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ON SOME CONSEQUENCES OF A THEOREM OF J. LUDWIG

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Abstract We prove some qualitative results about the p-adic Jacquet–Langlands correspondence defined by Scholze, in the $\mathrm{GL}_2(\mathbb{Q}_p)$ residually reducible case, using a vanishing theorem proved by Judith Ludwig. In particular, we show that in the cases under consideration, the global p-adic Jacquet–Langlands correspondence can also deal with automorphic forms with principal series representations at p in a nontrivial way, unlike its classical counterpart.

Keywords and phrases: p-adic Jacquet–Langlands correspondence; p-adic automorphic forms.

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1. Introduction

Let F be a finite extension of $\mathbb{Q}_p$, and let L be a further sufficiently large finite extension of F, which will serve as the field of coefficients. Let $\mathcal{O}$ be the ring of integers in L, ϖ be a uniformiser in $\mathcal{O}$ and $\mathcal{O}/\varpi = k$ be the residue field.

To a smooth admissible representation π of $\mathrm{GL}_n(F)$ on an $\mathcal{O}$-torsion module, Scholze in [47] attaches a sheaf $\mathcal{F}_\pi$ on the adic space $\mathbb{P}^{n-1}_{\mathbb{C}_p}$ and shows that the cohomology groups $H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p},\mathcal{F}_\pi)$ are admissible representations of $D_p^\times$, the group of units in a central division over F with invariant $1/n$, and carry a continuous commuting action of G_F, the absolute Galois group of F. His construction is expected to realise both p-adic local Langlands and p-adic Jacquet–Langlands correspondences. However, these groups seem to be very hard to compute, and even deciding whether they are zero or not is highly nontrivial.

In order to extend Scholze's construction to admissible unitary Banach space representations Π of $\mathrm{GL}_n(F)$, the following seems like a sensible thing to do: choose an open bounded $\mathrm{GL}_n(F)$-invariant lattice Θ in Π; then Θ/ϖ^m is an admissible smooth representation of $\mathrm{GL}_n(F)$ on an $\mathcal{O}$-torsion module and we can consider the limit $\varprojlim_m H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p},\mathcal{F}_{\Theta/\varpi^m})$ equipped with the p-adic topology. We would like to invert p and obtain a Banach space, but the $\mathcal{O}$-module might not be $\mathcal{O}$-torsion free, and once quotiented out by the torsion, it might not be Hausdorff. Hence, it seems sensible to define $\check{\mathcal{S}}^i(\Pi) := Q(\Theta)\otimes_{\mathcal{O}} L$, where $Q(\Theta)$ is the maximal Hausdorff, $\mathcal{O}$-torsion free quotient of $\varprojlim_m H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p},\mathcal{F}_{\Theta/\varpi^m})$. Even if we could compute $H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p},\mathcal{F}_{\Theta/\varpi})$ and show that it is nonzero, we cannot conclude that $\check{\mathcal{S}}^i(\Pi)$ is nonzero. However, it is easy to see that if the neighbouring groups $H^{i-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p},\mathcal{F}_{\Theta/\varpi})$ and $H^{i+1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p},\mathcal{F}_{\Theta/\varpi})$ both vanish, then the nonvanishing of $H^{i}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p},\mathcal{F}_{\Theta/\varpi})$ implies the nonvanishing of $\check{\mathcal{S}}^i(\Pi)$ (see Section 3.3).

If $F=\mathbb{Q}_p$ and $n=2$, and π is a principal series representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, then Judith Ludwig shows in [31] the vanishing of $H^i_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p},\mathcal{F}_\pi)$ when $i=2$. All the other groups for $i>2$ are known to vanish in this case due to Scholze. Moreover, it follows from Scholze's results that $H^0(\mathbb{P}^1_{\mathbb{C}_p},\mathcal{F}_\pi)$ also vanishes, if $\pi^{\mathrm{SL}_2(\mathbb{Q}_p)}$ is equal to zero. Thus if we restrict our attention only to those representations π which have all irreducible subquotients isomorphic to irreducible principal series representations, then the functor $\pi\mapsto H^1_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p},\mathcal{F}_\pi)$ is exact. Such subcategories of smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\mathcal{O}$-torsion modules have been studied in [37] and shown to be related to the reducible 2-dimensional mod p representations of $G_{\mathbb{Q}_p}$. The exactness of the functor $\pi\mapsto H^1_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p},\mathcal{F}_\pi)$ allows us to use arguments of Kisin [27], who used the exactness of Colmez's functor to make a connection between the deformation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$-representations and the deformation theory of 2-dimensional $G_{\mathbb{Q}_p}$-representations. For simplicity, we assume that $p>2$ in this article.

Theorem 1.1. *Let $r:G_{\mathbb{Q}_p}\to\mathrm{GL}_2(L)$ be a continuous representation with $\bar{r}^{\mathrm{ss}}=\chi_1\oplus\chi_2$, where $\chi_1,\chi_2:G_{\mathbb{Q}_p}\to k^\times$ are characters such that $\chi_1\chi_2^{-1}\neq\omega^{\pm1}$, where ω is the mod p cyclotomic character. Let Π be the admissible unitary L-Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ corresponding to r via the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. Then $\check{\mathcal{S}}^1(\Pi)\neq 0$.*

Previously, such a result was known only in the case when r is a (twist of a) potentially semistable, non-crystabelline representation lying on an automorphic component of a potentially semistable deformation ring, proved by Chojecki and Knight [12]. They prove it by patching and showing that locally algebraic vectors in $\check{\mathcal{S}}^1(\Pi)$ are nonzero. Their argument relies on the theorem of Emerton [19], which allows them to interpret classical automorphic forms as locally algebraic vectors in completed cohomology and enables them to handle only the representations which are 'discrete series at p'. For example, their argument does not work for representations which become crystalline after restriction to a Galois group of an abelian extension of $\mathbb{Q}_p$ (principal series at p), or for representations which do not become potentially semistable after twisting by a character (nonclassical).

Our argument works as follows: by the mod p local Langlands correspondence to $\bar{r}^{\mathrm{ss}}$, one may associate two principal series representations π_1 and π_2. We know from [15, 37] that the semisimplification of Π^0/ϖ is isomorphic to $\pi_1\oplus\pi_2$, where Π^0 is a unit ball in Π. Using the exactness results already described, it is enough to show that at least one of $H^1_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p},\mathcal{F}_{\pi_1})$, $H^1_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p},\mathcal{F}_{\pi_2})$ does not vanish. To show that, it is enough to find some π with all irreducible subquotients isomorphic to either π_1 or π_2, such that $H^1_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p},\mathcal{F}_\pi)$ does not vanish. As we have already mentioned, it seems impossible to compute $H^1_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p},\mathcal{F}_\pi)$ in general. However, Scholze manages to do so for certain representations coming from geometry. If $\pi=S(U^p,\mathcal{O}/\varpi^n)$ is a 0th completed cohomology group of a tower of zero-dimensional Shimura varieties associated to a quaternion algebra D_0 over[1] $\mathbb{Q}$ which is

[1] In the main body of the article, we work with a totally real field F, such that $F_{\mathfrak{p}}=\mathbb{Q}_p$ for a place $\mathfrak{p}$. We assume that $F=\mathbb{Q}$ for the purpose of this introduction.

split at p and ramified at ∞, Scholze shows in [47] that $H^1_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p},\mathcal{F}_\pi)$ is isomorphic as a $G_{\mathbb{Q}_p}\times D_p^\times$-representation to the 1st completed cohomology group $\widehat{H}^1(U^p,\mathcal{O}/\varpi^n)$ of a tower of Shimura curves associated to a quaternion algebra D, which is ramified at p and split at ∞ and has the same ramification as D_0 at all the other places. Here U^p denotes some fixed tame level. Scholze also shows that this isomorphism respects the action by Hecke operators on both sides. We show that a localisation of $S(U^p,\mathcal{O}/\varpi^n)$ at a maximal ideal $\mathfrak{m}$ of the Hecke algebra corresponding to an absolutely irreducible Galois representation $\bar\rho: G_{\mathbb{Q},S}\to \mathrm{GL}_2(k)$, such that the semisimplification of $\bar\rho|_{G_{\mathbb{Q}_p}}$ is equal to $\chi_1\oplus\chi_2$ as before, has all irreducible subquotients isomorphic to π_1 or π_2 as before. By applying Scholze's functor to this representation, we obtain $\widehat{H}^1(U^p,\mathcal{O}/\varpi^n)_{\mathfrak{m}}$, which can be shown to be nonzero using the classical Jacquet–Langlands correspondence.

Let K be any open uniform pro-p subgroup of $D_p^\times$; then its completed group ring $\mathcal{O}[[K]]$ and the localisation $\mathcal{O}[[K]]\otimes_{\mathcal{O}} L$ are both Auslander regular rings, and as a consequence there is a good dimension theory for their finitely generated modules, generalising the Krull dimension. If B is an admissible unitary Banach space representation of $D_p^\times$, then its Schikhof dual B^d is a finitely generated $\mathcal{O}[[K]]\otimes_{\mathcal{O}} L$-module, and we define the δ-dimension of B to be the dimension of B^d in this sense.

Theorem 1.2. *If* Π *is as in Theorem 1.1, then the* δ*-dimension of* $\check{\mathcal{S}}^1(\Pi)$ *is one.*

This theorem is proved by using the observation of Kisin in [27] that exact functors take flat modules to flat modules, and the results of Gee and Newton [23] on miracle flatness in a noncommutative setting.

One can show that a Banach space representation of $D_p^\times$ has δ-dimension zero if and only if it is finite-dimensional as an L-vector space. Moreover, the zero-dimensional representations build a Serre subcategory, and thus one may pass to a quotient category. Informally, this means that two 1-dimensional Banach space representations are isomorphic in the quotient category if they differ by a 0-dimensional Banach space representation. As Kohlhaase has pointed out to me, the quotient category of Banach space representation of δ-dimension at most one by the zero-dimensional Banach space representations is Noetherian and has an involution; hence it is also Artinian, and hence every object in the quotient category has finite length. From this it is easy to deduce the following:

Corollary 1.3. *If* Π *is as in Theorem 1.1, then* $\check{\mathcal{S}}^1(\Pi)$ *is of finite length in the category of admissible unitary* L*-Banach space representations of* $D_p^\times$ *if and only if it has finitely many irreducible subquotients, which are finite dimensional as* L*-vector spaces.*

We got quite excited about this corollary at first, since we hoped that it might imply that $\check{\mathcal{S}}^1(\Pi)$ is of finite length as a Banach space representation of $D_p^\times$ by some formal representation theoretic arguments. However, here is an example suggesting that one should be cautious. If $K=\mathbb{Z}_p$, then $\mathcal{O}[[K]]\cong\mathcal{O}[[x]]$, a commutative formal power series ring in one variable, and the Banach space of continuous functions on K has δ-dimension one and is irreducible in the quotient category, which is equivalent to the category of finite-dimensional vector spaces over the fraction field of $\mathcal{O}[[x]]$; its Schikhof dual is isomorphic to $\mathcal{O}[[x]][1/p]$, and all irreducible subquotients are finite-dimensional (see, however, Theorem 1.5).

1.1. Global p-adic Jacquet–Langlands correspondence

Theorem 1.1 has a global application, which we state in the introduction for $F = \mathbb{Q}$. The results are proved for a totally real field F such that $F_{\mathfrak{p}} = \mathbb{Q}_p$ (see Theorems 6.11 and 6.17 and Proposition 6.15). Scholze has shown in [47, Corollary 7.3] that

$$S(U^p,\mathcal{O})_{\mathfrak{m}} := \varprojlim_n S(U^p,\mathcal{O}/\varpi^n)_{\mathfrak{m}}, \quad \widehat{H}^1(U^p,\mathcal{O})_{\mathfrak{m}} := \varprojlim_n \widehat{H}^1(U^p,\mathcal{O}/\varpi^n)_{\mathfrak{m}}$$

have an action of the same Hecke algebra $\mathbb{T}(U^p)_{\mathfrak{m}}$. Moreover, there is a surjective homomorphism of local rings $R_{\bar{\rho}} \twoheadrightarrow \mathbb{T}(U^p)_{\mathfrak{m}}$, where $R_{\bar{\rho}}$ is the universal deformation ring of $\bar{\rho}$.

Theorem 1.4. *Let $x \in \text{m-Spec}\,\mathbb{T}(U^p)_{\mathfrak{m}}[1/p]$ be such that the restriction of the corresponding Galois representation $\rho_x : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\kappa(x))$ to $G_{\mathbb{Q}_p}$ is irreducible. Then $(\widehat{H}^1(U^p,\mathcal{O})_{\mathfrak{m}} \otimes L)[\mathfrak{m}_x]$ is nonzero if and only if $(S(U^p,\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} L)[\mathfrak{m}_x]$ is nonzero.*

In this case, there is an isomorphism of admissible unitary $\kappa(x)$-Banach space representations of $G_{\mathbb{Q}_p} \times D_p^{\times}$:

$$(\widehat{H}^1(U^p,\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} L)[\mathfrak{m}_x] \cong \check{\mathcal{S}}^1(\Pi)^{\oplus n}, \tag{1}$$

where Π is the absolutely irreducible $\kappa(x)$-Banach space representation corresponding to $\rho_x|_{G_{\mathbb{Q}_p}}$ via the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. In particular, the δ-dimension of $(\widehat{H}^1(U^p,\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} L)[\mathfrak{m}_x]$ is 1.

The first part of this theorem should be thought of as a global Jacquet–Langlands correspondence between the p-adic automorphic forms on $D_0^{\times}$ and $D^{\times}$. In particular, since D_0 is split at p, one can always find a classical automorphic form, which is a principal series at p, such that $(S(U^p,\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} L)[\mathfrak{m}_x]$ is nonzero, where $\mathfrak{m}_x$ is the corresponding maximal ideal. The theorem tells us that there is a p-adic automorphic form on $D^{\times}$ corresponding to it. Such a form cannot be classical, since the classical Jacquet–Langlands correspondence cannot cope with principal series.

It was pointed out to me by Sean Howe that in his 2017 University of Chicago PhD thesis [25], he proves an analogue of the first part of Theorem 1.4 in the setting when D_0 is a quaternion algebra over $\mathbb{Q}$ ramified at p and ∞ and $D = \mathrm{GL}_2$, for the maximal ideals corresponding to overconvergent Galois representations. He conjectures that the analogue of (1) holds in his setting.

Theorem 1.5. *Assume the setup of Theorem 1.4. Let $D_p^{\times,1}$ be the subgroup of $D_p^{\times}$ of elements with reduced norm equal to 1. Let $\check{\mathcal{S}}^1(\Pi)^{1\text{-alg}}$ be the subspace of locally algebraic vectors for the action of $D_p^{\times,1}$ on $\check{\mathcal{S}}^1(\Pi)$. Then $\check{\mathcal{S}}^1(\Pi)^{1\text{-alg}}$ is a finite-dimensional L-vector space. If it is nonzero, then $\rho_x|_{G_{\mathbb{Q}_p}}$ is a twist of a potentially semistable representation, which does not become crystalline after restriction to the Galois group of any abelian extension of $\mathbb{Q}_p$. The quotient $\check{\mathcal{S}}^1(\Pi)/\check{\mathcal{S}}^1(\Pi)^{1\text{-alg}}$ contains an irreducible closed subrepresentation of δ-dimension 1.*

To the best of my knowledge, the existence of irreducible admissible unitary Banach space representations of $D_p^\times$ of dimension 1 has not been known before.

If $F=\mathbb{Q}$, as we assume in this introduction, then, after twisting, Theorem 1.5 follows readily from Theorem 1.4 using Emerton's results on locally algebraic vectors in completed cohomology [19]. However, when F is a totally real field, p-adic automorphic representations must be dealt with, which have locally algebraic vectors at all places above p except for one, where they have locally algebraic vectors after a twist by a character which is not locally algebraic.

Let us point out that because $D_p^\times$ has an open normal pro-p subgroup, every smooth irreducible representation of $D_p^\times$ in characteristic p is finite-dimensional as a vector space. Moreover, if we fix a central character, there are only finitely many isomorphism classes. It follows from Theorem 1.2 that at least one of $H^1_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p},\mathcal{F}_{\pi_1})$ and $H^1_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p},\mathcal{F}_{\pi_2})$ must be an infinite-dimensional k-vector space and admissible as smooth representations of $D_p^\times$. This means that they are built together out of finite-dimensional pieces in some non-semisimple way. This non-semisimplicity makes the study of the mod-p Jacquet–Langlands correspondence complicated, and passing to semisimplification seems not to carry much information, since if the central character is fixed, only finitely many isomorphism classes of irreducible subquotients can appear.

If we work in characteristic zero, then this problem need not appear. One can show that any irreducible unitary Banach space representation of $D_p^\times$ on a finite-dimensional vector space is of the form $\operatorname{Sym}^b L^2\otimes\det^a\otimes\tau\otimes\eta\circ\operatorname{Nrd}$, where τ is a smooth irreducible representation and $\eta:\mathbb{Q}_p^\times\to L^\times$ is a unitary character. Such representations appear in the classical Jacquet–Langlands correspondence (up to a twist), and we speculate that if the Galois representation corresponding to Π does not become potentially semistable after twisting by a character, then such representations should not appear as subquotients of $\check{\mathcal{S}}^1(\Pi)$. It seems very reasonable to us in view of Corollary 1.3 and Theorem 1.5 to expect that $\check{\mathcal{S}}^1(\Pi)$ is of finite length as a Banach space representation of $D_p^\times$. This raises a natural question: whether one can construct the irreducible representations in Theorem 1.5 directly and prove local–global compatibility for them. We hope to pursue these questions in future work.

1.2. Patching

We have decided not to use patching in this article, since it would add another level of technicalities. However, let us indicate which parts of the article can be improved upon if one chooses to use patching.

In the miracle flatness theorem of Gee and Newton (see Proposition 4.1 for the form in which we use it), one needs the commutative ring to be regular, so we cannot apply it to $\mathbb{T}(U^p)_\mathfrak{m}$ and $S(U^p,\mathcal{O})_\mathfrak{m}$ directly. However, if one patches, under favourable assumptions one can arrange that the patched ring R_∞ is regular and then show that the patched module M_∞ is flat over R_∞ by the same argument. This would imply that $S(U^p,\mathcal{O})_\mathfrak{m}$ is flat over $\mathbb{T}(U^p)_\mathfrak{m}$. This is how Gee and Newton prove their theorem that big R is equal to big $\mathbb{T}$. As a consequence, one would know that the multiplicity n in (1) is independent of x, and one would not need to assume that $\rho_x|_{G_{\mathbb{Q}_p}}$ is irreducible. Instead, we use Cohen's

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structure theorem for complete local rings to prove the flatness of $S(U^p,\mathcal{O})_{\mathfrak{m}}$ over some (random) formally smooth subring of $\mathbb{T}(U^p)_{\mathfrak{m}}$ of the same dimension. As a consequence, we are forced to assume that $\rho_x|_{G_{\mathbb{Q}_p}}$ is irreducible, since we cannot exclude the possibility that at points corresponding to reducible Galois representations, the fibre contains only one of the two irreducible Banach spaces that should appear there. Since we can not exclude the possibility that one of $H^1_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p},\mathcal{F}_{\pi_1})$ or $H^1_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p},\mathcal{F}_{\pi_2})$ vanishes, this causes trouble. However, since we do not need precise knowledge of $(S(U^p,\mathcal{O})_{\mathfrak{m}}\otimes_{\mathcal{O}} L)[\mathfrak{m}_x]$, only that all irreducible subquotients of Π appear there, arguments of Breuil and Emerton [8] might be used here to get the same result without patching.

