PIETSCH INTEGRAL OPERATORS DEFINED ON INJECTIVE TENSOR PRODUCTS OF SPACES AND APPLICATIONS

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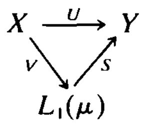
Abstract. For X and Y Banach spaces, let $X \otimes_\epsilon Y$, be the injective tensor product. If Z is also a Banach space and $U \in L(X \otimes_\epsilon Y, Z)$ we consider the operator

$$U^{\#}: X \to L(Y, Z), \qquad (U^{\#}x)(y) = U(x \otimes y), x \in X, y \in Y.$$

We prove that if $U \in PI(X \otimes_\epsilon Y, Z)$, then $U^{\#} \in I(X, PI(Y, Z))$. This result is then applied in the case of operators defined on the space of all X-valued continuous functions on the compact Hausdorff space T. We obtain also an affirmative answer to a problem of J. Diestel and J. J. Uhl about the RNP property for the space of all nuclear operators; namely if X^* and Y have the RNP and Y can be complemented in its bidual, then $N(X, Y)$ has the RNP.

An operator $U \in L(X, Y)$ is called a *Pietsch integral operator* if there exists a Y-valued vector measure with bounded variation on the Borel subsets of (U_{X^*}, weak^*) such that: $U(x) = \int_{U_{X^*}} x^*(x)\, dG(x^*)$ for each $x \in X$ and the Pietsch integral norm of U is: $\|U\|_{pint} = \inf |G|\,(U_{X^*})$. It is well known that the class of all Pietsch integral operators with the Pietsch integral norm is a normed ideal of operators in the sense of A. Pietsch, which in the sequel will be denoted by $(PI, \|\ \|_{pint})$. Also $U \in PI(X, Y)$ if and only if for each $\epsilon > 0$, U admits a factorisation of the form



where $V \in I(X, L_1(\mu))$, $S \in L(L_1(\mu), Y)$ and $\|V\|_{int} \le \|U\|_{pint} + \epsilon$, $\|S\| \le 1$; see [2] for details.

For the definition of integral operator, absolutely summing operator, nuclear operator and their basic properties see [2] or [4]. By $I(\ ,\ ), \|\ \|_{int}$, (resp. $(As, \|\ \|_{as})$, $(N, \|\ \|_{nuc})$ we denote the normed ideal of all integral operators (resp. absolutely summing operators, nuclear operators). For all notations and notions used and not defined we refer the reader to [2]. Given $U \in L(X \otimes_\epsilon Y, Z)$ we consider the operator $U^{\#}: X \to L(Y, Z)$ defined by $(U^{\#}x)(y) = U(x \otimes y)$, $x \in X$, $y \in Y$, that is evidently linear and continuous. Also for a given normed ideal of operators $\mathfrak{I}$ and $U \in \mathfrak{I}(X \otimes_\epsilon Y, Z)$ we have $U^{\#}x \in \mathfrak{I}(Y, Z)$, for any $x \in X$. Indeed, if $x \in X$, let $V_x \in L(Y, X \otimes_\epsilon Y)$ be the operator $V_x(y) = x \otimes y$, $y \in Y$. Since $U^{\#}x = UV_x$, by the ideal property of $\mathfrak{I}$ we obtain $U^{\#}x \in \mathfrak{I}(Y, Z)$. Hence for a normed ideal of operators $\mathfrak{I}$ and $U \in L(X \otimes_\epsilon Y, Z)$ we can consider the assertions

(a) $U \in \mathfrak{I}(X \otimes_\epsilon Y, Z)$,

(b) $U^{\#} \in \mathfrak{I}(X, \mathfrak{I}(Y, Z))$.

In the sequel for the normed ideal of Pietsch integral operators we study the connection between (a) and (b); see also [3], [6], [7] for corresponding work on other normed ideals.

