $\tilde{\mu}$ be a second vector measure on Σ' extending μ. By the uniqueness of the scalar measure extensions,

$$x^* \tilde{\mu} = x^* \bar{\mu} , \qquad \text{all } x^* \varepsilon X^* .$$

So for given $E \varepsilon \Sigma'$, $x^* \tilde{\mu}(E) = x^* \bar{\mu}(E)$ for all $x^* \varepsilon X^*$, and therefore $\tilde{\mu}(E) = \bar{\mu}(E)$.

REFERENCES

1. Gr. Arsene, S. Strătilă, Prolongement des mesures vectorielles. Rev. Roumaine Math. Pures appl. 10 (1965), 333-338.

2. N. Dinculeanu, On Regular Vector Measures. Acta Sci. Math. (Szeged) 24 (1963), 236-243.

3. Dunford-Schwartz, Linear Operators (part I). Interscience Publishers, New York (1957).

4. G. Fox, Note on the Borel method of measure extension. Canad. Math. Bull. 5 (1962), 285-296.

5. S. Găină, Extension of vector measures with finite variation. Rev. Roumaine Math. Pures appl. 8 (1963), 151-154.

6. P.R. Halmos, Measure Theory. D. Van Nostrand (1950).

https://doi.org/10.4153/CMB-1967-052-1 Published online by Cambridge University Press

7. L. Le Blanc and G. E. Fox, On the extension of measure by the method of Borel. Canad. Jour. Math. 8 (1956), 516-523.

8. M. Nicolescu, Mathematical Analysis. (Romanian) Vol. III Ed. Technica, Bucharest (1960).

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