A further improvement could be made in Theorem 1.5. Right now we use in an essential way the fact that Π corresponds to the restriction to $G_{\mathbb{Q}_p}$ of a global Galois representation. One could spread this result out by first showing that the patched module M_∞ is projective as a $\mathrm{GL}_2(\mathbb{Q}_p)$-representation in a suitable category, as in [11], so that if $r: G_{\mathbb{Q}_p}\to \mathrm{GL}_2(L)$ is an irreducible representation satisfying $\bar{r}^{\mathrm{ss}}=\chi_1\oplus\chi_2$, then the corresponding $\mathrm{GL}_2(\mathbb{Q}_p)$-representation Π can be obtained by specialising M_∞ at some closed point y of $R_\infty[1/p]$. If σ is a finite-dimensional $D_p^\times$-invariant subspace of $\check{\mathcal{S}}^1(\Pi)$, then it is of the form $\mathrm{Sym}^b L^2\otimes \det^a\otimes\tau\otimes\eta\circ\mathrm{Nrd}$. Let σ° be an $D_p^\times$-invariant $\mathcal{O}$-lattice in σ. Then $M'_\infty(\sigma^\circ):=\mathrm{Hom}_{D_p^\times}(\check{\mathcal{S}}^1(M_\infty),(\sigma^0)^d)^d$ is a finitely generated R_∞-module. Arguing as in [11, Lemma 4.18] and using Theorem 1.5, we see that the Galois representations corresponding to the closed points in $\text{m-Spec}\,R_\infty[1/p]$ in the support $M'_\infty(\sigma^\circ)$ have the same p-adic Hodge theoretic properties – so, for example, if η is trivial, then all of them are potentially semistable and have the same Hodge–Tate weights and inertial type. Moreover, the restriction of the associated Weil–Deligne representation to the Weil group of $\mathbb{Q}_p$ is of 'discrete series type'. As a part of patching, we obtain an ideal $\mathfrak{a}_\infty$ of R_∞ generated by an M_∞-regular sequence $y_1,\dots,y_h$ such that $M_\infty/\mathfrak{a}_\infty M_\infty\cong S(U^p,\mathcal{O})_{\mathfrak{m}}$. Then $\check{\mathcal{S}}^1(M_\infty)/\mathfrak{a}_\infty\check{\mathcal{S}}^1(M_\infty)\cong \widehat{H}^1(U^p,\mathcal{O})_{\mathfrak{m}}$ and $\varpi,y_1,\dots,y_h$ is a system of parameters for $M'_\infty(\sigma^\circ)$. Thus $M'_\infty(\sigma^\circ)/\mathfrak{a}_\infty M'_\infty(\sigma^\circ)[1/p]$ is nonzero, and we can convert this into a representation theoretic information using [39, Proposition 2.22] to deduce that there is some $x\in \text{m-Spec}\,\mathbb{T}(U^p)[1/p]$, such that $\mathrm{Hom}_{D_p^\times}(\sigma,(\widehat{H}^1(U^p,\mathcal{O})_{\mathfrak{m}}\otimes_{\mathcal{O}} L)[\mathfrak{m}_x])\neq 0$. Then, as in the proof of Theorem 6.17, the p-adic Hodge theoretic data of $\rho_x|_{G_{\mathbb{Q}_p}}$ (and hence of r) will determine σ. This allows us to conclude that if $\mathrm{Hom}_{D_p^\times}(\sigma,\check{\mathcal{S}}^1(\Pi))$ is nonzero, then $\check{\mathcal{S}}^1(\Pi)^{1\text{-alg}}$ is isomorphic to a finite direct sum of copies of σ and thus is a finite-dimensional L-vector space. Arguing as in the proof of Theorem 6.17, we conclude that $\check{\mathcal{S}}^1(\Pi)/\check{\mathcal{S}}^1(\Pi)^{1\text{-alg}}$ is nonzero, and any irreducible Banach space subrepresentation has δ-dimension equal to 1. The details of this argument will appear in a subsequent article.

1.3. Outline

We end this introduction with a brief outline of how the article is structured. Section 3 contains the local part of the article. We first review the results of Ludwig and Scholze, and then recall some results about the representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$ proved in [37]. In subsection 3.3, in order to make things transparent for the reader, we dualise everything at least twice. In section 4 we recall the notion of dimension for finitely generated modules over Auslander regular rings and the results of Gee and Newton on miracle flatness in a noncommutative setting. In section 5 we prove the local–global compatibility results. The main objective of this section is to be able to apply the results of [37] to the completed cohomology. These results were known to the authors of [11] at the time of writing that article. However, the results proved in section 5 form a technical backbone of this article, so it seems like a good idea to write down the details. The main ingredients are the theory of capture [15, §2.4], [40, §2.1], which is based on the ideas of Emerton in [18], the results of Berger and Breuil [5] on the universal unitary completions of locally algebraic principal series representations and the results of Colmez in [13] on the compatibility of the p-adic and classical local Langlands correspondences. In section 6 we prove the theorems stated in the introduction.

2. Notation

Our conventions on local class field theory, the classical local Langlands correspondence and p-adic Hodge theory agree with those of [11]. In particular, uniformisers correspond to geometric Frobenii, and the p-adic cyclotomic character $\varepsilon : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ has Hodge–Tate weight -1. We will denote its reduction modulo p by ω. We will denote by $\chi_{\mathrm{cyc}} : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ the global p-adic cyclotomic character.

If G is a p-adic analytic group, then we use the notation scheme introduced in [20] for the categories of its representations. In particular, $\mathrm{Mod}_G^{\mathrm{sm}}(\mathcal{O})$ denotes the category of smooth representations of G on $\mathcal{O}$-torsion modules, $\mathrm{Mod}_G^{\mathrm{adm}}(\mathcal{O})$ denotes the full subcategory of admissible representations and $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})$ denotes the full subcategory of locally admissible representations of G.

If $\zeta : Z(G) \to \mathcal{O}^\times$ is a continuous character of the centre of G, then we add a subscript ζ to indicate that we consider only those representations on which $Z(G)$ acts by the central character ζ. For example, $\mathrm{Mod}_{G,\zeta}^{\mathrm{l.adm}}(\mathcal{O})$ is the full subcategory of $\mathrm{Mod}_G^{\mathrm{sm}}(\mathcal{O})$ consisting of locally admissible representations on which $Z(G)$ acts by the character ζ.

The functor $M \mapsto M^\vee := \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(M, L/\mathcal{O})$ induces an antiequivalence of categories between the category of discrete $\mathcal{O}$-modules and the category of compact $\mathcal{O}$-modules. We will refer to this duality as Pontryagin duality and to $M^\vee$ as the Pontryagin dual of M.

Pontryagin duality induces an antiequivalence of categories between $\mathrm{Mod}_G^{\mathrm{sm}}(\mathcal{O})$ and a category which Emerton calls profinite augmented G-representations over $\mathcal{O}$ [20, Definition 2.1.6] and which we will denote by $\mathrm{Mod}_G^{\mathrm{pro}}(\mathcal{O})$. We will denote the full subcategory of $\mathrm{Mod}_G^{\mathrm{pro}}(\mathcal{O})$ antiequivalent to $\mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(\mathcal{O})$ by Pontryagin duality by $\mathrm{Mod}_{G,\zeta}^{\mathrm{pro}}(\mathcal{O})$. In particular, if G is compact, then $\mathrm{Mod}_{G,\zeta}^{\mathrm{pro}}(\mathcal{O})$ is the category of compact $\mathcal{O}[[G]]$-modules on which the centre acts by ζ^{-1}, where $\mathcal{O}[[G]]$ is the completed group algebra of G.

We would like to review Schikhof duality, which we learned from [46]. Let Π be an L-Banach space and let Θ be an open bounded lattice in Π. The Schikhof dual Θ^d is defined as $\mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(\Theta, \mathcal{O})$ equipped with the weak topology (or the topology of pointwise convergence). It is $\mathcal{O}$-torsion free and compact (see the proof of [46, Theorem 1.2]). Also let $\Pi^d := \mathrm{Hom}_L^{\mathrm{cont}}(\Pi, L)$ equipped with the weak topology. It follows from [44, Proposition 3.1] that $\Pi^d \cong \Theta^d \otimes_{\mathcal{O}} L$.

Conversely, if M is a linearly compact, torsion-free $\mathcal{O}$-module, then $\mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(M,L)$, equipped with a supremum norm, is an L-Banach space with the unit ball $M^d := \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(M,\mathcal{O})$. Note that M^d is complete for the p-adic topology. Since M is $\mathcal{O}$-torsion free, it is projective in the category of linearly compact $\mathcal{O}$-modules (see [46, Remark 1.1]). From this we obtain an isomorphism

$$M^d/\varpi^n \cong (M/\varpi^n)^\vee \tag{2}$$

(see the proof of [35, Lemma 5.4]). We thus obtain a homeomorphism

$$M^d \cong \varprojlim_n M^d/\varpi^n \cong \varprojlim_n (M/\varpi^n)^\vee. \tag{3}$$

It follows from [44, Remark 10.2] applied to Π and the gauge of Θ that Θ/ϖ^n is a free $\mathcal{O}/\varpi^n$-module with basis indexed by a choice of k-vector space basis of Θ/ϖ. Thus $\mathrm{Hom}_{\mathcal{O}}(\Theta/\varpi^n,\mathcal{O}/\varpi^n) \to \mathrm{Hom}_{\mathcal{O}}(\Theta/\varpi^n,\mathcal{O}/\varpi^m)$ is surjective for $n \geq m$. By passing to the limit, we deduce that $\mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(\Theta,\mathcal{O}) \to \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(\Theta,\mathcal{O}/\varpi^n)$ is surjective and induces a homeomorphism

$$\Theta^d/\varpi^n \cong (\Theta/\varpi^n)^\vee. \tag{4}$$

It follows from (3) applied to $M = \Theta^d$ and (4) that

$$(\Theta^d)^d \cong \varprojlim_n (\Theta^d/\varpi^n)^\vee \cong \varprojlim_n ((\Theta/\varpi^n)^\vee)^\vee \cong \varprojlim_n \Theta/\varpi^n \cong \Theta. \tag{5}$$

If $\Theta = M^d$, then (4) and (2) give

$$\Theta^d/\varpi^n \cong (M^d/\varpi^n)^\vee \cong ((M/\varpi^n)^\vee)^\vee \cong M/\varpi^n. \tag{6}$$

Since Θ^d and M are compact $\mathcal{O}$-modules, we obtain a homeomorphism

$$(M^d)^d \cong \varprojlim_n \Theta^d/\varpi^n \cong \varprojlim_n M/\varpi^n \cong M. \tag{7}$$

We are most interested in the situation when Π is a unitary L-Banach space representation of a p-adic analytic group G. In this case, Θ^d is a topological $\mathcal{O}[[K]]$-module, where K is a compact open subgroup of G (see [46, Section 2]). We say that Π is *admissible* if Θ^d is finitely generated as an $\mathcal{O}[[K]]$-module. The functor $\Pi \mapsto \Pi^d$ induces an antiequivalence of categories between the category of admissible unitary L-Banach space representations of K and the category of finitely generated $\mathcal{O}[[K]] \otimes_{\mathcal{O}} L$-modules [46, Theorem 3.5].

Colmez in [13, §IV] has defined an exact covariant functor $\mathbf{V}$ from the category of finite-length smooth representations of $G := \mathrm{GL}_2(\mathbb{Q}_p)$ with a central character on $\mathcal{O}$-torsion modules to the category of smooth-finite length representations of $G_{\mathbb{Q}_p}$ on $\mathcal{O}$-torsion modules. We modify this functor as follows. Let $\mathfrak{C}_\zeta(\mathcal{O})$ be the full subcategory of $\mathrm{Mod}_G^{\mathrm{pro}}(\mathcal{O})$ antiequivalent to $\mathrm{Mod}_{G,\zeta}^{\mathrm{l.adm}}(\mathcal{O})$ by the Pontryagin duality. If M is in $\mathfrak{C}_\zeta(\mathcal{O})$, then we can write $M = \varprojlim_n M_n$, where the projective limit is taken over all quotients of finite length. Then define $\check{\mathbf{V}}(M) := \varprojlim_n \mathbf{V}(M_n^\vee)^\vee(\zeta)$, where ζ is viewed as a character of $G_{\mathbb{Q}_p}$ via local class field theory. This normalisation differs from [37] by a twist of cyclotomic character and coincides with the normalisation of [11]. In particular, if π is a

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principal series representation $\mathrm{Ind}_B^G \omega\chi_1 \otimes \chi_2$, then $\check{\mathbf{V}}(\pi^\vee) = \chi_1$, viewed as a character of $G_{\mathbb{Q}_p}$ via the local class field theory.

Let Π be an admissible unitary L-Banach space representation of G with central character ζ, and let Θ be an open bounded G-invariant lattice in Π. It is easy to see that Θ^d is an object of $\mathfrak{C}_\zeta(\mathcal{O})$ (see [37, Lemma 4.11]). Thus we can apply the functor $\check{\mathbf{V}}$ to Θ^d to obtain a continuous $G_{\mathbb{Q}_p}$-representation on a compact $\mathcal{O}$-module. We define $\check{\mathbf{V}}(\Pi) := \check{\mathbf{V}}(\Theta^d) \otimes_{\mathcal{O}} L$. The definition of $\check{\mathbf{V}}(\Pi)$ does not depend on the choice of Θ, since any two are commensurable. The functor $\Pi \mapsto \check{\mathbf{V}}(\Pi)$ is contravariant. If Π is absolutely irreducible and occurs as a subquotient of a unitary parabolic induction of a unitary character, then we say that Π is *ordinary*. Otherwise, we say that Π is *nonordinary*. In this case it is shown in [15, 37] that $\check{\mathbf{V}}(\Pi)$ is a 2-dimensional representation of $G_{\mathbb{Q}_p}$ and (taking into account our normalisations) $\det \check{\mathbf{V}}(\Pi) = \zeta\varepsilon^{-1}$. A deep theorem of Colmez proved in [13] relates the existence of locally algebraic vectors in Π to the property of $\check{\mathbf{V}}(\Pi)$ being potentially semistable with distinct Hodge–Tate weights. With our conventions, Colmez's result says that $\mathrm{Hom}_U(\det^a \otimes \mathrm{Sym}^b L^2, \Pi) \neq 0$ for some open subgroup U of $\mathrm{GL}_2(\mathbb{Z}_p)$ if and only if $\check{\mathbf{V}}(\Pi)$ is potentially semistable with Hodge–Tate weights $(1-a, -a-b)$.

If A is a commutative ring, then denote by m-Spec A the set of its maximal ideals. If $x \in$ m-Spec A, then $\kappa(x)$ will denote its residue field. We will typically be considering m-Spec A when $A = R[1/p]$, where R is a complete local Noetherian $\mathcal{O}$-algebra with residue field k. In this case, $\kappa(x)$ is a finite extension of L.

3. Local part

Let F be a finite extension of $\mathbb{Q}_p$. Fix an algebraic closure $\overline{F}$ of F and let $G_F = \mathrm{Gal}(\overline{F}/F)$. Let $\mathbb{C}_p$ be the completion of $\overline{F}$ and let $\breve{F}$ be the completion of the maximal unramified extension of F in $\overline{F}$. Let $G = \mathrm{GL}_n(F)$ and let D/F be a central division algebra over F with invariant $1/n$.

3.1. Results of Ludwig and Scholze

We continue to denote by L a (sufficiently large) finite extension of F with the ring of integers $\mathcal{O}$, uniformiser ϖ and residue field k. To a smooth representation π of G on an $\mathcal{O}$-torsion module, Scholze associates a Weil-equivariant sheaf $\mathcal{F}_\pi$ on the étale site of the adic space $\mathbb{P}^{n-1}_{\breve{F}}$ (see [47, Proposition 3.1]). If π is admissible, then he shows that for any $i \geq 0$ the étale cohomology groups $H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi)$ carry a continuous $D^\times \times G_F$-action, which makes them into smooth admissible representations of $D^\times$. Moreover, they vanish for $i > 2(n-1)$ (see [47, Theorems 3.2 and 4.4]). Scholze's construction is expected to realise both the p-adic Jacquet–Langlands and p-adic local Langlands correspondences. The trouble is that these groups seem to be impossible to calculate in most cases. It is known that the natural map

$$H^0_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi^{\mathrm{SL}_n(F)}}) \hookrightarrow H^0_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi) \tag{8}$$

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is an isomorphism (see [47, Proposition 4.7]). In particular, if $\pi^{\mathrm{SL}_n(F)}=0$, then H^0 vanishes.

We want to extend the results of Scholze to the category of locally admissible representations. Recall that a smooth representation of G or $D^\times$ on an $\mathcal{O}$-torsion module is *locally admissible* if it is equal to the union of its admissible subrepresentations. In this case we can write $\pi=\varinjlim \pi'$, where the limit is taken over all admissible subrepresentations of π. The proof of [47, Proposition 3.1] shows that the natural map

$$\varinjlim \mathcal{F}_{\pi'} \to \mathcal{F}_\pi$$

is an isomorphism, since it induces an isomorphism on stalks at geometric points. In our setting, cohomology commutes with direct limits. For all $i\ge 0$ we have isomorphisms

$$\begin{aligned} H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p},\mathcal{F}_\pi) &\cong \varinjlim_K H^i((\mathbb{P}^{n-1}_{\mathbb{C}_p}/K)_{\text{ét}},\mathcal{F}_\pi)\\ &\cong \varinjlim_K \varinjlim_{\pi'} H^i((\mathbb{P}^{n-1}_{\mathbb{C}_p}/K)_{\text{ét}},\mathcal{F}_{\pi'})\\ &\cong \varinjlim_{\pi'} H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p},\mathcal{F}_{\pi'}), \end{aligned} \tag{9}$$

where the first isomorphism follows from [47, Proposition 2.8] and the limit is taken over all compact open subgroups of $D^\times$; the second isomorphism follows from the fact that the site $(\mathbb{P}^{n-1}_{\mathbb{C}_p}/K)_{\text{ét}}$ is coherent (see [47, Lemma 2.7(v)]), so cohomology commutes with filtered direct limits [3, Exposé VI, Corollary 5.2]; and the last isomorphism is [47, Proposition 2.8] again. Since quotients of admissible representations of p-adic analytic groups are admissible, from (9) and Scholze's results we deduce that $H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p},\mathcal{F}_\pi)$ is a locally admissible representation of $D^\times$, which vanishes for $i>2(n-1)$.

If $n=2$, then the cohomology vanishes for $i>2$. If we additionally assume that $F=\mathbb{Q}_p$ and π is a principal series representation, Ludwig has shown in [31, Theorem 4.6] that $H^2_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p},\mathcal{F}_\pi)$ vanishes. Thus we get the following:

Corollary 3.1. *If π is a locally admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ such that all its irreducible subquotients are principal series, then*

$$H^i_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p},\mathcal{F}_\pi)=0,\quad \forall i\neq 1.$$

3.2. Representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$

From now on we assume that $n=2$ and $F=\mathbb{Q}_p$, so that $G=\mathrm{GL}_2(\mathbb{Q}_p)$. Recall some representation theory of G. The category $\mathrm{Mod}^{\mathrm{l.adm}}_G(\mathcal{O})$ of smooth locally admissible G-representations on $\mathcal{O}$-torsion modules decomposes into a direct sum of indecomposable subcategories called blocks. Blocks containing an absolutely irreducible k-representation of G correspond to semisimple representations $\bar{r}^{\mathrm{ss}}:G_{\mathbb{Q}_p}\to\mathrm{GL}_2(k)$, such that $\bar{r}^{\mathrm{ss}}$ is either absolutely irreducible or a direct sum of 1-dimensional representations.

Let us describe these blocks explicitly. Denote by $[\pi]$ the isomorphism class of a representation π of G. If $\bar{r}^{\mathrm{ss}}$ is absolutely irreducible, then let $\mathfrak{B}_{\bar{r}^{\mathrm{ss}}}=\{[\pi]\}$, where π is the supersingular representation of G corresponding to $\bar{r}^{\mathrm{ss}}$ under the semisimple mod p local Langlands correspondence [6]. If $\bar{r}^{\mathrm{ss}}=\chi_1\oplus\chi_2$, then χ_1 and χ_2 are considered as characters of $\mathbb{Q}_p^\times$ via the local class field theory, and let

$$\pi_1 := \mathrm{Ind}_B^G \chi_1\omega \otimes \chi_2, \quad \pi_2 := \mathrm{Ind}_B^G \chi_2\omega \otimes \chi_1, \tag{10}$$

where $\varepsilon : \mathbb{Q}_p^\times \to L^\times$ is the character, which maps x to $x|x|$, and $\omega : \mathbb{Q}_p^\times \to k^\times$ is its reduction modulo ϖ. Then let $\mathfrak{B}_{\bar{r}^{\mathrm{ss}}}$ be the set of isomorphism classes of irreducible subquotients of π_1 and π_2. It can have from one up to four elements, depending on $\chi_1\chi_2^{-1}$.

Let $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})_{\bar{r}^{\mathrm{ss}}}$ be the full subcategory of the category of locally admissible representations $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})$ of G, such that π is in $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})_{\bar{r}^{\mathrm{ss}}}$ if and only if the isomorphism classes of all the irreducible subquotients of π lie in $\mathfrak{B}_{\bar{r}^{\mathrm{ss}}}$. It follows from the Ext^1-calculations in [37, Proposition 5.34] and [38, Corollary 1.2] that $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})_{\bar{r}^{\mathrm{ss}}}$ is a direct summand of the category $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})$, in the sense that every $\pi \in \mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})$ can be written uniquely as

$$\pi = \pi_{\bar{r}^{\mathrm{ss}}} \oplus \pi^{\bar{r}^{\mathrm{ss}}}, \tag{11}$$

where $\pi_{\bar{r}^{\mathrm{ss}}}$ is the maximal G-invariant subspace of π, such that the isomorphism classes of all its irreducible subquotients lie in $\mathfrak{B}_{\bar{r}^{\mathrm{ss}}}$, and $\pi^{\bar{r}^{\mathrm{ss}}}$ is the maximal G-invariant subspace of π such that none of its irreducible subquotients lie in $\mathfrak{B}_{\bar{r}^{\mathrm{ss}}}$.