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THEOREM 1. *If $U \in PI(X \otimes_\epsilon Y, Z)$, then $U^\# \in I(X, PI(Y,Z))$ and $\|U\|_{int} \leq \|U\|_{pint}$.*

Proof. We make first a remark. If W can be complemented in its bidual by a norm one projection, then $I(X \otimes_\epsilon Y, W) = I(X, I(Y, W))$, which follows easily using Theorem 2.1 from [**3**]. Now if $U \in PI(X \otimes_\epsilon Y, Z)$, then for each fixed $\epsilon > 0$, U admits a factorisation

$$\begin{array}{ccc} X \otimes_\epsilon Y & \xrightarrow{U} & Z \\ & {}_V\searrow \quad \nearrow_S & \\ & L_1(\mu) & \end{array}$$

where $V \in I(X \otimes_\epsilon Y, L_1(\mu))$, $S \in L(L_1(\mu), Z)$ and $\|V\|_{int} \leq \|U\|_{pint} + \epsilon$, $\|S\| \leq 1$. (Here μ is a regular Borel measure on some compact Hausdorff space Ω.). See [**2**, Theorem 11, p. 168]. Using the above remark for $W = L_1(\mu)$ we obtain that $V^\# \in I(X, I(Y, L_1(\mu)))$ and $\|V^\#\|_{int} = \|V\|_{int}$. However Grothendieck's theorem shows that $I(., L_1(\mu)) = PI(., L_1(\mu))$ and $\| \ \|_{int} = \| \ \|_{pint}$, (See [**2**, Theorem p. 558].) Thus we have the factorisation

$$\begin{array}{ccc} X & \xrightarrow{U^\#} & PI(Y,Z) \\ & {}_{V^\#}\searrow \quad \nearrow_{S^\#} & \\ & PI(Y, L_1(\mu)) & \end{array}$$

where $S^\#(A) = SA$, $A \in PI(Y, L_1(\mu))$ and, by the ideal property of the class of all integral operators, we obtain $U^\# \in I(X, PI(Y,Z))$ and $\|U^\#\|_{int} \leq \|V^\#\|_{int} \|S^\#\| \leq \|V^\#\|_{int} \|S\| \leq \|V\|_{int}$.

Thus $\|U^\#\|_{int} \leq \|U\|_{pint} + \epsilon$, hence $\|U^\#\|_{int} \leq \|U\|_{pint}$.

In the sequel, by T we denote a compact Hausdorff space and $C(T, X)$ will be the Banach space of all X-valued continuous functions on T under the supremum norm. For $X = \mathbf{R}$ (or $\mathbf{C}$) we note that $C(T, X) = C(T)$. By Σ we denote the σ-field of all Borel subsets of T. It is well known [**2**, p. 182] that any $U \in L(C(T, X), Y)$ has a representing finitely additive vector measure $G : \Sigma \to L(X, Y^{**})$. For $U \in L(C(T, X), Y)$, we consider the operator

$$U^\# : C(T) \to L(X, Y), \qquad (U^\# \varphi)(x) = U(\varphi x), \qquad \varphi \in C(T), \qquad x \in X.$$

Since $C(T, X) = C(T) \otimes_\epsilon X$, from Theorem 1 we obtain the following corollary.

COROLLARY 2. *Let $U \in L(C(T, X), Y))$, $U^\#$ be as above and G be the representing measure of U. We consider the following assertions*:

(a) *$U \in PI(C(T, X), Y)$;*

(b) *$U^\# \varphi \in PI(X, Y)$ for each $\varphi \in C(T)$ and $U^\# \in PI(C(T), PI(X, Y))$;*

(c) *$G(E) \in PI(X, Y)$ for each $E \in \Sigma$ and $G : \Sigma \to PI(X, Y)$ has bounded variation with respect to the Pietsch integral norm on $PI(X, Y)$.*

Then we have (a)$\Rightarrow$(b)$\Rightarrow$(c) and, in this case the following inequality holds: $\|U^\#\|_{\text{pint}} = |G|_{\text{pint}}(T) \leq \|U\|_{\text{pint}}$.

Proof. For the implication (a)$\Rightarrow$(b) we use Theorem 1 and the well known facts $PI(C(T), .) = As(C(T), .)$ and $\| \ \|_{pint} = \| \ \|_{as}$. See Theorem 12 of [**2**, p. 169]. For the implication (b)$\Rightarrow$(c) we again use Theorem 12 of [**2**, p. 69] and the obvious fact that the representing measure of U in the hypothesis of (b) coincides with that of U. The relations: $\|U^{\#}\|_{pint} = |G|_{pint}(T) \leq \|U\|_{pint}$ are also true. In this way arises the following conjecture.

Conjecture 3. If $U \in L(C(T,X),Y)$ has the representing measure G which satisfies the conditions

(1) $G(E) \in PI(X,Y)$ for each $E \in \Sigma$ and

(2) $G:\Sigma \to PI(X,Y)$ has bounded variation with respect to the Pietsch integral norm,

then it follows that $U \in PI(C(T,X),Y)$.