In this article we am especially interested in the case when $\bar{r}^{\mathrm{ss}} = \chi_1 \oplus \chi_2$ and $\chi_1\chi_2^{-1} \neq \omega^{\pm 1}$. In this case, both representations π_1 and π_2 in (10) are irreducible principal series representations, and so every $\pi \in \mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})_{\bar{r}^{\mathrm{ss}}}$ satisfies the hypothesis of Corollary 3.1. Let $\mathrm{Mod}_{G_{\mathbb{Q}_p}\times D^\times}^{\mathrm{l.adm}}(\mathcal{O})$ be the category of locally admissible representations of $D^\times$ on $\mathcal{O}$-torsion modules with a continuous commuting $G_{\mathbb{Q}_p}$-action. We immediately deduce the following:

Corollary 3.2. *If $\bar{r}^{\mathrm{ss}} = \chi_1 \oplus \chi_2$ and $\chi_1\chi_2^{-1} \neq \omega^{\pm 1}$, then the functor*

$$\mathcal{S}^1 : \mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})_{\bar{r}^{\mathrm{ss}}} \longrightarrow \mathrm{Mod}_{G_{\mathbb{Q}_p}\times D^\times}^{\mathrm{l.adm}}(\mathcal{O}), \quad \pi \mapsto H^1_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_\pi)$$

is exact and covariant.

3.3. Dual categories and Banach space representations

Let $\mathfrak{C}_G(\mathcal{O})$ be the category antiequivalent to $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})$ via Pontryagin duality, and let $\mathfrak{C}_{G_{\mathbb{Q}_p}\times D^\times}(\mathcal{O})$ be the category antiequivalent to $\mathrm{Mod}_{G_{\mathbb{Q}_p}\times D^\times}^{\mathrm{l.adm}}(\mathcal{O})$ via the Pontryagin duality. Define a covariant homological δ-functor $\{\check{\mathcal{S}}^i\}_{i\geq 0}$ by

$$\check{\mathcal{S}}^i : \mathfrak{C}_G(\mathcal{O}) \to \mathfrak{C}_{G_{\mathbb{Q}_p}\times D^\times}(\mathcal{O}), \quad M \mapsto H^i_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_{M^\vee})^\vee,$$

where $M^\vee = \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(M, L/\mathcal{O})$ denotes the Pontryagin dual of M. Note that $(M^\vee)^\vee \cong M$. We introduce these dual categories because it is much more convenient to work with compact torsion-free $\mathcal{O}$-modules than with discrete divisible $\mathcal{O}$-modules.

Denote the category of unitary admissible L-Banach space representations of G by $\mathrm{Ban}_G^{\mathrm{adm}}(L)$. If Π is in $\mathrm{Ban}_G^{\mathrm{adm}}(L)$ and Θ is an open bounded G-invariant lattice in Π, then it follows from [37, Lemma 4.4] that the Schikhof dual

$$\Theta^d := \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(\Theta, \mathcal{O})$$

equipped with the weak topology is an object of $\mathfrak{C}_G(\mathcal{O})$. We refer the reader to Section 2 for properties of the Schikhof dual. We thus can apply the functors $\check{\mathcal{S}}^i$ to it to obtain a compact $\mathcal{O}$-module $\check{\mathcal{S}}^i(\Theta^d)$ with a continuous $G_{\mathbb{Q}_p} \times D^\times$-action. Denote the Schikhof dual of this module by $\check{\mathcal{S}}^i(\Theta^d)^d$ and equip it with the p-adic topology.

Lemma 3.3. *There is an exact sequence of topological $\mathcal{O}$-modules*

$$0 \to \varprojlim_m (H^{i-1}_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_{\Theta\otimes_{\mathcal{O}} L/\mathcal{O}})/\varpi^m) \to \varprojlim_m H^i_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_{\Theta/\varpi^m}) \to \check{\mathcal{S}}^i(\Theta^d)^d \to 0$$

which identifies $\check{\mathcal{S}}^i(\Theta^d)^d$ with the maximal Hausdorff $\mathcal{O}$-torsion-free quotient of $\varprojlim_m H^i_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_{\Theta/\varpi^m})$.

Proof. Since

$$\Theta^d/\varpi^n\Theta^d \cong (\Theta/\varpi^n\Theta)^\vee, \quad \forall n \geq 1, \tag{12}$$

we have a natural isomorphism $(\Theta^d)^\vee \cong \Theta \otimes_{\mathcal{O}} L/\mathcal{O}$, and thus

$$\check{\mathcal{S}}^i(\Theta^d) \cong H^i_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_{\Theta\otimes_{\mathcal{O}} L/\mathcal{O}})^\vee \cong (\varinjlim_n H^i_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_{\Theta/\varpi^n}))^\vee. \tag{13}$$

Hence

$$\begin{aligned} \check{\mathcal{S}}^i(\Theta^d)^d &\cong \varprojlim_m \operatorname{Hom}^{\text{cont}}_{\mathcal{O}}(\check{\mathcal{S}}^i(\Theta^d), \mathcal{O}/\varpi^m) \cong \varprojlim_m (\check{\mathcal{S}}^i(\Theta^d)/\varpi^m)^\vee \\ &\cong \varprojlim_m ((H^i_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_{\Theta\otimes_{\mathcal{O}} L/\mathcal{O}})[\varpi^m])^\vee)^\vee) \cong \varprojlim_m (H^i_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_{\Theta\otimes_{\mathcal{O}} L/\mathcal{O}})[\varpi^m]), \end{aligned}$$

where the transition maps in the projective system are induced by multiplication by ϖ. The short exact sequence $0 \to \Theta/\varpi^m \to \Theta \otimes_{\mathcal{O}} L/\mathcal{O} \xrightarrow{\varpi^m} \Theta \otimes_{\mathcal{O}} L/\mathcal{O} \to 0$ yields an exact sequence:

$$0 \to H^{i-1}_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_{\Theta\otimes_{\mathcal{O}} L/\mathcal{O}})/\varpi^m \to H^i_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_{\Theta/\varpi^m}) \to H^i_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_{\Theta\otimes_{\mathcal{O}} L/\mathcal{O}})[\varpi^m] \to 0.$$

The exact sequence is obtained by passing to the limit by noting that the system satisfies the Mittag-Leffler condition.

If we let $M := H^{i-1}_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_{\Theta\otimes_{\mathcal{O}} L/\mathcal{O}})$, $\widehat{M}$ its p-adic completion and M' the image of M in $\widehat{M}$, then M' is a dense $\mathcal{O}$-torsion submodule of $\widehat{M}$. Equip $\widehat{M}/M'$ with the quotient topology. Then for all topological $\mathcal{O}$-modules N which are Hausdorff and $\mathcal{O}$-torsion free, we have

$$\operatorname{Hom}^{\text{cont}}_{\mathcal{O}}(\widehat{M}, N) \cong \operatorname{Hom}^{\text{cont}}_{\mathcal{O}}(\widehat{M}/M', N) = 0.$$

Since $\check{\mathcal{S}}^i(\Theta^d)^d$ is Hausdorff and $\mathcal{O}$-torsion free, this implies the last assertion. $\square$

Define

$$\check{\mathcal{S}}^i(\Pi) := \check{\mathcal{S}}^i(\Theta^d)^d \otimes_{\mathcal{O}} L.$$

The definition does not depend on the choice of Θ, since any two are commensurable. To motivate this definition, observe that (12) implies that we have natural isomorphisms

$$((\Theta\otimes_{\mathcal{O}} L/\mathcal{O})^\vee)^d \cong \Theta, \quad ((\Theta\otimes_{\mathcal{O}} L/\mathcal{O})^\vee)^d\otimes_{\mathcal{O}} L\cong \Pi.$$

Lemma 3.4. *If Π is an admissible unitary L-Banach space representation of G, then $\check{\mathcal{S}}^i(\Pi)$ is an admissible unitary L-Banach space representation of $D^\times$ for all $i\geq 0$.*

Proof. By applying $\{\check{\mathcal{S}}^i\}_{i\geq 0}$ to the exact sequence $0\to\Theta^d\overset{\varpi}{\to}\Theta^d\to\Theta^d/\varpi\Theta^d\to 0$, we obtain a long exact sequence:

$$0\to\check{\mathcal{S}}^2(\Theta^d)\overset{\varpi}{\to}\check{\mathcal{S}}^2(\Theta^d)\to\check{\mathcal{S}}^2(\Theta^d/\varpi)\to\check{\mathcal{S}}^1(\Theta^d)\overset{\varpi}{\to}\check{\mathcal{S}}^1(\Theta^d)\\ \to\check{\mathcal{S}}^1(\Theta^d/\varpi)\to\check{\mathcal{S}}^0(\Theta^d)\overset{\varpi}{\to}\check{\mathcal{S}}^0(\Theta^d)\to\check{\mathcal{S}}^0(\Theta^d/\varpi)\to 0. \quad (14)$$

The terms for $i\geq 3$ vanish, because of the results of Scholze explained in the previous section. Moreover, since Π is an admissible Banach space representation, we know that Θ/ϖ is an admissible smooth G-representation. Scholze's result implies that $H^i_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p},\mathcal{F}_{\Theta/\varpi})$ are admissible smooth $D^\times$-representations. Thus the Pontryagin dual is a finitely generated $\mathcal{O}[[K]]$-module, where K is any compact open subgroup of $D^\times$. Since $\mathcal{O}[[K]]$ is Noetherian, we deduce from (14) that $\check{\mathcal{S}}^i(\Theta^d)/\varpi$ is a finitely generated $\mathcal{O}[[K]]$-module, and the topological Nakayama's lemma for compact $\mathcal{O}[[K]]$-modules implies that $\check{\mathcal{S}}^i(\Theta^d)$ is a finitely generated $\mathcal{O}[[K]]$-module (see [10, Lemma 1.4, Corollary 1.5]). The assertion follows from the theory of Schneider and Teitelbaum [46, Theorem 3.5]. □

Now note that even if we could show that $\check{\mathcal{S}}^1(\Theta^d/\varpi)$ is nonzero, we cannot rule out using (14) alone that $\check{\mathcal{S}}^i(\Pi)=0$ for all $i\geq 0$. A priori it could happen that $\check{\mathcal{S}}^2(\Theta^d)$ and $\check{\mathcal{S}}^0(\Theta^d)$ are both zero and $\check{\mathcal{S}}^1(\Theta^d)$ is killed by some power of ϖ. This is where Ludwig's theorem enters.

Let $\bar{r}^{\text{ss}}$ be as in the previous section. Let $\text{Ban}^{\text{adm}}_G(L)_{\bar{r}^{\text{ss}}}$ be the full subcategory of the category of $\text{Ban}^{\text{adm}}_G(L)$ such that Π is in $\text{Ban}^{\text{adm}}_G(L)_{\bar{r}^{\text{ss}}}$ if and only if the isomorphism classes of all the irreducible subquotients of Θ/ϖ lie in $\mathfrak{B}_{\bar{r}^{\text{ss}}}$. Note that this last condition depends only on Π, and not on the choice of open bounded G-invariant lattice Θ (see [37, Lemma 4.3]). One can deduce from the result for $\text{Mod}^{\text{l.adm}}_G(\mathcal{O})$ that $\text{Ban}^{\text{adm}}_G(L)_{\bar{r}^{\text{ss}}}$ is a direct summand of $\text{Ban}^{\text{adm}}_G(L)$ (see [37, Proposition 5.36]).

Proposition 3.5. *Assume that $\bar{r}^{\text{ss}}=\chi_1\oplus\chi_2$ with $\chi_1\chi_2^{-1}\neq\omega^{\pm 1}$, and let π_1 and π_2 be the principal series representation defined in (10), so that $\mathfrak{B}_{\bar{r}^{\text{ss}}}$ consists of the isomorphism classes of π_1 and π_2. Then the following assertions are equivalent:*

(i) *Both $H^1_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p},\mathcal{F}_{\pi_1})$ and $H^1_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p},\mathcal{F}_{\pi_2})$ vanish.*

(ii) *$H^1_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p},\mathcal{F}_{\pi})=0$ for all π in $\text{Mod}^{\text{l.adm}}_G(\mathcal{O})_{\bar{r}^{\text{ss}}}$.*

(iii) *$\check{\mathcal{S}}^1(\Pi)=0$ for all $\Pi\in\text{Ban}^{\text{adm}}_G(L)_{\bar{r}^{\text{ss}}}$.*

(iv) *If $\Pi\in\text{Ban}^{\text{adm}}_G(L)_{\bar{r}^{\text{ss}}}$ is absolutely irreducible and nonordinary, then $\check{\mathcal{S}}^1(\Pi)=0$.*

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Proof. Any locally admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ is equal to the union of its subrepresentations of finite length (see [20, Theorem 2.3.8]). If (i) holds, then (ii) follows from (9) and Corollary 3.2. If (ii) holds, then (13) implies (iii), which trivially implies (iv). For the proof that (iv) implies (i), recall that Π is nonordinary if and only if it does not occur as a subquotient of a unitary parabolic induction of any unitary character of the maximal torus in G. If Θ is an open bounded G-invariant lattice in Π, then the semisimplification of Θ/ϖ is isomorphic to $\pi_1 \oplus \pi_2$ by [37, Theorem 11.1]. Ludwig's theorem implies that $H^2_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_{\Theta/\varpi}) = 0$, and the isomorphism (8) implies that the same holds for H^0. The topological Nakayama's lemma together with (14) implies that $\check{\mathcal{S}}^i(\Theta^d) = 0$ for $i = 0$ and $i = 2$. We deduce that we have a short exact sequence:

$$0 \to \check{\mathcal{S}}^1(\Theta^d) \overset{\varpi}{\to} \check{\mathcal{S}}^1(\Theta^d) \to H^1_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_{\Theta/\varpi})^\vee \to 0. \tag{15}$$

Thus $\check{\mathcal{S}}^1(\Theta^d)$ is $\mathcal{O}$-torsion free. Since $\check{\mathcal{S}}^1(\Theta^d)$ is a compact $\mathcal{O}$-module, this implies that $\check{\mathcal{S}}^1(\Theta^d)$ is isomorphic to a product of copies of $\mathcal{O}$. In particular, if $\check{\mathcal{S}}^1(\Theta^d) \neq 0$, then $\check{\mathcal{S}}^1(\Pi) \neq 0$. Thus (iv) implies that $\check{\mathcal{S}}^1(\Theta^d) = 0$ and (15) implies that $H^1_{\text{ét}}(\mathbb{P}^1_{\mathbb{C}_p}, \mathcal{F}_{\Theta/\varpi}) = 0$. Since the semisimplification of Θ/ϖ is isomorphic to $\pi_1 \oplus \pi_2$, Corollary 3.2 implies (i). $\square$

Remark 3.6. We will show later that part (i) of Proposition 3.5 does not hold by showing that completed cohomology gives a counterexample to (ii). However, we can not rule out that one of the groups can vanish in (i) (unless of course $\pi_1 \cong \pi_2$). Most likely both groups are nonzero, since there is no natural way to distinguish one of the principal series in the block.

Let us record a further consequence of Corollary 3.2. Let $\mathfrak{C}_G(\mathcal{O})_{\bar{r}^{\text{ss}}}$ be the full subcategory of $\mathfrak{C}_G(\mathcal{O})$, which is antiequivalent to $\mathrm{Mod}^{\text{l.adm}}_G(\mathcal{O})_{\bar{r}^{\text{ss}}}$ via Pontryagin duality. We refer the reader to [10] for the basics on pseudocompact rings and completed tensor products.

Proposition 3.7. *Assume that $\bar{r}^{\text{ss}} = \chi_1 \oplus \chi_2$ with $\chi_1\chi_2^{-1} \neq \omega^{\pm 1}$. Let $M \in \mathfrak{C}_G(\mathcal{O})_{\bar{r}^{\text{ss}}}$ and let A be a pseudocompact ring together with a continuous action on M via a ring homomorphism $A \to \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M)$. Then for all right pseudocompact A-modules m, we have a natural isomorphism:*

$$\check{\mathcal{S}}^1(\mathrm{m} \widehat{\otimes}_A M) \cong \mathrm{m} \widehat{\otimes}_A \check{\mathcal{S}}^1(M).$$

In particular, if M is a pro-flat A-module, in the sense that the functor $\mathrm{m} \mapsto \mathrm{m} \widehat{\otimes}_A M$ is exact, then so is $\check{\mathcal{S}}^1(M)$.

Proof. Since the functor $\mathcal{S}^1 : \mathrm{Mod}^{\text{l.adm}}_G(\mathcal{O})_{\bar{r}^{\text{ss}}} \longrightarrow \mathrm{Mod}^{\text{l.adm}}_{G_{\mathbb{Q}_p} \times D^\times}(\mathcal{O})$ commutes with direct sums and is exact by Corollary 3.2, the functor $\check{\mathcal{S}}^1 : \mathfrak{C}_G(\mathcal{O})_{\bar{r}^{\text{ss}}} \to \mathfrak{C}_{G_{\mathbb{Q}_p} \times D^\times}(\mathcal{O})$ commutes with products and is also exact. Since any pseudocompact A-module m can be presented as

$$\prod_{i \in I} A \to \prod_{j \in J} A \to \mathrm{m} \to 0$$

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for some sets I and J, the proposition is a formal consequence of these two properties (see the proof of [39, Proposition 2.4], which is based on ideas of Kisin in [27]). □

Lemma 3.8. *Let A be a complete local Noetherian $\mathcal{O}$-algebra with residue field k and let M be a pseudocompact A-module. Then M is pro-flat if and only if it is flat.*

Proof. Let $\widehat{\mathrm{Tor}}_i^A(*,M)$ be the ith left derived functor of $*\widehat{\otimes}_A M$. If m is a finitely generated A-module, then, since A is Noetherian, m has a resolution by free A-modules of finite rank. Since $A^n \otimes_A M \cong M^n \cong A^n \widehat{\otimes}_A M$, we conclude that $\widehat{\mathrm{Tor}}_i^A(\mathrm{m},M) \cong \mathrm{Tor}_i^A(\mathrm{m},M)$ for all finitely generated m. If M is pro-flat, then applying this observation to $\mathrm{m} = A/I$ for any ideal I of A, we deduce that the map $I \otimes_A M \to M$ is injective and hence M is flat. If M is flat, then taking $\mathrm{m} = k$ we obtain that $\widehat{\mathrm{Tor}}_1^A(k,M) = 0$, and a standard application of the topological Nakayama's lemma shows that M is topologically free, and hence pro-flat (see [17, Proposition 0.3.8 in Exposé VII_B]). □

4. Fibres and flatness

In this section we explain a variation on [23, Proposition A.30], where the authors prove a version of 'miracle flatness' in a noncommutative setting. Let Λ be an Auslander regular ring (we refer the reader to [23, §A.1] or [49] for the definition). If M is a finitely generated Λ-module, then the grade of M over Λ is defined as

$$j_\Lambda(M) := \inf\{i : \mathrm{Ext}^i_\Lambda(M,\Lambda) \neq 0\}$$

and the dimension of M over Λ is defined as

$$\delta_\Lambda(M) := \mathrm{gld}(\Lambda) - j_\Lambda(M),$$

where $\mathrm{gld}(\Lambda)$ is the global dimension of Λ. We say that M is *Cohen–Macaulay* if $\mathrm{Ext}^i_\Lambda(M,\Lambda)$ is nonzero for a single degree i. In particular, finite free modules are Cohen–Macaulay.

Let K be a compact torsion-free p-adic analytic pro-p group. Then the rings $k[[K]]$, $\mathcal{O}[[K]]$, $\mathcal{O}[[K]] \otimes_{\mathcal{O}} L$ are Auslander regular of global dimension $\dim K$, $1+\dim K$ and $\dim K$, respectively, where $\dim K$ is the dimension of K as a p-adic manifold. If K is any compact p-adic analytic group, then the ring $\mathcal{O}[[K]]$ might not be Auslander regular in general, but we can choose an open pro-p subgroup K_1 and, for a finitely generated $\mathcal{O}[[K]]$-module M, define $\delta_{\mathcal{O}[[K]]}(M) := \delta_{\mathcal{O}[[K_1]]}(M)$. Then $\delta_{\mathcal{O}[[K]]}(M)$ does not depend on a choice of K_1 and has the expected properties of a dimension function. We will sometimes omit $\mathcal{O}[[K]]$ from the notation and write just $\delta(M)$.

Proposition 4.1. *Let $A = \mathcal{O}[[x_1,\dots,x_r]]$ and let M be an $A[[K]]$-module which is finitely generated as an $\mathcal{O}[[K]]$-module. Then*

$$\delta_{\mathcal{O}[[K]]}(k \otimes_A M) + \dim A \geq \delta_{\mathcal{O}[[K]]}(M). \tag{16}$$

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If M is A-flat, then

$$\delta_{\mathcal{O}[[K]]}(k\otimes_A M)+\dim A=\delta_{\mathcal{O}[[K]]}(M). \tag{17}$$

If M is Cohen–Macaulay as an $\mathcal{O}[[K]]$-module, then (17) implies that M is A-flat.

Proof. The first assertion follows from [23, Lemma A.15], which says that if we mod out one relation then the dimension either goes down by one or stays the same. The other assertions follow from [23, Proposition A.30] by observing that $\operatorname{gld}(A[[K]])=\dim A+\operatorname{gld}(k[[K]])$ and $\delta_{A[[K]]}(M)=\delta_{\mathcal{O}[[K]]}(M)$, $\delta_{k[[K]]}(k\otimes_A M)=\delta_{\mathcal{O}[[K]]}(k\otimes_A M)$ by [23, Lemma A.19]. Note that the ring $A[[K]]$ is again Auslander regular. $\square$

Corollary 4.2. *Let R be a complete local Noetherian $\mathcal{O}$-algebra with residue field k. Let M be an $R[[K]]$-module which is finitely generated over $\mathcal{O}[[K]]$. Assume that M is $\mathcal{O}$-torsion free, M is Cohen–Macaulay over $\mathcal{O}[[K]]$ and the action of R on M is faithful. Then*

$$\delta_{\mathcal{O}[[K]]}(k\otimes_R M)+\dim R\geq\delta_{\mathcal{O}[[K]]}(M), \tag{18}$$

and equality holds if and only if there is a subring $A\subset R$ such that R is a finite A-module, A is formally smooth over $\mathcal{O}$ and M is A-flat.

Proof. Since R acts faithfully on M and M is $\mathcal{O}$-torsion free, R is $\mathcal{O}$-torsion free. By Cohen's structure theorem for complete local rings, there is a subring $A\subset R$ such that R is finite over A and $A\cong\mathcal{O}[[x_1,\dots,x_r]]$ (see [32, Theorem 29.4 (iii)] and the remark following it). Note that this implies that $\dim A=\dim R$.

Since $k\otimes_A M\cong(k\otimes_A R)\otimes_R M$, it admits $k\otimes_R M$ as a quotient, and since $k\otimes_A R$ is an R-module of finite length, it has a filtration of finite length with graded pieces isomorphic to subquotients of $k\otimes_R M$. [23, Lemma A.8] implies that $\delta_{\mathcal{O}[[K]]}(k\otimes_A M)=\delta_{\mathcal{O}[[K]]}(k\otimes_R M)$. The corollary now follows from the proposition. $\square$

Lemma 4.3. *Let M be a finitely generated $k[[K]]$-module. Then $\delta_{k[[K]]}(M)=0$ if and only if M is a finite-dimensional k-vector space.*

Proof. We can assume that K is a uniform pro-p group of dimension d. Let $\mathfrak{m}$ be the maximal ideal of $k[[K]]$; then the graded ring $\operatorname{gr}^\bullet_{\mathfrak{m}}(k[[K]])$ is a polynomial ring in d-variables [50, Theorem 8.7.10], and $\operatorname{gr}^\bullet_{\mathfrak{m}}(M)$ is a finitely generated $\operatorname{gr}^\bullet_{\mathfrak{m}}(k[[K]])$-module of dimension equal to $\delta_{k[[K]]}(M)$ [2, Proposition 5.4]. In particular, the degree of the Hilbert polynomial of $\operatorname{gr}^\bullet_{\mathfrak{m}}(M)$ is equal to $\delta_{k[[K]]}(M)$. If M is a finite-dimensional k-vector space, then $\delta_{k[[K]]}(M)=0$. Conversely, if $\delta_{k[[K]]}(M)=0$, then the Hilbert polynomial of $\operatorname{gr}^\bullet_{\mathfrak{m}}(M)$ is constant and thus $\mathfrak{m}^jM=\mathfrak{m}^{j+1}M$ for some j. Nakayama's lemma implies that $\mathfrak{m}^jM=0$. Since M is finitely generated over $k[[K]]$, we deduce that M is a finite-dimensional k-vector space. $\square$

5. Local–global compatibility

Let p be a prime and let F be a totally real number field with a fixed place $\mathfrak{p}$ above p, such that $F_\mathfrak{p} = \mathbb{Q}_p$, and a fixed infinite place ∞_F. Let D_0 be a quaternion algebra with centre F, ramified at all the infinite places of F and split at $\mathfrak{p}$. Let Σ be a set of finite ramification places of D_0. Fix a maximal order $\mathcal{O}_{D_0}$ of D_0, and for each finite place $v \notin \Sigma$ an isomorphism $(\mathcal{O}_{D_0})_v \cong M_2(\mathcal{O}_{F_v})$. For each finite place v of F, we will denote by $\mathbf{N}(v)$ the order of the residue field at v, and by $\varpi_v \in F_v$ a uniformiser.

Denote by $\mathbb{A}_F^f \subset \mathbb{A}_F$ the finite adeles and the adeles, respectively. Let $U = \prod_v U_v$ be a compact open subgroup contained in $\prod_v (\mathcal{O}_{D_0})_v^\times$. We assume that $U_\mathfrak{p} = \mathrm{GL}_2(\mathbb{Z}_p) = K$ and that U_v is a pro-p group at other places above p. We can write

$$(D_0 \otimes_F \mathbb{A}_F^f)^\times = \coprod_{i \in I} D_0^\times t_i U (\mathbb{A}_F^f)^\times \tag{19}$$

for some $t_i \in (D_0 \otimes_F \mathbb{A}_F^f)^\times$ and a finite index set I, where we have identified $(\mathbb{A}_F^f)^\times$ with the centre of $(D_0 \otimes_F \mathbb{A}_F^f)^\times$. In the arguments that follow, we are free to replace U by a smaller open subgroup by shrinking U_v at any place v different from $\mathfrak{p}$. In particular, we can assume that

$$(U(\mathbb{A}_F^f)^\times \cap t_i^{-1} D_0^\times t_i)/F^\times = 1 \tag{20}$$

for all $i \in I$ (see, e.g., [40, Lemma 3.2]). We will use standard notation for subgroups of U, so, for example, $U_p^\mathfrak{p} = \prod_{v|p, v \neq \mathfrak{p}} U_v$, $U^p = \prod_{v \nmid p} U_v$.

If A is a topological $\mathcal{O}$-algebra, let $S(U^p, A)$ be be the space of continuous functions

$$f : D_0^\times \backslash (D_0 \otimes_F \mathbb{A}_F^f)^\times / U^p \to A.$$

The group $(D_0 \otimes \mathbb{Q}_p)^\times$ acts continuously on $S(U^p, A)$ by right translations. It follows from (20) that the map $f \mapsto [u \mapsto \sum_{i \in I} f(t_i u)]$ induces an isomorphism of $U(\mathbb{A}_F^f)^\times$-representations

$$S(U^p, A) \xrightarrow{\cong} \bigoplus_{i \in I} C(U^p F^\times \backslash U(\mathbb{A}_F^f)^\times, A), \tag{21}$$

where C denotes the space of continuous functions. Let $\psi : (\mathbb{A}_F^f)^\times / F^\times \to \mathcal{O}^\times$ be a continuous character such that ψ is trivial of $(\mathbb{A}_F^f)^\times \cap U^p$. We can consider ψ as an A-valued character, via $\mathcal{O}^\times \to A^\times$. Let $S_\psi(U^p, A)$ be the A-submodule of $S(U^p, A)$ consisting of functions such that $f(gz) = \psi(z) f(g)$ for all $z \in (\mathbb{A}_F^f)^\times$. The isomorphism (21) induces an isomorphism of U_p-representations:

$$S_\psi(U^p, A) \xrightarrow{\cong} \bigoplus_{i \in I} C_\psi(U_p, A), \tag{22}$$

where C_ψ denotes the continuous functions on which the centre acts by the character ψ. One may think of $S_\psi(U^p, A)$ as the space of algebraic automorphic forms on $D_0^\times$ with tame level U^p and no restrictions on the level or weight at places dividing p. We want to introduce a variant by fixing the level and weight at places dividing p, different from $\mathfrak{p}$. Let λ be a continuous representation $U_p^\mathfrak{p}$ on a free $\mathcal{O}$-module of finite rank, such that

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$(\mathbb{A}_F^f)^\times \cap U_p^{\mathfrak{p}}$ acts on λ by the restriction of ψ to this group. Let

$$S_{\psi,\lambda}(U^{\mathfrak{p}},A) := \operatorname{Hom}_{U_p^{\mathfrak{p}}}(\lambda, S_\psi(U^p,A)).$$

We will omit λ as an index if it is the trivial representation. Note that a presentation $\mathcal{O}[[U_p^{\mathfrak{p}}]]^{\oplus n} \to \mathcal{O}[[U_p^{\mathfrak{p}}]]^{\oplus m} \twoheadrightarrow \lambda$ gives us an exact sequence:

$$0 \to S_{\psi,\lambda}(U^{\mathfrak{p}},A) \to S_\psi(U^p,A)^{\oplus m} \to S_\psi(U^p,A)^{\oplus n}. \tag{23}$$

If A is an $\mathcal{O}/\varpi^n$-module, then there is an open subgroup $V_p^{\mathfrak{p}}$ of $U_p^{\mathfrak{p}}$ which acts trivially on λ/ϖ^n. If we let $V^{\mathfrak{p}} := U_p V_p^{\mathfrak{p}}$, then by taking $V_p^{\mathfrak{p}}$-invariants of (23) we have an exact sequence

$$0 \to S_{\psi,\lambda}(U^{\mathfrak{p}},A) \to S_\psi(V^{\mathfrak{p}},A)^{\oplus m} \to S_\psi(V^{\mathfrak{p}},A)^{\oplus n}. \tag{24}$$

If the topology on A is discrete – for example, if $A = L/\mathcal{O}$ or $A = \mathcal{O}/\varpi^n$ – then we have

$$S_\psi(U^p,A) \cong \varinjlim_{U_p^{\mathfrak{p}}} S_\psi(U^{\mathfrak{p}},A). \tag{25}$$

The action of $(D_0 \otimes \mathbb{Q}_p)^\times$ on $S_\psi(U^p,A)$ by right translations induces a continuous action of $(D_0 \otimes_F F_{\mathfrak{p}})^\times \cong \mathrm{GL}_2(\mathbb{Q}_p) = G$ on $S_\psi(U^p,A)$ and $S_{\psi,\lambda}(U^{\mathfrak{p}},A)$. Let $\zeta : \mathbb{Q}_p^\times \to \mathcal{O}^\times$ be the character obtained by restricting ψ to $F_{\mathfrak{p}}^\times$.

Lemma 5.1. *The representations $S_\psi(U^p,L/\mathcal{O})$, $S_{\psi,\lambda}(U^{\mathfrak{p}},L/\mathcal{O})$ lie in $\operatorname{Mod}^{\mathrm{l.adm}}_{G,\zeta}(\mathcal{O})$. Moreover, $S_{\psi,\lambda}(U^{\mathfrak{p}},L/\mathcal{O})$ is admissible.*

Proof. The first assertion follows from (22), which also implies that we have an isomorphism of K-representations

$$S_{\psi,\lambda}(U^{\mathfrak{p}},L/\mathcal{O}) \cong \bigoplus_{i\in I} \lambda^\vee \otimes_{\mathcal{O}} C_\zeta(K,L/\mathcal{O}), \tag{26}$$

where $\lambda^\vee := \operatorname{Hom}_{\mathcal{O}}^{\mathrm{cont}}(\lambda, L/\mathcal{O})$ denotes the Pontryagin dual of λ. Since the set I is finite and λ is a finite $\mathcal{O}$-module, we deduce the second assertion. $\square$

Lemma 5.2. *The restrictions of $S_\psi(U^p,L/\mathcal{O})$ and $S_{\psi,\lambda}(U^{\mathfrak{p}},L/\mathcal{O})$ to K are injective in $\operatorname{Mod}^{\mathrm{sm}}_{K,\zeta}(\mathcal{O})$.*

Proof. This follows from (22) and (26). $\square$

Let S be a finite set of places of F containing Σ, all the places above p, all the infinite places and all the places v, where U_v is not maximal and all the ramification places of ψ. Let $\mathbb{T}_S^{\mathrm{univ}} = \mathcal{O}[T_v, S_v]_{v\notin S}$ be a commutative polynomial ring in the indicated formal variables. If A is a topological $\mathcal{O}$-algebra, then $S_\psi(U^p,A)$ and $S_{\psi,\lambda}(U^{\mathfrak{p}},A)$ become $\mathbb{T}_S^{\mathrm{univ}}$-modules with S_v acting via the double coset $U_v\left(\begin{smallmatrix}\varpi_v & 0\\ 0 & \varpi_v\end{smallmatrix}\right)U_v$ and T_v acting via the double coset $U_v\left(\begin{smallmatrix}\varpi_v & 0\\ 0 & 1\end{smallmatrix}\right)U_v$.

Let $G_{F,S}$ be the absolute Galois group of the maximal extension of F in $\overline{F}$ which is unramified outside S. Let $\bar{\rho} : G_{F,S} \to \mathrm{GL}_2(k)$ be a continuous absolutely irreducible representation. Let $\mathfrak{m}$ be the maximal ideal of $\mathbb{T}_S^{\mathrm{univ}}$ generated by ϖ and all elements which reduce modulo ϖ to $T_v - \operatorname{tr}\bar{\rho}(\mathrm{Frob}_v), S_v\mathbf{N}(v) - \det\bar{\rho}(\mathrm{Frob}_v)$ for all $v \notin S$. Denote by $S_\psi(U^p,L/\mathcal{O})_\mathfrak{m}$ and $S_{\psi,\lambda}(U^\mathfrak{p},L/\mathcal{O})_\mathfrak{m}$ the localisations at $\mathfrak{m}$, and assume that they are nonzero for fixed ψ and λ.

Lemma 5.3. *The representations $S_\psi(U^p,L/\mathcal{O})_\mathfrak{m}$ and $S_{\psi,\lambda}(U^\mathfrak{p},L/\mathcal{O})_\mathfrak{m}$ are direct summands of $S_\psi(U^p,L/\mathcal{O})$ and $S_{\psi,\lambda}(U^\mathfrak{p},L/\mathcal{O})$, respectively. In particular, their restrictions to K are injective in $\mathrm{Mod}^{\mathrm{sm}}_{K,\zeta}(\mathcal{O})$. Moreover, if $S(U^p,L/\mathcal{O})_\mathfrak{m}$ (resp. $S_\lambda(U^p,L/\mathcal{O})_\mathfrak{m}$) is nonzero, then $S_\psi(U^p,L/\mathcal{O})_\mathfrak{m}$ (resp. $S_{\psi,\lambda}(U^\mathfrak{p},L/\mathcal{O})_\mathfrak{m}$) is nonzero if and only if $\det\bar{\rho} \equiv \psi\chi_{\mathrm{cyc}} (\mathrm{mod}\ \varpi)$.*

Proof. If U_p' is an open subgroup of U_p, then $S_\psi(U^p,\mathcal{O}/\varpi^n)^{U_p'}$ is a finitely generated $\mathcal{O}/\varpi^n$-module and the Chinese remainder theorem implies that $(S_\psi(U^p,\mathcal{O}/\varpi^n)^{U_p'})_\mathfrak{m}$ is a direct summand of $S_\psi(U^p,\mathcal{O}/\varpi^n)^{U_p'}$. By passing to a direct limit over all such U_p' and $n \geq 1$, we obtain the assertion for $S_\psi(U^p,L/\mathcal{O})$. The argument for $S_{\psi,\lambda}(U^\mathfrak{p},L/\mathcal{O})$ is the same. The second assertion follows from Lemma 5.2. Since $S_\psi(U^p,L/\mathcal{O})$ is a a union of $\mathbb{T}_S^{\mathrm{univ}}$-submodules, which are $\mathcal{O}$-modules of finite length, its localisation at $\mathfrak{m}$ is nonzero if and only if the $\mathfrak{m}$-torsion subspace $S_\psi(U^p,L/\mathcal{O})[\mathfrak{m}]$ is nonzero. The same argument applied to $S(U^p,L/\mathcal{O})$ shows that $S(U^p,L/\mathcal{O})[\mathfrak{m}]$ is nonzero. Since the Hecke operators S_v act on $S(U^p,L/\mathcal{O})[\mathfrak{m}]$ with eigenvalue $\det\bar{\rho}(\mathrm{Frob}_v)\mathbf{N}(v)^{-1}$ for all $v \notin S$, we deduce from Chebotarev's density theorem that $(\mathbb{A}_F^f)^\times$ acts on $S(U^p,L/\mathcal{O})[\mathfrak{m}]$ by the character $\chi_{\mathrm{cyc}}\det\bar{\rho}$. Hence, $S_\psi(U^p,L/\mathcal{O})[\mathfrak{m}]$ is nonzero if and only if $\det\bar{\rho} \equiv \psi\chi_{\mathrm{cyc}} (\mathrm{mod}\ \varpi)$. The same argument applies to $S_{\psi,\lambda}(U^\mathfrak{p},L/\mathcal{O})_\mathfrak{m}$. □

Let

$$S_\psi(U^p,\mathcal{O})_\mathfrak{m} := \varprojlim_n S_\psi(U^p,\mathcal{O}/\varpi^n)_\mathfrak{m}, \quad S_{\psi,\lambda}(U^\mathfrak{p},\mathcal{O})_\mathfrak{m} := \varprojlim_n S_{\psi,\lambda}(U^\mathfrak{p},\mathcal{O}/\varpi^n)_\mathfrak{m},$$

equipped with the p-adic topology. It follows from (22) that for all $n \geq 1$, the map $S_\psi(U^p,\mathcal{O})_\mathfrak{m}/\varpi^n \to S_\psi(U^p,\mathcal{O}/\varpi^n)_\mathfrak{m}$ is an isomorphism, and (26) implies that the same holds for $S_{\psi,\lambda}(U^\mathfrak{p},\mathcal{O})_\mathfrak{m}$. It follow from the discussion in Section 3.3 that we have natural homeomorphisms

$$S_\psi(U^p,\mathcal{O})_\mathfrak{m} \cong ((S_\psi(U^p,L/\mathcal{O})_\mathfrak{m})^\vee)^d, \quad S_{\psi,\lambda}(U^\mathfrak{p},\mathcal{O})_\mathfrak{m} \cong ((S_{\psi,\lambda}(U^\mathfrak{p},L/\mathcal{O})_\mathfrak{m})^\vee)^d. \tag{27}$$

Let $\bar{r}$ denote the restriction of $\bar{\rho}$ to the decomposition group at $\mathfrak{p}$, which we identify with the absolute Galois group of $\mathbb{Q}_p$.

Proposition 5.4. *$S_\psi(U^p,L/\mathcal{O})_\mathfrak{m}$ and $S_{\psi,\lambda}(U^\mathfrak{p},L/\mathcal{O})_\mathfrak{m}$ lie in $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\zeta}(\mathcal{O})_{\bar{r}^{\mathrm{ss}}}$.*

Proof. We first observe that the centre of G acts on both $S_\psi(U^p,L/\mathcal{O})_\mathfrak{m}$ and $S_{\psi,\lambda}(U^\mathfrak{p},L/\mathcal{O})_\mathfrak{m}$ by the restriction of ψ to $F_\mathfrak{p}^\times$, which is equal to ζ by definition.

Since for any $\pi \in \mathrm{Mod}^{\mathrm{l.adm}}_G(\mathcal{O})$, $\mathrm{soc}_G\,\pi \hookrightarrow \pi$ is essential, where soc_G denotes the maximal semisimple subrepresentation, it follows from (11) that it is enough to show that all irreducible subrepresentations of $S_{\psi,\lambda}(U^\mathfrak{p},L/\mathcal{O})_\mathfrak{m}$ and $S_\psi(U^p,L/\mathcal{O})_\mathfrak{m}$ lie in $\mathfrak{B}_{\bar{r}^{\mathrm{ss}}}$. Since any such representation is killed by ϖ, we can replace $L/\mathcal{O}$ with k.

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It is enough to prove the statement for $S_\psi(U^\mathfrak{p},k)_\mathfrak{m}$ with $U_p^\mathfrak{p}$ arbitrary small, since then (24) and (25) imply the assertion in general.