If Y can be complemented in its bidual, then it is well known that we have $I(.,Y) = PI(.,Y)$ Corollary 10 of [**2**, p. 235] and hence using the result of P. Saab from [**6**] we obtain that this with supplementary hypothesis about Y Conjecture 3 is true. In the sequel we describe the Question 5 from the paper of P. Saab [**6**] as the Saab conjecture.

Saab conjecture. If Y has the *RNP* and $U \in L(C(T,X),Y)$ has the representing measure G which satisfies the conditions

(1) $G(E) \in N(X,Y)$ for each $E \in \Sigma$ and

(2) $G:\Sigma \to N(X,Y)$ has bounded variation with respect to the nuclear norm,

it follows that $U \in N(C(T,X),Y)$.

Recall also the following open problem of Diestel and Uhl. See [**2**, p. 258].

Diestel-Uhl conjecture. If X^* and Y have the *RNP*, then the space of all nuclear operators from X to Y also has the *RNP*.

The following theorem establishes a connection between these problems.

THEOREM 4. *Conjecture* 3 *is true implies Saab conjecture is true implies Diestel-Uhl conjecture is true.*

Proof. Conjecture 3 is true implies Saab conjecture is true; it is obvious since, if Y has the *RNP*, then $PI(.,Y) = N(.,Y)$ and $\| \ \|_{pint} = \| \ \|_{nuc}$. See Theorem 2 of [**2**, p. 175].)

Saab conjecture is true implies Diestel-Uhl conjecture is true. Let X and Y be Banach spaces such that X^* and Y have the *RNP*. Let Σ be the Borel subsets of $[0,1]$ and $G \in rcabv(\Sigma, N(X,Y), \| \ \|_{nuc})$. Let $U:C([0,1],X) \to Y$ be the operator $U(f) = \int_0^1 f\,dG$, $f \in C(T,X)$. Then U is a linear and continuous operator and G is its representing measure.

Since Y has the *RNP* and the Saab conjecture is true then, U will be a nuclear operator. Since X^* has the *RNP* from [**5**, Theorem 1] or [**7**, Theorem 6] we obtain that $G:\Sigma \to (N(X,Y), \| \ \|_{nuc})$ has a Bochner integrable derivative $g \in L_1(\mu, N(X,Y), \| \ \|_{nuc})$, where $\mu = |G|_{nuc}$. Thus $N(X,Y)$ has the *RNP*.

In [**2**, Theorem 5 p. 249] and [**1**, Theorem 7 p. 119] are given positive answers to the Diestel-Uhl conjecture, with supplementary hypotheses about X or Y. Since as we have seen the Conjecture 3 is true when Y can be complemented in its bidual from Theorem 4 we obtain the following corollary which is another positive answer to the Diestel-Uhl conjecture different from those given in [**1**] and [**2**].

COROLLARY 5. *If X and Y are Banach spaces such that X^* and Y have the RNP and Y can be complemented in its bidual, then $N(X,Y)$ also has the RNP.*

REFERENCES

1. K. T. Andrews, The Radon Nikodym property for spaces of nuclear operators, *J. London Math. Soc.* **28** (1983), 113–122.
2. J. Diestel and J. J. Uhl, *Vector measures*, *Math Surveys* No 15, (A.M.S., 1977).
3. S. Montgomery-Smith and Paulette Saab, p-summing operators on injective tensor products of spaces, *Proc. Royal Soc. Edinburgh Sect.* **120** (1992), 283–296.
4. A. Pietsch, *Operator ideals* (Veb. Deutscher Verlag der Wiss., Berlin, 1978).
5. D. Popa, Nuclear operators in $C(T,X)$, *Studii si Cercet. Mat.* **42**(1) (1990), 47–50.
6. P. Saab, Integral operators on spaces of continuous vector valued functions, *Proc. Amer. Math. Soc.* **111** (1991) 1003–1013.
7. P. Saab and B. Smith, Nuclear operators on spaces of continuous vector valued functions, *Glasgow Math.*, **33** (1991), 223–230.
8. Charles Swartz, Absolutely summing and dominated operators on spaces of vector-valued continuous functions, *Trans. Amer. Math. Soc.* **179** (1973), 123–131.

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