Let π be an irreducible subrepresentation of $S_\psi(U^\mathfrak{p},k)_\mathfrak{m}$. After enlarging L, we can assume that π is absolutely irreducible. Let σ be an irreducible K-subrepresentation of π. Then $\bar\sigma$ is isomorphic to $\mathrm{Sym}^b k^2 \otimes \det^a$ for uniquely determined integers $0 \le b \le p-1$, $0 \le a \le p-2$. It follows from [4] that $\mathrm{End}_G(\text{c-Ind}_K^G \sigma) \cong k[T,S^{\pm 1}]$, where the Hecke operators T and S correspond to the double cosets $K\left(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right)K$ and $K\left(\begin{smallmatrix} p & 0 \\ 0 & p\end{smallmatrix}\right)K$, respectively. Moreover, there are $\lambda,\mu \in k$ such that π is a quotient of

$$\text{c-Ind}_K^G \sigma/(T-\lambda, S-\mu).$$

We claim that λ, μ and the possible values of (a,b) can be read off from the restriction of $\bar\rho$ to the decomposition group at $\mathfrak{p}$. It then follows from [4, 6], which describe the irreducible quotients of $\text{c-Ind}_K^G \sigma/(T-\lambda, S-\mu)$, that π lies in $\mathfrak{B}_{\bar r^{\mathrm{ss}}}$. These arguments are by now fairly standard and appear in the weight part of Serre's conjecture, so we give only a sketch.

We first modify the setting slightly: if $\psi' : (\mathbb{A}_F^f)^\times/F^\times \to \mathcal{O}^\times$ is a character congruent to ψ modulo ϖ, λ' is a representation of $U_p^\mathfrak{p}$ on a finite $\mathcal{O}$-module with central character ψ' and $U_p^\mathfrak{p}$ is a pro-p group, then the $U_p^\mathfrak{p}$-invariants of λ'/ϖ are nonzero, and thus $S_\psi(U^\mathfrak{p},k)_\mathfrak{m}$ is a G-invariant subspace of $S_{\psi',\lambda'}(U^\mathfrak{p},k)_\mathfrak{m}$. We can choose $\psi' = \chi_{\mathrm{cyc}}^{b+2a}\alpha$, where a and b are as before, α is the Teichmüller lift of the character $\psi\chi_{\mathrm{cyc}}^{-b-2a} (\mathrm{mod}\ \varpi)$, χ_{cyc} is the p-adic cyclotomic character and $\lambda' = \otimes_\iota(\mathrm{Sym}^b \mathcal{O}^2 \otimes \det^a)$, where the tensor product is taken over all embeddings $\iota : F \hookrightarrow L$ which do not factor through $F_\mathfrak{p}$. Note that since $U_p^\mathfrak{p}$ is assumed to be pro-p, the character $\psi\chi_{\mathrm{cyc}}^{-b-2a} (\mathrm{mod}\ \varpi)$ is trivial on $U_p^\mathfrak{p} \cap (\mathbb{A}_F^f)^\times$, which implies that α is trivial on $U_p^\mathfrak{p} \cap (\mathbb{A}_F^f)^\times$ and thus λ' has central character ψ'. Note that since ζ is also the central character of $\bar\sigma$, we have $\zeta(x) \equiv x^{b+2a} (\mathrm{mod}\ \varpi)$ for all $x \in \mathbb{Z}_p^\times$. To ease the notation, we drop the superscript $'$ from ψ' and λ'.

Since $S_{\psi,\lambda}(U^\mathfrak{p},k)_\mathfrak{m}$ is an admissible representation by Lemma 5.1, the k-vector space

$$\mathrm{Hom}_G(\pi, S_{\psi,\lambda}(U^\mathfrak{p},k)_\mathfrak{m})$$

is finite-dimensional. We can thus assume that π is contained in $S_{\psi,\lambda}(U^\mathfrak{p},k)[\mathfrak{m}]$. Let

$$\sigma^\circ := \mathrm{Sym}^b \mathcal{O}^2 \otimes \det^a, \quad \sigma := \mathrm{Sym}^b L^2 \otimes \det^a.$$

Since $S_{\psi,\lambda}(U^\mathfrak{p},L/\mathcal{O})_\mathfrak{m}$ is admissible and injective in $\mathrm{Mod}_{K,\zeta}^{\mathrm{sm}}(\mathcal{O})$ by Lemmas 5.1 and 5.2, using (27) we see that $\mathrm{Hom}_K(\sigma^\circ, S_{\psi,\lambda}(U^\mathfrak{p},\mathcal{O})_\mathfrak{m})$ is a free $\mathcal{O}$-module of finite rank, which is congruent to $\mathrm{Hom}_K(\bar\sigma, S_{\psi,\lambda}(U^\mathfrak{p},k)_\mathfrak{m})$ modulo ϖ. It follows from [16, Lemme 6.11] that after L is replaced with a finite extension,

$$\mathrm{Hom}_K(\sigma^\circ, S_{\psi,\lambda}(U^\mathfrak{p},\mathcal{O})_\mathfrak{m}) \otimes_\mathcal{O} L$$

contains an eigenvector ϕ for all the Hecke operators in $\mathbb{T}_S^{\mathrm{univ}}$ with eigenvalues lifting those given by $\mathfrak{m}$. By evaluating ϕ, we obtain an automorphic form f on $D_0^\times$ such that the associated Galois representation ρ_f lifts $\bar\rho$ and its restriction to the decomposition group at $\mathfrak{p}$ is crystalline with Hodge–Tate weights $(1-a, -a-b)$, where we adopt the conventions of [11], so that the cyclotomic character has Hodge–Tate weight equal to -1.

Note that the difference of the two Hodge–Tate weights is equal to $b+1$, which is between 1 and p. In particular, $\bar{\sigma}$ is a Serre weight for $\bar{r}$. The possibilities for the pair (a,b) are listed in the proof of [11, Lemma 2.15]. The compatibility of local and global Langlands correspondences implies that the G-subrepresentation of $S_{\psi,\lambda}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} L$ generated by the image of ϕ is of the form $\Psi \otimes \sigma$, where Ψ is a smooth unramified principal series representation. Moreover, the Satake parameters of Ψ can be read off from the Weil–Deligne representation associated to $\rho_f|_{G_{F_{\mathfrak{p}}}}$ (see [11, Proposition 2.9]). It follows from [7] that λ and μ are reductions modulo ϖ of the Satake parameters of Ψ, rescaled by a suitable power of p (see the proof of [11, Lemma 2.15]). It then follows from [4, 6], which describe the irreducible quotients of $\text{c-Ind}_K^G \sigma/(T-\lambda, S-\mu)$, that π lies in $\mathfrak{B}_{\bar{r}^{\text{ss}}}$. The last part of the argument can be summed up as the compatibility of p-adic and mod-p Langlands correspondences. □

Let $R^{\text{ps}}_{\text{tr}\,\bar{r}}$ be the universal deformation ring parameterising 2-dimensional pseudocharacters of $G_{\mathbb{Q}_p}$ lifting $\text{tr}\,\bar{r}$. We have two actions of $R^{\text{ps}}_{\text{tr}\,\bar{r}}$ on $S_\psi(U^{\mathfrak{p}},L/\mathcal{O})_{\mathfrak{m}}$. The first action is given by the composition

$$\theta_1 : R^{\text{ps}}_{\text{tr}\,\bar{r}} \to R^{\text{univ}}_{\bar{\rho}} \to \mathbb{T}(U^{\mathfrak{p}})_{\mathfrak{m}} \to \text{End}^{\text{cont}}_G(S_\psi(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}),$$

where $R^{\text{univ}}_{\bar{\rho}}$ is the universal global deformation ring of $\bar{\rho}$, and the first arrow is obtained by considering the restriction of the trace of the universal deformation of $\bar{\rho}$ to $G_{F_{\mathfrak{p}}}$. The second arrow is obtained by associating Galois representations to automorphic forms (see [47, Proposition 5.7]). The third arrow is given by [47, Corollary 7.3]. The second action,

$$\theta_2 : R^{\text{ps}}_{\text{tr}\,\bar{r}} \to \text{End}^{\text{cont}}_G(S_\psi(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}),$$

is given by interpreting $R^{\text{ps}}_{\text{tr}\,\bar{r}}$ as the centre of the category $\text{Mod}^{\text{l.adm}}_G(\mathcal{O})_{\bar{r}^{\text{ss}}}$ using [37]. In order to apply the results of [37], if $\bar{r}^{\text{ss}} = \chi \oplus \chi\omega$ for some character $\chi : G_{\mathbb{Q}_p} \to k^\times$ then we assume that $p \geq 5$. In [37] we worked with a fixed central character, but this can be unfixed by using the ideas of [11, §6.5, Corollary 6.23]. Since the centre of the category acts naturally on every object in the category, Proposition 5.4 implies that $R^{\text{ps}}_{\text{tr}\,\bar{r}}$ acts naturally on $S_\psi(U^{\mathfrak{p}},L/\mathcal{O})_{\mathfrak{m}}$, and using (27) we obtain a natural action of $R^{\text{ps}}_{\text{tr}\,\bar{r}}$ on $S_\psi(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}$, which we denote by θ_2.

Proposition 5.5. *Assume that ψ is of finite prime-to-p order and ψ is trivial on the subgroup $U^{\mathfrak{p}} \cap (\mathbb{A}_F^f)^\times$. Then the two actions θ_1 and θ_2 of $R^{\text{ps}}_{\text{tr}\,\bar{r}}$ on $S_\psi(U^{\mathfrak{p}},L/\mathcal{O})_{\mathfrak{m}}$ coincide.*

Proof. The result follows from three ingredients: the theory of capture [15, §2.4], [40, §2.1], which is based on the ideas of Emerton in [18]; the results of Berger and Breuil [5] on the universal unitary completions of locally algebraic principal series representations; and the results of Colmez in [13] on the compatibility of the p-adic and classical local Langlands correspondence.

It follows from (27) that it is enough to show that the two actions coincide on $S_\psi(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}$. Let $\Pi(U^{\mathfrak{p}})$ be the L-Banach space representation of G defined by

$$\Pi(U^{\mathfrak{p}}) := S_\psi(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} L.$$

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Since $S_\psi(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}$ is $\mathcal{O}$-torsion free, it is enough to show that θ_1 and θ_2 agree on $\Pi(U^{\mathfrak{p}})$. Since the Schikhof dual of $S_\psi(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}$ is isomorphic to $S_\psi(U^{\mathfrak{p}},L/\mathcal{O})^\vee_{\mathfrak{m}}$, Lemmas 5.2 and 5.1 imply that $S_\psi(U^{\mathfrak{p}},\mathcal{O})^d_{\mathfrak{m}}$ is projective in the category $\mathrm{Mod}^{\mathrm{pro}}_{K,\zeta}(\mathcal{O})$ and is finitely generated over $\mathcal{O}[[K]]$.

Let $\eta:\mathbb{Z}_p^\times \to \mathcal{O}^\times$ be a nontrivial character of order 2. Let $\alpha:\mathbb{Z}_p^\times \to \mathcal{O}^\times$ be a character of order p. Let $J := \left(\begin{smallmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^2\mathbb{Z}_p & \mathbb{Z}_p^\times \end{smallmatrix}\right)$ and let χ be a character of J defined by

$$\chi(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) := \zeta(a)\alpha(a)\alpha(d)^{-1}.$$

Let $\tau := \mathrm{Ind}_J^K \chi$ and let $V_a := \tau \otimes_L \mathrm{Sym}^{2a} L^2 \otimes \det^{-a}$, $a \ge 0$. Note that the central character of V_a is equal to ζ. It follows from the proof of [40, Proposition 2.7] that if φ is a continuous K-equivariant endomorphism of $\Pi(U^{\mathfrak{p}})$ which kills

$$\mathrm{Hom}_K(V_a,\Pi(U^{\mathfrak{p}})) \quad \text{and} \quad \mathrm{Hom}_K(V_a\otimes \eta\circ\det,\Pi(U^{\mathfrak{p}}))$$

for all $a \ge 0$, then φ is zero. Thus it is enough to show that the two actions of $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$ on the previously noted modules coincide, so that $\theta_1(r)-\theta_2(r)$ annihilates them for all $r \in R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$ and all $a \ge 0$.

In the following, let $V = V_{\mathrm{sm}} \otimes V_{\mathrm{alg}}$, where $V_{\mathrm{alg}} = \mathrm{Sym}^{2a} L^2 \otimes \det^{-a}$ and $V_{\mathrm{sm}} = \tau$ or $\tau\otimes\eta\circ\det$. Since V is a locally algebraic representation of K, we have $\mathrm{Hom}_K(V,\Pi(U^{\mathfrak{p}})) = \mathrm{Hom}_K(V,\Pi(U^{\mathfrak{p}})^{\mathrm{l.alg}})$, where $\Pi(U^{\mathfrak{p}})^{\mathrm{l.alg}}$ is the subspace of locally algebraic vectors in $\Pi(U^{\mathfrak{p}})$. This subspace can be identified with a subspace of classical automorphic forms on $D_0^\times$ (see [19, §3], [28, §3.1.14] and [48, Lemma 1.3]). In particular, the action of $\mathbb{T}(U^p)_{\mathfrak{m}}[1/p]$ on $\Pi(U^{\mathfrak{p}})^{\mathrm{l.alg}}$, and hence on $\mathrm{Hom}_K(V,\Pi(U^{\mathfrak{p}}))$, is semisimple. Since $S_\psi(U^{\mathfrak{p}},\mathcal{O})^d_{\mathfrak{m}}$ is finitely generated as an $\mathcal{O}[[K]]$-module, the vector space $\mathrm{Hom}_K(V,\Pi(U^{\mathfrak{p}}))$ is finite-dimensional. Thus for a fixed V, after replacing L by a finite extension we can assume that $\mathrm{Hom}_K(V,\Pi(U^{\mathfrak{p}}))$ has a basis of eigenvectors for the action of $\mathbb{T}(U^{\mathfrak{p}})_{\mathfrak{m}}[1/p]$.

Let $\phi \in \mathrm{Hom}_K(V,\Pi(U^{\mathfrak{p}}))$ be such an eigenvector. Then it is enough to show that ϕ is an eigenvector for the action $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$ via θ_2 and that the annihilators of ϕ for the two actions coincide, since then $\theta_1(r)-\theta_2(r)$ will kill ϕ for all $r \in R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$.

Let $x \in \text{m-Spec}\, R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}[1/p]$ be the kernel for the action $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}[1/p]$ on ϕ via θ_1. Then by unravelling the definition of θ_1, we get that x corresponds to the pseudocharacter $\mathrm{tr}(\rho_f|_{G_{F_{\mathfrak{p}}}})$, where ρ_f is the Galois representation attached to the automorphic form f, corresponding to the Hecke eigenvalues given by the action of $\mathbb{T}(U^p)_{\mathfrak{m}}[1/p]$ on ϕ. Moreover, by the same argument as in the proof of Proposition 5.4, $\rho_f|_{G_{F_{\mathfrak{p}}}}$ is potentially semistable with Hodge–Tate weights $(1+a,-a)$. As in the proof of Proposition 5.4, the G-subrepresentation of $\Pi(U^{\mathfrak{p}})$ generated by the image of ϕ is of the form $\Psi\otimes V_{\mathrm{alg}}$, where Ψ is an irreducible smooth representation with $\mathrm{Hom}_K(V_{\mathrm{sm}},\Psi)\neq 0$. The theory of types (see [24]) implies that $\Psi \cong (\mathrm{Ind}_B^G \psi_1\otimes\psi_2|\cdot|^{-1})_{\mathrm{sm}}$, where $\psi_1,\psi_2:\mathbb{Q}_p^\times \to L^\times$ are smooth characters, such that if $V_{\mathrm{sm}}=\tau$ then $\psi_1|_{\mathbb{Z}_p^\times}=\zeta\alpha$, $\psi_2|_{\mathbb{Z}_p^\times}=\alpha^{-1}$, and if $V_{\mathrm{sm}}=\tau\otimes\eta\circ\det$ then $\psi_1|_{\mathbb{Z}_p^\times}=\zeta\alpha\eta$, $\psi_2|_{\mathbb{Z}_p^\times}=\alpha^{-1}\eta$. Hence $\rho_f|_{G_{F_{\mathfrak{p}}}}$ is potentially crystalline. Moreover, Ψ determines the Weil–Deligne representation associated to $\rho_f|_{G_{F_{\mathfrak{p}}}}$ via the classical local Langlands correspondence.

The universal unitary completion of $\Psi \otimes V_{\mathrm{alg}}$ is absolutely irreducible, by [5, 5.3.4] and [8, 2.2.1], and will coincide with the closure of $\Psi \otimes V_{\mathrm{alg}}$ in $\Pi(U^{\mathfrak{p}})$, which we denote by Π. The action of $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}[1/p]$ on $\Pi(U^{\mathfrak{p}})$ via θ_2 preserves Π, since it acts as the centre of the category. Schur's lemma implies that the annihilator of Π is a maximal ideal of $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}[1/p]$, which we denote by y. Note that we have shown that the action of $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}[1/p]$ via θ_2 preserves ϕ and the annihilator is equal to y. So we have to show that $x = y$.

If Π is nonordinary, then $r := \check{\mathbf{V}}(\Pi)$ is an absolutely irreducible two-dimensional potentially crystalline representation of $G_{\mathbb{Q}_p}$ lifting $\bar{r}$, and it follows from [37, Proposition 11.3] that y corresponds to $\mathrm{tr}\,r$. The compatibility of p-adic and classical Langlands correspondences, proved by Colmez in [13], implies that r and $\rho_f|_{G_{F_{\mathfrak{p}}}}$ have the same Weil–Deligne representations. Moreover, the Hodge–Tate weights of r are determined by V_{alg} and are equal to those of ρ_f. Since the Hodge–Tate weights of r and $\rho_f|_{G_{F_{\mathfrak{p}}}}$ are equal, it follows from [14, §4.5] that $r \cong \rho_f|_{G_{F_{\mathfrak{p}}}}$ and so $x = y$.

If Π is ordinary, then $\check{\mathbf{V}}(\Pi)$ is one-dimensional, and we denote the character by δ. It follows from [37, Corollaries 8.15 and 9.37, Proposition 10.107] that y corresponds to the pseudocharacter $\delta \oplus \delta^{-1}\zeta\varepsilon^{-1}$. Since Ψ is irreducible, we can assume that $\mathrm{val}(\psi_1(p)) \geq \mathrm{val}(\psi_2(p))$, since interchanging the characters will give an isomorphic representation. Then it follows from [37, Lemma 12.5] that Π is isomorphic to a parabolic induction of a unitary character $\gamma_1 \otimes \gamma_2$, such that $\gamma_1(x) = \psi_1(x)x^{-a}$, $\gamma_2(x) = \psi_2(x)|x|^{-1}x^a$ for all $x \in \mathbb{Q}_p^\times$. It follows from the definition of $\check{\mathbf{V}}$ that $\check{\mathbf{V}}(\Pi) \cong \gamma_1\varepsilon^{-1}$, so that $\delta \oplus \delta^{-1}\zeta\varepsilon^{-1} = \gamma_1\varepsilon^{-1} \oplus \gamma_2$, where for ease of notation we use local class field theory to identify characters of $G_{\mathbb{Q}_p}$ with unitary characters of $\mathbb{Q}_p^\times$. Note that to match the conventions of [11] we twist the definition of $\check{\mathbf{V}}$ in [37, §5.7] by the inverse of the cyclotomic character. The cyclotomic character ε corresponds to the character $x \mapsto x|x|$ via local class field theory. We have that $\gamma_1\varepsilon^{-1} = \psi_1|\cdot|^a\varepsilon^{-a-1}$ is crystalline with Hodge–Tate weight $a+1$ and Weil–Deligne representation $\psi_1|\cdot|^{-1}$; similarly, γ_2 is crystalline with Hodge–Tate weight $-a$ and Weil–Deligne representation equal to $\psi_2|\cdot|^{-1}$. The Weil–Deligne representation attached to ρ_f is the Weil–Deligne representation attached to Ψ by the classical Langlands correspondence, which is equal to $\psi_1|\cdot|^{-1} \oplus \psi_2|\cdot|^{-1}$, see the calculation in the proof of Proposition 2.9 in [11]. It follows from [14, §4.5] that r and $\rho_f|_{G_{F_{\mathfrak{p}}}}$ have the same semisimplification. Hence $x = y$. $\square$

Corollary 5.6. *The two actions of $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$ on $S_\psi(U^p, L/\mathcal{O})_{\mathfrak{m}}$ and $S_{\psi,\lambda}(U^{\mathfrak{p}}, L/\mathcal{O})_{\mathfrak{m}}$ via θ_1 and θ_2 coincide.*

Proof. If $\chi : (\mathbb{A}_F^f)^\times / F^\times \to \mathcal{O}^\times$ is a continuous character trivial on $U^p \cap (\mathbb{A}_F^f)^\times$ and trivial modulo ϖ, then the map $f \mapsto [g \mapsto f(g)\chi(\det(g))]$ induces an isomorphism of $\mathbb{T}(U^p)_{\mathfrak{m}}[G]$-modules

$$S_\psi(U^p, L/\mathcal{O}) \otimes \chi \circ \det \xrightarrow{\cong} S_{\psi\chi^2}(U^p, L/\mathcal{O}).$$

This induces an isomorphism between the G-endomorphism rings

$$\varphi : \mathrm{End}_G(S_\psi(U^p, L/\mathcal{O})) \xrightarrow{\cong} \mathrm{End}_G(S_{\psi\chi^2}(U^p, L/\mathcal{O})).$$

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Moreover, for $i = 1,2$ we have $\phi \circ \theta_i = \theta_i \circ \mathrm{tw}_\chi$, where tw_χ is the automorphism of $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$ obtained by sending a deformation to its twist by χ. Thus if the assertion for ψ is proved, we can deduce the assertion for $\psi\chi^2$ for all χ as before.

Let $\bar{\psi}$ be the reduction of ψ modulo ϖ and let $[\bar{\psi}]$ be the Teichmüller lift of $\bar{\psi}$; then the character $\psi^{-1}[\bar{\psi}]$ takes values in $1+\mathfrak{p}$ and thus we can take its square root by the usual binomial formula. If $\chi := \sqrt{\psi^{-1}[\bar{\psi}]}$, then $\psi\chi^2 = [\bar{\psi}]$ has order prime to p. Proposition 5.5 implies that for all open pro-p subgroups $V_p^{\mathfrak{p}}$ of $U_p^{\mathfrak{p}}$, the two actions on $S_{[\bar{\psi}]}(U^p, L/\mathcal{O})_{\mathfrak{m}}^{V_p^{\mathfrak{p}}}$ coincide. By passing to the direct limit we obtain that the two actions on $S_{[\bar{\psi}]}(U^p, L/\mathcal{O})$ coincide, and by twisting we obtain the same result with ψ instead of $[\bar{\psi}]$. □

Corollary 5.7. *Let $x \in \text{m-Spec}\,\mathbb{T}(U^p)_{\mathfrak{m}}[1/p]$ be such that the restriction of the corresponding Galois representation $\rho_x : G_{F,S} \to \mathrm{GL}_2(\kappa(x))$ to $G_{F_{\mathfrak{p}}}$ is irreducible. Then there is an isomorphism of L-Banach space representations of G,*

$$(S_{\psi,\lambda}(U^{\mathfrak{p}}, \mathcal{O}) \otimes_{\mathcal{O}} L)[\mathfrak{m}_x] \cong \Pi^{\oplus n} \tag{28}$$

for some integer $n \geq 0$, where $\Pi \in \mathrm{Ban}^{\mathrm{adm}}_G(L)$ corresponds to $\rho_x|_{G_{F_{\mathfrak{p}}}}$ via the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

Proof. Let $y : R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}} \to L$ be the homomorphism corresponding to $\mathrm{tr}(\rho_x|_{G_{F_{\mathfrak{p}}}})$. It follows from [37, Corollaries 6.8, 8.14 and 9.36, Proposition 10.107] that (28) holds for any $\Pi' \in \mathrm{Ban}^{\mathrm{adm}}_G(L)_{\bar{r}^{\mathrm{ss}}}$ of finite length on which $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$ acts as the centre of the category via y. Corollary 5.6 and Lemma 5.8 imply that we can apply this observation to $\Pi' = (S_{\psi,\lambda}(U^{\mathfrak{p}}, \mathcal{O}) \otimes_{\mathcal{O}} L)[\mathfrak{m}_x]$. □

Lemma 5.8. *Let $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$ act on $\tau \in \mathrm{Mod}^{\mathrm{l.adm}}_G(\mathcal{O})_{\bar{r}^{\mathrm{ss}}}$ as the centre of the category $\mathrm{Mod}^{\mathrm{l.adm}}_G(\mathcal{O})_{\bar{r}^{\mathrm{ss}}}$. If the G-socle of τ is of finite length, then $k \otimes_{R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}} \tau^\vee$ is of finite length in $\mathfrak{C}(\mathcal{O})$. In particular, the assertion holds for any admissible representation τ in $\mathrm{Mod}^{\mathrm{l.adm}}_G(\mathcal{O})_{\bar{r}^{\mathrm{ss}}}$.*

Proof. There are only finitely many isomorphism classes of irreducible objects in $\mathfrak{B}_{\bar{r}^{\mathrm{ss}}}$. Let $\pi_1, \dots, \pi_k$ be a set of representatives. For each i, let $\pi_i \hookrightarrow J_i$ be an injective envelope of π_i in $\mathrm{Mod}^{\mathrm{l.adm}}_G(\mathcal{O})$. Dually, let $P_i := J_i^\vee$ so that $P_i \twoheadrightarrow \pi_i^\vee$ is a projective envelope of $\pi_i^\vee$ in $\mathfrak{C}(\mathcal{O})$. Since the category $\mathrm{Mod}^{\mathrm{l.adm}}_G(\mathcal{O})_{\bar{r}^{\mathrm{ss}}}$ is locally finite, we can embed τ into an injective envelope of its G-socle which is isomorphic to a direct sum of finitely many copies of J_i. This implies that dually there is a surjection $\oplus_{i=1}^k P_i^{\oplus m_i} \twoheadrightarrow \tau^\vee$, where m_i are finite multiplicities. Thus it is enough to prove the statement for $\tau = J_i$, $1 \leq i \leq k$. It follows from [37, Lemma 3.3] that $k \otimes_{R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}} P_i$ is of finite length if and only if $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P_j, k \otimes_{R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}} P_i)$ is a finite-dimensional k-vector space for $1 \leq j \leq k$. Since P_j is projective, the natural map

$$k \otimes_{R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}} \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P_j, P_i) \to \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P_j, k \otimes_{R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}} P_i)$$

is an isomorphism. Thus it is enough to show that $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P_j, P_i)$ is a finitely generated $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$-module for $1 \leq i, j \leq k$. This assertion follows from [37, Proposition 6.3 and

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Corollary 8.11 in conjunction with Proposition B.26, Corollaries 9.25 and 9.27 and Lemma 10.90]. □

Lemma 5.9. *Let $\tau \in \mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})_{\bar{r}^{\mathrm{ss}}}$ and suppose that we are given a faithful action on τ of a local $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$-algebra R with residue field k via $R \hookrightarrow \mathrm{End}_G(\tau)$. Assume that the composition $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}} \to R \to \mathrm{End}_G(\tau)$ coincides with the action of $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$ on τ as the centre of category $\mathrm{Mod}_G^{\mathrm{l.adm}}(\mathcal{O})_{\bar{r}^{\mathrm{ss}}}$. If the G-socle of τ is of finite length, then R is finite over $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$.*

Proof. Let P_i be as in the proof of Lemma 5.8 and let $P = \oplus_{i=1}^n P_i$. Then P is a projective generator for the category $\mathfrak{C}(\mathcal{O})_{\bar{r}^{\mathrm{ss}}}$, and thus R acts faithfully on $\mathrm{m} := \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P, \tau^\vee)$. The proof of Lemma 5.8 shows that $\mathrm{m} \otimes_{R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}} k$ is a finite-dimensional k-vector space. Since m is a compact $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$-module, we deduce that m is finitely generated over $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$ and hence also over R. If $v_1, \dots, v_n$ are generators of m as an R-module, then the map $r \mapsto (rv_1, \dots, rv_n)$ induces an embedding of R-modules $R \hookrightarrow \mathrm{m}^{\oplus n}$. Since $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$ is Noetherian, we deduce that R is a finitely generated $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$-module. □

Lemma 5.10. *Let J be an injective envelope of an irreducible representation in $\mathrm{Mod}_{G,\zeta}^{\mathrm{l.adm}}(k)$. Then J does not admit an admissible quotient.*

Proof. The action of $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$ on J as the centre of the category factors through the quotient $R^{\mathrm{ps},\psi}_{\mathrm{tr}\,\bar{r}}/\varpi$. If σ is a smooth irreducible representation of K, then $\mathrm{Hom}_K(\sigma, J)^\vee$ is either zero or a finitely generated Cohen–Macaulay module of $R^{\mathrm{ps},\psi}_{\mathrm{tr}\,\bar{r}}/\varpi$ of dimension 1; in most cases this follows from [39, Theorem 5.2], since it implies that the element denoted by x in that theorem is a regular parameter. The rest of the cases are handled in [26, Proposition 3.9] by a similar argument. It is proved in [37, Proposition 5.16] that J is injective in $\mathrm{Mod}_{G,\zeta}^{\mathrm{sm}}(k)$. Since restriction is right adjoint to compact induction, which is an exact functor, we deduce that J is injective in $\mathrm{Mod}_{K,\zeta}^{\mathrm{sm}}(k)$. Thus if $\sigma \in \mathrm{Mod}_{K,\zeta}^{\mathrm{sm}}(k)$ is of finite length, then $\mathrm{Hom}_K(\sigma, J)^\vee$ is either zero or a successive extension of 1-dimensional Cohen–Macaulay modules corresponding to the irreducible subquotients σ' of σ for which $\mathrm{Hom}_K(\sigma', J)$ is nonzero. Hence if $\mathrm{Hom}_K(\sigma, J)^\vee$ is nonzero, then it is Cohen–Macaulay of dimension 1. Note that by taking $\sigma = \mathrm{Ind}_{K_n(Z\cap K)}^K \zeta$, where K_n is the nth congruence subgroup of K, $(J^{K_n})^\vee$ is Cohen–Macaulay of dimension 1.

If τ is a quotient of J, then we can choose $n \ge 1$ such that the map $J^{K_n} \to \tau^{K_n}$ is nonzero. By taking Pontryagin duals, we obtain a nonzero map of $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$-modules $(\tau^{K_n})^\vee \to (J^{K_n})^\vee$. If τ is admissible, then τ^{K_n} is a finite-dimensional k-vector space, and hence $(J^{K_n})^\vee$ will contain a nonzero submodule killed by the maximal ideal of $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$. This is not possible, as $(J^{K_n})^\vee$ is Cohen–Macaulay of dimension 1. □

Proposition 5.11. *Let τ and $R \hookrightarrow \mathrm{End}_G(\tau)$ be as in Lemma 5.9. Assume that τ is killed by ϖ and has a central character. If τ is admissible, then the Krull dimension of R is less than or equal to 2.*

Proof. Since τ is admissible, its G-socle is of finite length. Lemma 5.9 implies that R is a finite $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$-module. Since we are interested only in the Krull dimension, we can

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assume that R is the image of $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$ in $\mathrm{End}_G(\tau)$. Since τ has a central character, the map $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}} \to \mathrm{End}_G(\tau)$ factors through the quotient $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}} \twoheadrightarrow R^{\mathrm{ps},\psi}_{\mathrm{tr}\,\bar{r}}$, which parameterises pseudocharacters with a fixed determinant. Since ϖ kills τ, we get a surjection $R^{\mathrm{ps},\psi}_{\mathrm{tr}\,\bar{r}}/\varpi \twoheadrightarrow R$. Now $R^{\mathrm{ps},\psi}_{\mathrm{tr}\,\bar{r}}/\varpi$ is an integral domain of Krull dimension 3 (see [37, §A]), so we have to show that $R^{\mathrm{ps},\psi}_{\mathrm{tr}\,\bar{r}}/\varpi$ does not act faithfully on τ.

Let $\pi_1,\dots,\pi_k$ be as in the proof of Lemma 5.8, but let $\pi_i \hookrightarrow J_i$ now denote an injective envelope of π_i in $\mathrm{Mod}^{\mathrm{l.adm}}_{G,\zeta}(k)$, so that the rest of the proof works with a fixed central character. Let $P := \oplus_{i=1}^{k} P_i$, where $P_i := J_i^{\vee}$, and let $E = \mathrm{End}_{\mathfrak{C}(k)}(P)$. Then P is a projective generator of $\mathfrak{C}_\zeta(k)_{\bar{r}^{\mathrm{ss}}}$, and if we let $\mathrm{m} := \mathrm{Hom}_{\mathfrak{C}(k)}(P,\tau^{\vee})$, then evaluation induces a natural isomorphism $\mathrm{m}\,\widehat{\otimes}_E P \xrightarrow{\cong} \tau^{\vee}$. Thus it is enough to show that $R^{\mathrm{ps},\psi}_{\mathrm{tr}\,\bar{r}}/\varpi$ does not act faithfully on m.

The action of $R^{\mathrm{ps},\psi}_{\mathrm{tr}\,\bar{r}}/\varpi$ on m is faithful if and only if $\mathrm{m}\otimes_{R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}} Q$ is nonzero, where Q is the quotient field of $R^{\mathrm{ps},\psi}_{\mathrm{tr}\,\bar{r}}/\varpi$. Let us assume that this is the case. After replacing m by a subquotient, we can assume that m is a cyclic E-module and the map $\mathrm{m} \to \mathrm{m}\otimes_{R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}} Q$ is injective. We claim that the algebra $E\otimes_{R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}} Q$ is semisimple. Granting the claim, we deduce that the surjection $E\otimes_{R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}} Q \twoheadrightarrow \mathrm{m}\otimes_{R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}} Q$ has a section of $E\otimes_{R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}} Q$-modules. By composing it with the injection $\mathrm{m}\hookrightarrow \mathrm{m}\otimes_{R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}} Q$, we obtain an injection $\mathrm{m}\hookrightarrow E\otimes_{R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}} Q$ of E-modules. Since m is finitely generated over E, we can multiply this embedding by an element of $R^{\mathrm{ps},\psi}_{\mathrm{tr}\,\bar{r}}$ to obtain an injection of E-modules $\mathrm{m}\hookrightarrow E$. Since P is a projective generator for the block, applying $\widehat{\otimes}_E P$ gives an injection $\tau^{\vee}\hookrightarrow P$ and dually a surjection $J \twoheadrightarrow \tau$. Thus one of the J_i's admits an admissible quotient contradicting Lemma 5.10.

To prove the claim, observe that if $\bar{r}$ does not have scalar semisimplification, then $E\otimes_{R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}} Q$ is a matrix algebra over Q. If $\bar{r}$ is irreducible, then this follows from [37, Proposition 6.3]; if $\bar{r}$ is reducible generic, then it follows from [37, Corollary 8.11] together with [41, Propositions 4.3, 3.12]; and if $\bar{r}^{\mathrm{ss}} = \chi\oplus\chi\omega$, then it follows from the explicit description of the endomorphism ring of a projective generator of $\mathfrak{C}_\zeta(\mathcal{O})_{\bar{r}^{\mathrm{ss}}}$ given after [37, Corollary 10.94]: both generators c_0 and c_1 of the reducible locus in $R^{\mathrm{ps},\psi}_{\mathrm{tr}\,\bar{r}}$ are nonzero in $R^{\mathrm{ps},\psi}_{\mathrm{tr}\,\bar{r}}/\varpi$ and hence in Q.

Let us assume that $\bar{r}$ has scalar semisimplification. After twisting, we can assume that ζ is trivial and use [37, Corollary 9.27] to identify E with the ring opposite to R/ϖ, where R is the ring defined in [37, Equation (125)]. Then $R^{\mathrm{ps},\psi}_{\mathrm{tr}\,\bar{\rho}}$ gets identified with the subring denoted by $\mathcal{O}[[t_1,t_2,t_3]]$ in [37]. Let C be the finite $R^{\mathrm{ps},\psi}_{\mathrm{tr}\,\bar{\rho}}$-algebra defined in [37, Definition 9.7], let x be a generic point of $C\otimes_{R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{\rho}}} Q$ and let $\kappa(x)$ be its residue field. We claim that $E\otimes_{R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{\rho}}}\kappa(x)\cong M_2(\kappa(x))$. The proof of [37, Lemma 9.20] shows that the element denoted by $(uv-vu)(uv-vu)^*$ is nonzero in Q. The proof of [37, Lemma 9.21] goes on to show that the specialisation at x of the representation $\rho : E\to M_2(C)$, constructed in [37, Proposition 9.8], is absolutely irreducible over $\kappa(x)$. The double centraliser theorem implies that the map $E\otimes_{R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{\rho}}}\kappa(x)\to M_2(\kappa(x))$ is surjective. Since the algebra $E\otimes_{R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{\rho}}}\kappa(x)$ is 4-dimensional as a $\kappa(x)$-vector space (see [37, Lemma 9.18]), we obtain the claim. □

Remark 5.12. A different proof of Proposition 5.11 could be given using [26, Proposition 5.6].

Corollary 5.13. *Let τ and $R \hookrightarrow \mathrm{End}_G(\tau)$ be as in Lemma 5.9. Assume that τ has a central character and is ϖ-divisible (equivalently, $\tau^\vee$ is ϖ-torsion free). Then the Krull dimension of R is at most* 3.

Proof. Since $\tau^\vee$ is ϖ-torsion free, we have an exact sequence $0 \to \tau^\vee \overset{\varpi}{\to} \tau^\vee \to \tau^\vee/\varpi \to 0$. Let P be the projective generator of $\mathfrak{C}_\zeta(\mathcal{O})_{\bar{r}^{\mathrm{ss}}}$ as in the proof of Proposition 5.11. We have an exact sequence of R-modules

$$0 \to \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P,\tau^\vee) \overset{\varpi}{\to} \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P,\tau^\vee) \to \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P,\tau^\vee/\varpi) \to 0.$$

As explained in the proof of Lemma 5.9, $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P,\tau^\vee)$ is a finitely generated faithful R-module. If we let $\overline{R}$ be the quotient of R which acts faithfully on $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P,\tau^\vee/\varpi)$, then its Krull dimension is equal to the dimension of the support of $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P,\tau^\vee/\varpi)$ in $\mathrm{Spec}\, R$, which is one less than the dimension of the support of $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P,\tau^\vee)$ in $\mathrm{Spec}\, R$, as ϖ is regular on the module, and thus is equal to $\dim R - 1$, as the action of R on $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P,\tau^\vee)$ is faithful.

It follows from Proposition 5.11 applied to $\overline{R}$ and $\tau[\varpi]$ that the Krull dimension of $\overline{R}$ is at most 2, and thus the Krull dimension of R is at most 3. □

Corollary 5.14. *The image of $\mathbb{T}(U^p)_{\mathfrak{m}}$ in $\mathrm{End}_G^{\mathrm{cont}}(S_{\psi,\lambda}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}})$ is a finite $R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}$-algebra of Krull dimension at most* 3.

Proof. This follows from Lemma 5.9 and Corollary 5.13 applied to $\tau = S_{\psi,\lambda}(U^{\mathfrak{p}},L/\mathcal{O})_{\mathfrak{m}}$. □

Remark 5.15. The proof of Corollary 5.14 goes back to the Workshop on Galois Representations and Automorphic Forms at Princeton in 2011, and was motivated by discussions with Matthew Emerton there. It remained unpublished, since after we communicated the result to Frank Calegari [42], he and Patrick Allen proved much more general results concerning the finiteness of global deformation rings over local deformation rings [1]. However, one advantage of our argument is that we do not have to assume anything about the image of the global Galois representation $\bar{\rho}$.

Lemma 5.16. *If $\pi \in \mathrm{Mod}^{\mathrm{l.adm}}_{G,\zeta}(\mathcal{O})$ is irreducible and not a character, then*

$$\delta_{\mathcal{O}[[K]]}(\pi^\vee) = 1.$$

Proof. This follows from the proof of [43, Corollary 7.5]. Alternatively, one can use [29, Propositions 5.4, 5.7, Theorem 5.13]. However, the proof dearest to my heart is to show that $\dim \pi^{K_n}$ grows as Cp^n, for some constant C, as K_n runs over principal congruence subgroups of K. If π is principal series, then this can be done by hand; the result for special series can be deduced from this. The most interesting case, when π is supersingular, can be deduced from the exact sequence in [36, Theorem 6.3]. We sketch the argument using the notation of that theorem. Indeed, using the exact sequence, it is enough to estimate the growth of $\dim M^{K_n}$. [36, Lemma 4.10] implies that it is enough to estimate the growth of $\dim M_{\sigma}^{K_n}$ and $\dim M_{\tilde{\sigma}}^{K_n}$. By [36, Proposition 4.7], the restrictions of M_σ and $M_{\tilde{\sigma}}$ to $N_0 := \left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$ are isomorphic to the space of smooth functions from N_0 to k. This implies that

$$\dim M_{\sigma}^{K_n \cap N_0} = \dim M_{\tilde{\sigma}}^{K_n \cap N_0} = p^n.$$

This gives an upper bound on the growth of K_n-invariants, which has to be of the right order, since both M_σ and $M_{\tilde{\sigma}}$ are infinite-dimensional k-vector spaces.

Yet another alternative is to use results of Morra [33], who actually computes the dimensions $\dim \pi^{I_n}$, where I_n is a certain filtration of Iwahori subgroups by open normal subgroups. □

Proposition 5.17. *Let R be the image of $\mathbb{T}(U^p)_{\mathfrak{m}} \to \mathrm{End}_G^{\mathrm{cont}}(S_{\psi,\lambda}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}})$. Then there is a subring $A \subset R$, such that $A \cong \mathcal{O}[[x,y]]$, R is finite over A and $S_{\psi,\lambda}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}^d$ is a flat A-module.*

Proof. Lemmas 5.1 and 5.3 and (27) imply that $S_{\psi,\lambda}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}^d$ is projective in $\mathrm{Mod}_{K,\zeta}^{\mathrm{pro}}(\mathcal{O})$ and finitely generated over $\mathcal{O}[[K]]$. Thus $S_{\psi,\lambda}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}^d/\varpi$ is isomorphic as an $\mathcal{O}[[K_1]]$-module to a finite direct sum of copies of $k[[K_1/Z_1]]$, where Z_1 is the centre of K_1. This implies that $S_{\psi,\lambda}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}^d$ is a Cohen–Macaulay $\mathcal{O}[[K_1]]$-module of dimension 4. The fibre $\mathcal{F} := k \otimes_R S_{\psi,\lambda}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}^d$ is a quotient of $k \otimes_{R^{\mathrm{ps}}_{\mathrm{tr}\,\bar{r}}} S_{\psi,\lambda}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}^d$. Corollary 5.6 and Lemma 5.8 imply that $\mathcal{F}$ is of finite length as a G-representation. Lemma 5.16 implies that the fibre has dimension less than or equal to one. If it is zero, then all irreducible subquotients of $\mathcal{F}$ are characters, and by looking at the graded pieces of the $\mathfrak{m}$-adic filtration on $S_{\psi}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}^d$, where $\mathfrak{m}$ is the maximal ideal of R, we can deduce that all the irreducible subquotients of $S_{\psi}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}^d$ are characters. Since the central character is fixed and $p > 2$, there are no nontrivial extensions between 1-dimensional G-representations over k. This would imply that $\mathrm{SL}_2(\mathbb{Q}_p)$ acts trivially on $S_{\psi,\lambda}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}}^d$, which is impossible, since it would imply that $\mathrm{SL}_2(\mathbb{Z}_p)$ acts trivially on a projective object in $\mathrm{Mod}_{K,\zeta}^{\mathrm{pro}}(\mathcal{O})$. Hence, the dimension of the fibre is 1. It follows from Corollary 5.14 that the Krull dimension of R is at most 3. The assertion follows from Corollary 4.2. □

6. Main result

Keep the notation of the previous section. Let D be the quaternion algebra over F, which is ramified at $\mathfrak{p}$ and split at ∞_F and has the same ramification behaviour as D_0 at all the other places. Fix an isomorphism

$$D_0 \otimes_F \mathbb{A}_F^{\mathfrak{p},\infty_F} \cong D \otimes_F \mathbb{A}_F^{\mathfrak{p},\infty_F}.$$

This allows us to view the subgroup $U^{\mathfrak{p}}$ of $(D_0 \otimes_F \mathbb{A}_F^f)^\times$, considered in the previous section, as a subgroup of $(D \otimes_F \mathbb{A}_F^f)^\times$. Let $D_{\mathfrak{p}} := D \otimes_F F_{\mathfrak{p}}$. Then $D_{\mathfrak{p}}$ is the nonsplit quaternion algebra over $F_{\mathfrak{p}} = \mathbb{Q}_p$. Let $U_{\mathfrak{p}} = \mathcal{O}_{D_{\mathfrak{p}}}^\times$ and $U = U_{\mathfrak{p}} U^{\mathfrak{p}}$. If K is an open subgroup of $U := U_{\mathfrak{p}} U^{\mathfrak{p}}$, then let $X(K)$ be the corresponding Shimura curve for D/F defined over

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F. Let

$$\widehat{H}^i(U^{\mathfrak{p}}, L/\mathcal{O}) := \varinjlim_{K_{\mathfrak{p}}} H^i_{\text{ét}}(X(K_{\mathfrak{p}}U^{\mathfrak{p}})_{\overline{F}}, L/\mathcal{O}),$$

where the limit is taken over open subgroups of $U_{\mathfrak{p}}$. [47, Theorem 6.2] gives an isomorphism of $G_{\mathbb{Q}_p} \times D_{\mathfrak{p}}^{\times}$-representations:

$$\mathcal{S}^i(S(U^{\mathfrak{p}}, L/\mathcal{O})) \cong \widehat{H}^i(U^{\mathfrak{p}}, L/\mathcal{O}), \tag{29}$$

where $\mathcal{S}^i$ is the functor $\pi \mapsto H^i_{\text{ét}}(\mathbb{P}^1, \mathcal{F}_\pi)$. Let v be a finite place of F different from $\mathfrak{p}$ and let $g_v \in (D_0 \otimes_F F_v)^{\times}$. We view g_v also as an element of $(D \otimes_F F_v)^{\times}$ using the identification above. Multiplication with g_v induces an isomorphism $S(U^{\mathfrak{p}}, L/\mathcal{O}) \cong S(g_v^{-1} U^{\mathfrak{p}} g_v, L/\mathcal{O})$. It follows from the identifications explained at the end of the proof of [47, Proposition 6.5] that the following diagram of $G_{\mathbb{Q}_p} \times D_{\mathfrak{p}}^{\times}$-representations commutes:

$$\begin{tikzcd}
\mathcal{S}^i(S(U^{\mathfrak{p}}, L/\mathcal{O})) \arrow[r, "\mathcal{S}^i(\cdot g_v)"] \arrow[d, "\cong", "(29)"'] & \mathcal{S}^i(S(g_v^{-1} U^{\mathfrak{p}} g_v, L/\mathcal{O})) \arrow[d, "\cong", "(29)"'] \\
\widehat{H}^i(U^{\mathfrak{p}}, L/\mathcal{O}) \arrow[r, "\cdot g_v"] & \widehat{H}^i(g_v U^{\mathfrak{p}} g_v^{-1}, L/\mathcal{O}).
\end{tikzcd}$$

Thus, if we let

$$\widehat{H}^i(U^{p}, L/\mathcal{O}) := \varinjlim_{K_p} H^i_{\text{ét}}(X(K_p U^p)_{\overline{F}}, L/\mathcal{O}),$$

where the limit is taken over open subgroups of U_p, then we deduce that (29) induces an isomorphism of $\mathbb{T}_S^{\text{univ}}[G_{\mathbb{Q}_p} \times (D \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\times}]$-modules:

$$\mathcal{S}^i(S(U^p, L/\mathcal{O})) \cong \widehat{H}^i(U^p, L/\mathcal{O}). \tag{30}$$

Let $\bar{\rho} : G_{F,S} \to \mathrm{GL}_2(k)$ be an absolutely irreducible representation as in the previous section, and let $\mathfrak{m}$ be the corresponding maximal ideal in $\mathbb{T}_S^{\text{univ}}$. Let

$$\bar{r} := \bar{\rho}|_{G_{F_{\mathfrak{p}}}}$$

and recall the assumption that $F_{\mathfrak{p}} = \mathbb{Q}_p$. It follows from in [47, Corollary 7.5] that (30) induces an isomorphism of $\mathbb{T}(U^p)_{\mathfrak{m}}[G_{\mathbb{Q}_p} \times (D \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\times}]$-modules:

$$\mathcal{S}^1(S(U^p, L/\mathcal{O})_{\mathfrak{m}}) \cong \widehat{H}^1(U^p, L/\mathcal{O})_{\mathfrak{m}}. \tag{31}$$

Lemma 6.1. *Let λ be a continuous representation of $U_p^{\mathfrak{p}}$ on a finite free $\mathcal{O}$-module and let I be an ideal of $\mathbb{T}(U^p)_{\mathfrak{m}}$. Then (31) induces an injection*

$$\mathcal{S}^1(\mathrm{Hom}_{U_p^{\mathfrak{p}}}(\lambda, S(U^p, L/\mathcal{O})_{\mathfrak{m}}[I]) \subset \mathrm{Hom}_{U_p^{\mathfrak{p}}}(\lambda, \widehat{H}^1(U^p, L/\mathcal{O})_{\mathfrak{m}})[I], \tag{32}$$

such that the subgroup of $\mathcal{O}_{D_{\mathfrak{p}}}^{\times}$ of elements of reduced norm 1 acts trivially on the cokernel. Moreover, if the semisimplification of $\bar{r}$ is not of the form $\chi \oplus \chi\omega$, then the cokernel is zero.

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Proof. If λ is the trivial representation, then the first assertion is [47, Proposition 7.7]. The assertion for general λ follows from this by presenting λ as an $\mathcal{O}[[U_p^{\mathfrak{p}}]]$-module and arguing as in (23) and (24).

If (32) is not an isomorphism, then it follows from the proof of [47, Proposition 7.7] that (after $U^{\mathfrak{p}}$ is replaced with an open subgroup) there is a subquotient of $S(U^{\mathfrak{p}}, L/\mathcal{O})_{\mathfrak{m}}$ with nonzero $\mathrm{SL}_2(\mathbb{Q}_p)$-invariants. Since this representation is locally admissible, we deduce that $S_\psi(U^{\mathfrak{p}}, L/\mathcal{O})_{\mathfrak{m}}$ has a subquotient with nonzero $\mathrm{SL}_2(\mathbb{Q}_p)$-invariants, for some character $\psi : (\mathbb{A}_F^f)^\times / F^\times \to \mathcal{O}^\times$. Proposition 5.4 implies that the block corresponding to $\bar{r}^{\mathrm{ss}}$ contains a character. Thus we are in the setting of [37, §10], and so $\bar{r}^{\mathrm{ss}} \cong \chi \oplus \chi\omega$. $\square$

Let $\psi : (\mathbb{A}_F^f)^\times / F^\times \to \mathcal{O}^\times$ be a continuous character such that ψ is trivial of $(\mathbb{A}_F^f)^\times \cap U^p$. To ease the notation, we will use the same symbol to denote the restriction of ψ to the intersection of $(\mathbb{A}_F^f)^\times$ with various subgroups of $(D \otimes_F \mathbb{A}_F^f)^\times$, with the exception of $\zeta := \psi|_{F_{\mathfrak{p}}^\times}$. We will also view ψ as a character of $G_{F,S}$ via the class field theory and denote by the same letter its restriction to various decomposition groups.

Let $\widehat{H}^1_\psi(U^p, L/\mathcal{O})$ be the maximal submodule of $\widehat{H}^1(U^p, L/\mathcal{O})$ on which $(\mathbb{A}_F^f)^\times$ acts by the character ψ. By Chebotarev's density theorem, this coincides with the common eigenspace of all Hecke operators S_v for the eigenvalue $\psi(\mathrm{Frob}_v)$.

If λ is a continuous representation of $U_p^{\mathfrak{p}}$ on a finite free $\mathcal{O}$-module with central character ψ, then let

$$\widehat{H}^1_{\psi,\lambda}(U^{\mathfrak{p}}, L/\mathcal{O})_{\mathfrak{m}} := \mathrm{Hom}_{U_p^{\mathfrak{p}}}(\lambda, \widehat{H}^1_\psi(U^p, L/\mathcal{O})_{\mathfrak{m}}).$$

Lemma 6.2. *The map (31) induces isomorphisms of $\mathbb{T}(U^p)_{\mathfrak{m}}[G_{\mathbb{Q}_p} \times D_{\mathfrak{p}}^\times]$-modules*

$$\mathcal{S}^1(S_{\psi,\lambda}(U^{\mathfrak{p}}, L/\mathcal{O})_{\mathfrak{m}}) \cong \widehat{H}^1_{\psi,\lambda}(U^{\mathfrak{p}}, L/\mathcal{O})_{\mathfrak{m}} \tag{33}$$

and of $\mathbb{T}(U^p)_{\mathfrak{m}}[G_{\mathbb{Q}_p} \times (D \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times]$-modules

$$\mathcal{S}^1(S_\psi(U^p, L/\mathcal{O})) \cong \widehat{H}^1_\psi(U^p, L/\mathcal{O}). \tag{34}$$

Proof. It follows from [34, Proposition 5.6] that $\widehat{H}^1_\psi(U^p, L/\mathcal{O})_{\mathfrak{m}}$ is injective in the category $\mathrm{Mod}^{\mathrm{sm}}_{U_p,\psi}(\mathcal{O})$. Arguing as in the proof of Lemma 5.2, we obtain that $\widehat{H}^1_{\psi,\lambda}(U^{\mathfrak{p}}, L/\mathcal{O})_{\mathfrak{m}}$ is injective in $\mathrm{Mod}^{\mathrm{sm}}_{U_{\mathfrak{p}},\zeta}(\mathcal{O})$. Hence, if H is an open pro-p subgroup of

$$D_{\mathfrak{p}}^{\times,1} := \{g \in D_{\mathfrak{p}}^\times : \mathrm{Nrd}(g) = 1\}$$

which intersects the centre of $D_{\mathfrak{p}}^\times$ trivially, then $(\widehat{H}^1_{\psi,\lambda}(U^{\mathfrak{p}}, L/\mathcal{O})_{\mathfrak{m}})^\vee$ is a free $\mathcal{O}[[H]]$-module of finite rank. Since $D_{\mathfrak{p}}^\times$ is a p-adic analytic group, we can choose H to be torsion free, in which case $\mathcal{O}[[H]]$ is an integral domain and thus does not contain a nonzero $\mathcal{O}$-submodule on which H acts trivially. Thus, the cokernel of (31), applied with I equal to the ideal generated by $S_v - \psi(\mathrm{Frob}_v)$ for all $v \notin S$, is zero, and we get (33). The last isomorphism follows from (33) using (25). $\square$

Arguing as in [18, Lemma 5.3.8], we obtain isomorphisms

$$(\widehat{H}^1_\psi(U^p,L/\mathcal{O})_\mathfrak{m})^{K_p}[\varpi^n] \cong H^1_\psi(X(U^pK_p),\mathcal{O}/\varpi^n)_\mathfrak{m}, \tag{35}$$

$$(\widehat{H}^1_\psi(U^p,L/\mathcal{O})_\mathfrak{m})^{K_p^\mathfrak{p}}[\varpi^n] \cong \widehat{H}^1_\psi(U^pK_p^\mathfrak{p},\mathcal{O}/\varpi^n)_\mathfrak{m}. \tag{36}$$

Thus if we let

$$\widehat{H}^1_\psi(U^p,\mathcal{O})_\mathfrak{m} := \varprojlim_n \widehat{H}^1_\psi(U^p,\mathcal{O}/\varpi^n)_\mathfrak{m}, \quad \widehat{H}^1_{\psi,\lambda}(U^\mathfrak{p},\mathcal{O})_\mathfrak{m} := \varprojlim_n \widehat{H}^1_{\psi,\lambda}(U^p,\mathcal{O}/\varpi^n)_\mathfrak{m},$$

then these are $\mathcal{O}$-torsion free, and by inverting p we obtain admissible unitary Banach space representations of $(D\otimes_\mathbb{Q}\mathbb{Q}_p)^\times$ and $D_\mathfrak{p}^\times$, respectively. Moreover, we have natural homeomorphisms of $\mathcal{O}$-modules

$$(\widehat{H}^1_\psi(U^p,\mathcal{O})_\mathfrak{m})^d \cong (\widehat{H}^1_\psi(U^p,L/\mathcal{O})_\mathfrak{m})^\vee, \quad (\widehat{H}^1_{\psi,\lambda}(U^\mathfrak{p},\mathcal{O})_\mathfrak{m})^d \cong (\widehat{H}^1_{\psi,\lambda}(U^\mathfrak{p},L/\mathcal{O})_\mathfrak{m})^\vee. \tag{37}$$

In the following, to ease the notation we will omit the outer brackets in the duals.

Proposition 6.3. *There is an isomorphism of $\mathbb{T}(U^p)_\mathfrak{m}[G_{\mathbb{Q}_p}\times D_\mathfrak{p}^\times]$-modules*

$$\check{\mathcal{S}}^1(S_{\psi,\lambda}(U^\mathfrak{p},\mathcal{O})^d_\mathfrak{m}) \cong \widehat{H}^1_{\psi,\lambda}(U^\mathfrak{p},\mathcal{O})^d_\mathfrak{m}.$$

Proof. This follows from the definition of the functor $\check{\mathcal{S}}^1$, together with (33) and (37). $\square$

Proposition 6.4. *$\widehat{H}^1_\psi(U^p,\mathcal{O})^d_\mathfrak{m}$ is a finitely generated $\mathcal{O}[[U_p]]$-module which is projective in $\mathrm{Mod}^{\mathrm{pro}}_{U_p,\psi}(\mathcal{O})$. $\widehat{H}^1_{\psi,\lambda}(U^\mathfrak{p},\mathcal{O})^d_\mathfrak{m}$ is a finitely generated $\mathcal{O}[[U_\mathfrak{p}]]$-module which is projective in $\mathrm{Mod}^{\mathrm{pro}}_{U_\mathfrak{p},\zeta}(\mathcal{O})$.*

Proof. The projectivity and finite generation follow from (37), together with the injectivity and admissibility statements for $\widehat{H}^1_\psi(U^p,L/\mathcal{O})_\mathfrak{m}$ and $\widehat{H}^1_{\psi,\lambda}(U^\mathfrak{p},L/\mathcal{O})_\mathfrak{m}$ already explained. $\square$

Assume that ψ and $\bar{\rho}$ are such that $S_\psi(U^p,L/\mathcal{O})_\mathfrak{m}\neq 0$ (see Lemma 5.3). After twisting by a character, we can assume that the restriction of ψ to $(\mathbb{A}_F^f)^\times\cap U_p$ is a locally algebraic character.

Theorem 6.5. *The functor $\mathcal{S}^1$ is not identically zero on $\mathrm{Mod}^{\mathrm{l.adm}}_G(\mathcal{O})_{\bar{r}^{\mathrm{ss}}}$.*

Proof. Let ψ be such that $S_\psi(U^p,L/\mathcal{O})_\mathfrak{m}$ is nonzero and the restriction of ψ to $(\mathbb{A}_F^f)^\times\cap U_p$ is a locally algebraic character. We can write $\psi=\psi_{\mathrm{sm}}\psi_{\mathrm{alg}}$, where ψ_{sm} is a smooth character of $(\mathbb{A}_F^f)^\times\cap U_p$ and ψ_{alg} is the restriction to $(\mathbb{A}_F^f)^\times\cap U_p$ of the central character of an irreducible algebraic representation W_{alg} evaluated at L of the algebraic group $(\mathrm{Res}^F_\mathbb{Q} D_0^\times)\otimes_\mathbb{Q} L$, where Res denotes the restriction of scalars.

Since $S_\psi(U^p,L/\mathcal{O})_\mathfrak{m}$ is an object of $\mathrm{Mod}^{\mathrm{l.adm}}_G(\mathcal{O})_{\bar{r}^{\mathrm{ss}}}$ by Proposition 5.4, using (34) it is enough to check that $\widehat{H}^1_\psi(U^p,L/\mathcal{O})_\mathfrak{m}$ is nonzero, and (37) implies that it is enough to check that $\widehat{H}^1_\psi(U^p,\mathcal{O})_\mathfrak{m}$ is nonzero. The locally algebraic vectors in the Banach spaces $\widehat{H}^1_\psi(U^p,\mathcal{O})_\mathfrak{m}\otimes_\mathcal{O} L$ and $S_\psi(U^p,\mathcal{O})_\mathfrak{m}\otimes_\mathcal{O} L$ are related to classical automorphic forms on

$D^\times$ and $D_0^\times$, respectively (see [19, §3] and [34, Theorem 5.3]). The assumption that $S_\psi(U^p,L/\mathcal{O})_\mathfrak{m}\neq 0$ implies that $S_\psi(U^p,\mathcal{O})_\mathfrak{m}\otimes_\mathcal{O} L$ is nonzero, and if V_p is an open pro-p subgroup of U_p, then it follows from Lemma 5.3 that $S_\psi(U^p,\mathcal{O})_\mathfrak{m}\otimes_\mathcal{O} L$ as a V_p-representation is isomorphic to a finite direct sum of copies of $C_\psi(V_p,L)$ in the notation of Section 5. Thus if γ is a representation of V_p on a finite-dimensional L-vector space with the central character ψ, then $\mathrm{Hom}_{V_p}(\gamma,S_\psi(U^p,\mathcal{O})_\mathfrak{m}\otimes_\mathcal{O} L)$ is nonzero.

We choose V_p of the form $V_p^\mathfrak{p}\times V_\mathfrak{p}$, such that ψ_{sm} is trivial on $V_p\cap(\mathbb{A}_F^f)^\times$ and $V_\mathfrak{p}=\left(\begin{smallmatrix}1+\mathfrak{p}^m & \mathfrak{p}^{m-1}\\ \mathfrak{p}^m & 1+\mathfrak{p}^m\end{smallmatrix}\right)$, for some $m\geq 1$. Let $\theta:V_\mathfrak{p}\to L^\times$ be the character which maps $\left(\begin{smallmatrix}1+p^m a & p^{m-1}b\\ p^m c & 1+p^m d\end{smallmatrix}\right)$ to $\alpha(b+c)$, where $\alpha:\mathcal{O}_{F_\mathfrak{p}}/\mathfrak{p}^{m-1}\to L^\times$ is any nontrivial additive character. Then θ is trivial on $V_\mathfrak{p}\cap(\mathbb{A}_F^f)^\times$ and is a supercuspidal type, by which we mean that if π is a smooth irreducible representation of $\mathrm{GL}_2(F_\mathfrak{p})$ and $\mathrm{Hom}_{V_\mathfrak{p}}(\theta,\pi)\neq 0$, then π is supercuspidal (see [21, Proposition 3.19] or [47, Proposition 7.1] for a more general setting). Extend θ to a character of V_p by mapping $V_p^\mathfrak{p}$ to 1. If we let $\gamma:=\theta\otimes W_{\mathrm{alg}}$ as a representation of V_p, then it has a central character ψ by construction and thus $\mathrm{Hom}_{V_p}(\gamma,S_\psi(U^p,\mathcal{O})_\mathfrak{m}\otimes_\mathcal{O} L)$ is nonzero, as already explained. An eigenvector for the Hecke operators on this finite-dimensional vector space will give a classical automorphic form on $D_0^\times$. Since θ is a supercuspidal type, the automorphic form will be supercuspidal at $\mathfrak{p}$ (see the proof of [21, Theorem 5.1] or the proof of [47, Corollary 7.3]). Since the automorphic form is supercuspidal at $\mathfrak{p}$, we can transfer it to an automorphic form on $D^\times$ by the classical Jacquet–Langlands correspondence, which in turn gives a nonzero vector in the locally algebraic vectors of $\widehat{H}^1_\psi(U^p,\mathcal{O})_\mathfrak{m}\otimes_\mathcal{O} L$. $\square$

Remark 6.6. If $\bar{r}$ is irreducible, then $\mathfrak{B}_{\bar{r}}$ consists of one isomorphism class of a supersingular representation π. It follows from Theorem 6.5 that $\mathcal{S}^1(\pi)\neq 0$. This implies that $\mathcal{S}^1(\pi')\neq 0$ for any $\pi'\in\mathrm{Mod}^{\mathrm{l.adm}}_G(\mathcal{O})_{\bar{r}}$.

From now on, assume that $\bar{r}^{\mathrm{ss}}=\chi_1\oplus\chi_2$, with $\chi_1\chi_2^{-1}\neq\omega^{\pm 1}$. Let π_1, π_2 be the principal series representations defined in (10). Proposition 3.5 implies that at least one of $\check{\mathcal{S}}^1(\pi_1^\vee)$ and $\check{\mathcal{S}}^1(\pi_2^\vee)$ is nonzero.

For a finitely generated $\mathcal{O}[[U_\mathfrak{p}]]$-module M, abbreviate $\delta(M):=\delta_{\mathcal{O}[[U_\mathfrak{p}]]}(M)$. If B is an admissible unitary Banach space representation of $D_\mathfrak{p}^\times$, then we choose an open bounded $D_\mathfrak{p}^\times$-invariant lattice Θ in B and an open uniform pro-p group K of $D_\mathfrak{p}^\times$. Then $\Theta^d\otimes_\mathcal{O} L$ is a finitely generated module over the Auslander regular ring $\mathcal{O}[[K]]\otimes_\mathcal{O} L$. Let $\delta(\mathrm{B})$ be the dimension of $\Theta^d\otimes_\mathcal{O} L$ over $\mathcal{O}[[K]]\otimes_\mathcal{O} L$ and note that it is equal to $\delta_{\mathcal{O}[[K]]}(\Theta^d)-1$. We will sometimes refer to $\delta(M)$ and $\delta(\mathrm{B})$ as the δ-*dimension*.

Proposition 6.7. *The maximum of* $\delta(\check{\mathcal{S}}^1(\pi_1^\vee))$ *and* $\delta(\check{\mathcal{S}}^1(\pi_2^\vee))$ *is equal to* 1.

Proof. Let A be the ring in Proposition 5.17 and let K be an open uniform pro-p subgroup of $D_\mathfrak{p}^\times$. It follows from Propositions 3.7 and 6.3 that $\widehat{H}^1_{\psi,\lambda}(U^\mathfrak{p},\mathcal{O})^d_\mathfrak{m}$ is A-flat. Since it is nonzero, the fibre $\mathcal{F}:=k\otimes_A\widehat{H}^1_{\psi,\lambda}(U^\mathfrak{p},\mathcal{O})^d_\mathfrak{m}$ is also nonzero, and (17) implies that its δ-dimension is equal to $\delta(\widehat{H}^1_{\psi,\lambda}(U^\mathfrak{p},\mathcal{O})^d_\mathfrak{m})-3$. It follows from Proposition 6.4 that the δ-dimension of $\widehat{H}^1_{\psi,\lambda}(U^\mathfrak{p},\mathcal{O})^d_\mathfrak{m}$ is equal to 4 (see the argument in the proof of the analogous

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statement for $S_{\psi,\lambda}(U^{\mathfrak{p}},\mathcal{O})^d_{\mathfrak{m}}$ in Proposition 5.17). Hence the δ-dimension of the fibre is equal to 1.

As explained in the proof of Proposition 5.17, the fibre $k\otimes_A S_{\psi,\lambda}(U^{\mathfrak{p}},\mathcal{O})^d_{\mathfrak{m}}$ is of finite length in $\mathfrak{C}(\mathcal{O})_{\bar{r}^{\mathrm{ss}}}$ and all irreducible subquotients are isomorphic to either $\pi_1^\vee$ or $\pi_2^\vee$. Moreover, both $\pi_1^\vee$ and $\pi_2^\vee$ occur as subquotients. Since $\check{\mathcal{S}}^1$ is exact, by Corollary 3.2 we deduce that $\mathcal{F}$ has a filtration of finite length with graded pieces isomorphic to either $\check{\mathcal{S}}^1(\pi_1^\vee)$ or $\check{\mathcal{S}}^1(\pi_2^\vee)$, which implies the assertion. □

Corollary 6.8. *Let $r: G_{\mathbb{Q}_p}\to \mathrm{GL}_2(L)$ be a continuous representation with $\det r=\psi\varepsilon^{-1}$ and $\bar{r}^{\mathrm{ss}}=\chi_1\oplus\chi_2$. Let $\Pi\in \mathrm{Ban}^{\mathrm{adm}}_G(L)$ correspond to r via the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. Then $\check{\mathcal{S}}^1(\Pi)\neq 0$ and $\delta(\check{\mathcal{S}}^1(\Pi))=1$.*

Proof. Let Θ be an open bounded G-invariant lattice in Π. Then $(\Theta\otimes_{\mathcal{O}}k)^{\mathrm{ss}}\cong\pi_1\oplus\pi_2$ (see [37, §11]). We can assume that $\Theta\otimes_{\mathcal{O}}k$ is an extension of π_1 by π_2. Then $\check{\mathcal{S}}^1(\Theta^d/\varpi)$ is an extension of $\check{\mathcal{S}}^1(\pi_2^\vee)$ by $\check{\mathcal{S}}^1(\pi_1^\vee)$. Proposition 6.7 implies that $\delta(\check{\mathcal{S}}^1(\Theta^d/\varpi))=1$. Proposition 3.7 applied with $A=\mathcal{O}$ implies that $\check{\mathcal{S}}^1(\Theta^d)$ is $\mathcal{O}$-torsion free and $\delta(\check{\mathcal{S}}^1(\Theta^d))=\delta(\check{\mathcal{S}}^1(\Theta^d/\varpi))+1=2$. Hence, $\delta(\check{\mathcal{S}}^1(\Pi))=1$. □

Lemma 6.9. *Let B be an admissible unitary $D_{\mathfrak{p}}^\times$-representation. Then B is a finite dimensional L-vector space if and only if its δ-dimension is 0.*

Proof. Let K be an open uniform pro-p subgroup of $D_{\mathfrak{p}}^\times$ and let Θ be an open bounded $D_{\mathfrak{p}}^\times$-invariant lattice in B. Then

$$\delta(\mathrm{B})=\delta(\Theta^d)-1=\delta(\Theta^d/\varpi),$$

and the assertion follows from Lemma 4.3. □

Corollary 6.10. *If Π is as in Corollary 6.8, then $\check{\mathcal{S}}^1(\Pi)$ is of finite length in the category of admissible unitary L-Banach space representations of $D_{\mathfrak{p}}^\times$ if and only if it has finitely many irreducible subquotients which are finite-dimensional as L-vector spaces.*

Proof. Since $\check{\mathcal{S}}^1(\Pi)$ is admissible, it has an irreducible subrepresentation, which we denote by B_1 (see [35, Lemma 5.8]). If $\check{\mathcal{S}}^1(\Pi)$ is not of finite length, then by repeating the argument we obtain an ascending chain of Banach space subrepresentations $\{\mathrm{B}_i\}_{i\geq 0}$ such that $\mathrm{B}_0=0$ and the quotients $\mathrm{B}_{i+1}/\mathrm{B}_i$ for $i\geq 0$ are irreducible. Let K be a compact open subgroup of $D_{\mathfrak{p}}^\times$ without torsion. Then $M_i:=(\check{\mathcal{S}}^1(\Pi)/\mathrm{B}_i)^d$ for $i\geq 0$ is a descending chain of finitely generated modules over a Noetherian Auslander regular ring $\mathcal{O}[[K]]\otimes_{\mathcal{O}}L$. According to [30, Theorem 4.2], there is n_0 such that

$$\delta(M_i/M_{i+1})\leq\delta(M_0)-1=0,\quad \forall i\geq n_0,$$

where the last equality follows from Corollary 6.8. Lemma 6.9 implies that the quotients $\mathrm{B}_{i+1}/\mathrm{B}_i$ for $i\geq n_0$ are finite-dimensional L-vector spaces. This contradicts the assumption that there are only finitely many such subquotients. □

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Theorem 6.11. *Let $x \in \text{m-Spec}\,\mathbb{T}(U^p)_{\mathfrak{m}}[1/p]$ be such that the restriction of the corresponding Galois representation $\rho_x : G_{F,S} \to \mathrm{GL}_2(\kappa(x))$ to $G_{F_{\mathfrak{p}}}$ is irreducible. Then $(\widehat{H}^1_{\psi,\lambda}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} L)[\mathfrak{m}_x]$ is nonzero if and only if $(S_{\psi,\lambda}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} L)[\mathfrak{m}_x]$ is nonzero.*

In this case, there is an isomorphism of admissible unitary $\kappa(x)$-Banach space representations of $G_{F_{\mathfrak{p}}} \times D_{\mathfrak{p}}^{\times}$:

$$(\widehat{H}^1_{\psi,\lambda}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} L)[\mathfrak{m}_x] \cong \check{\mathcal{S}}^1(\Pi)^{\oplus n}, \tag{38}$$

where Π is the absolutely irreducible $\kappa(x)$-Banach space representation corresponding to $\rho_x|_{G_{F_{\mathfrak{p}}}}$ via the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. In particular, the δ-dimension of $(\widehat{H}^1_{\psi,\lambda}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} L)[\mathfrak{m}_x]$ is 1.

Proof. The assumption on $\bar{r}$ implies that (32) is an isomorphism. It follows from Proposition 6.3 that

$$\check{\mathcal{S}}^1((S_{\psi,\lambda}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} L)[\mathfrak{m}_x]) \cong (\widehat{H}^1_{\psi,\lambda}(U^{\mathfrak{p}},\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} L)[\mathfrak{m}_x],$$

as $G_{F_{\mathfrak{p}}} \times D_{\mathfrak{p}}^{\times}$-representations. The isomorphism (38) is obtained by applying $\check{\mathcal{S}}^1$ to (28). The assertion about the dimension follows from Corollary 6.8. □

Remark 6.12. Since $\mathfrak{m}_x$ contains the ideal generated by $S_v - \psi(\mathrm{Frob}_v)$ for all $v \notin S$, Theorem 6.11 implies Theorem 1.4.

Proposition 6.13. *A unitary $D_{\mathfrak{p}}^{\times}$-representation B on a finite-dimensional L-vector space with a central character is semisimple. Moreover,*

$$\mathrm{B} \cong \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} \mathrm{Sym}^{a_i} L^2 \otimes \det{}^{b_i} \otimes \tau_j \otimes \eta_{ij} \circ \mathrm{Nrd},$$

where $a_i \in \mathbb{Z}_{\geq 0}$, $b_i \in \mathbb{Z}$, τ_j is irreducible smooth and $\eta_{ij} : \mathbb{Q}_p^{\times} \to \mathcal{O}^{\times}$ is a character. If the central character ζ_{B} is locally algebraic, then we can take η_{ij} to be either trivial or $\eta_{ij}(x) = \sqrt{\mathrm{pr}(x|x|)}$, where $\mathrm{pr} : \mathbb{Z}_p^{\times} \to 1 + p\mathbb{Z}_p$ denotes the projection.

Proof. We closely follow the proof of [45, Proposition 3.2]. Note first that any continuous action of a p-adic analytic group on a finite-dimensional L-vector space is automatically locally analytic, and hence induces an action of the universal enveloping algebra of its Lie algebra. Let $D_{\mathfrak{p}}^{\times,1} = \{g \in D_{\mathfrak{p}}^{\times} : \mathrm{Nrd}(g) = 1\}$ and let $\mathfrak{h}$ be its Lie algebra. The Lie algebra of $D_{\mathfrak{p}}^{\times}$ is just $D_{\mathfrak{p}}$. Assume L to be sufficiently large, so that $D_{\mathfrak{p}}$ splits over L. Thus we have an isomorphism of L-Lie algebras $\mathfrak{h} \otimes_{\mathbb{Q}_p} L \cong \mathfrak{sl}_2$. This induces an isomorphism of enveloping algebras $U(\mathfrak{h}) \otimes_{\mathbb{Q}_p} L \cong U(\mathfrak{sl}_2)$ and an embedding $D_{\mathfrak{p}}^{\times,1} \hookrightarrow \mathrm{SL}_2(L)$. Let J be the kernel of the map $U(\mathfrak{sl}_2) \to \mathrm{End}_L(\mathrm{B})$. Since B is a finite-dimensional L-vector space, the codimension of J is finite, and as explained in the proof of [45, Proposition 3.2], the algebra $U(\mathfrak{sl}_2)/J$ is semisimple, and evaluation induces an isomorphism of $U(\mathfrak{sl}_2)$-modules:

$$\bigoplus_{a \geq 0} \mathrm{Sym}^a L^2 \otimes_L \mathrm{Hom}_{U(\mathfrak{sl}_2)}(\mathrm{Sym}^a L^2, \mathrm{B}) \xrightarrow{\cong} \mathrm{B}. \tag{39}$$

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We can upgrade this isomorphism to an isomorphism of $D_{\mathfrak{p}}^{\times,1}$-representations as follows. Let $D_{\mathfrak{p}}^{\times,1}$ act on $\mathrm{Sym}^a L^2$ via the embedding $D_{\mathfrak{p}}^{\times,1} \hookrightarrow \mathrm{SL}_2(L)$ defined earlier and on $\mathrm{Hom}_L(\mathrm{Sym}^a L^2, \mathrm{B})$ by conjugation. Since $\mathrm{Hom}_L(\mathrm{Sym}^a L^2, \mathrm{B})$ is a finite-dimensional L-vector space, the corresponding representation is locally analytic. It follows from [45, Proposition 2.1] that the smooth vectors for this action are equal to $\mathrm{Hom}_{U(\mathfrak{sl}_2)}(\mathrm{Sym}^a L^2, \mathrm{B})$, which makes it into a smooth representation of $D_{\mathfrak{p}}^{\times,1}$. If we put the diagonal action of $D_{\mathfrak{p}}^{\times,1}$ on $\mathrm{Sym}^a L^2 \otimes_L \mathrm{Hom}_{U(\mathfrak{sl}_2)}(\mathrm{Sym}^a L^2, \mathrm{B})$, then the evaluation map is $D_{\mathfrak{p}}^{\times,1}$-equivariant and hence the same holds for (39). Since $D_{\mathfrak{p}}^{\times,1}$ is a compact group, the category of smooth representations on L-vector spaces is semisimple. Moreover, if τ is an irreducible smooth representation of $D_{\mathfrak{p}}^{\times,1}$, then $\mathrm{Sym}^a L^2 \otimes \tau$ is irreducible by [45, Proposition 3.4]. Thus we have shown that the restriction of B to $D_{\mathfrak{p}}^{\times,1}$ is semisimple. Since the centre $F_{\mathfrak{p}}^{\times}$ acts by a central character by assumption, the restriction of B to $D_{\mathfrak{p}}^{\times,1} F_{\mathfrak{p}}^{\times}$ is semisimple. Since $D_{\mathfrak{p}}^{\times,1} F_{\mathfrak{p}}^{\times}$ is of finite index in $D_{\mathfrak{p}}^{\times}$, this implies that B is a semisimple representation of $D_{\mathfrak{p}}^{\times}$.

Assume that B is absolutely irreducible, and let a be such that

$$\mathrm{Hom}_{U(\mathfrak{sl}_2)}(\mathrm{Sym}^a L^2, \mathrm{B}) \neq 0.$$

Let $\mathrm{pr} : \mathcal{O}^{\times} \to 1 + \mathfrak{p}$ denote the projection to the principal units. We use the same symbol for $\mathbb{Z}_p^{\times} \to 1 + p\mathbb{Z}_p$. Since $p > 2$, we can define a continuous square root on $1 + \mathfrak{p}$ by the usual binomial formula. Let $\eta : \mathbb{Q}_p^{\times} \to \mathcal{O}^{\times}$ be the character $\eta(x) = \sqrt{\mathrm{pr}(\zeta_{\mathrm{B}}(x))^{-1} \mathrm{pr}(x|x|)^a}$, where ζ_{B} is the central character of B. Then the restriction of the central character of $\mathrm{B} \otimes \eta \circ \mathrm{Nrd}$ to $1 + p\mathbb{Z}_p$ is equal to $x \mapsto x^a$. If H is an open subgroup of $D_{\mathfrak{p}}^{\times,1}$, then $(1 + p\mathbb{Z}_p)H$ is an open subgroup of $D_{\mathfrak{p}}^{\times}$ and so their Lie algebras will coincide. Hence, $\mathrm{Hom}_{U(\mathfrak{gl}_2)}(\mathrm{Sym}^a L^2, \mathrm{B} \otimes \eta \circ \mathrm{Nrd}) \neq 0$, and arguing as before, we conclude that $\mathrm{B} \otimes \eta \circ \mathrm{Nrd} \cong \mathrm{Sym}^a L^2 \otimes \tau$, where τ is a smooth irreducible representation of $D_{\mathfrak{p}}^{\times}$.

If ζ_{B} is locally algebraic, then $\zeta_{\mathrm{B}}(x) = x^c \zeta_{\mathrm{sm}}(x)$ for all $x \in \mathbb{Q}_p^{\times}$, where ζ_{sm} is a smooth character. After possibly twisting B by the character $x \mapsto \sqrt{x|x|}$ (or its inverse), we can assume that $a - c$ is even. Then $\mathrm{Hom}_{U(\mathfrak{gl}_2)}(\mathrm{Sym}^a L^2 \otimes \det^b, \mathrm{B}) \neq 0$, where $a - c = 2b$, and we can conclude as before. □

Remark 6.14. The characters appearing in Proposition 6.13 are not uniquely determined, since we can write the character $\det^a$ as a product of a unitary character $\mathrm{pr}(\det^a)$ and a smooth character $(\det^a \mathrm{pr}(\det^a)^{-1})$.

We say that an irreducible component of a potentially semistable deformation ring is of *discrete series type* if the closed points in the generic fibre corresponding to Galois representations, which do not become crystalline after restricting to the Galois group of an abelian extension, are Zariski dense in that component.

Proposition 6.15. *Assume the setup of Theorem 6.11. Let $\check{\mathcal{S}}^1(\Pi)^{1\text{-alg}}$ be the subset of $D_{\mathfrak{p}}^{\times,1}$-locally algebraic vectors in $\check{\mathcal{S}}^1(\Pi)$, where $D_{\mathfrak{p}}^{\times,1}$ is the subgroup of $D_{\mathfrak{p}}^{\times}$ of elements with reduced norm equal to 1. Then $\check{\mathcal{S}}^1(\Pi)^{1\text{-alg}}$ is a finite-dimensional L-vector space.*

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Moreover, if $\check{\mathcal{S}}^1(\Pi)^{\text{1-alg}}$ is nonzero, then a twist of $\rho_x|_{G_{F_{\mathfrak{p}}}}$ by a character defines a point lying on an irreducible component of discrete series type of some potentially semistable deformation ring of $\bar{r}$.

Proof. After twisting by a character $\chi : (\mathbb{A}_F^f)^\times/F^\times \to 1+\mathfrak{p}$, which is trivial on $U^p \cap (\mathbb{A}_F^f)^\times$, we can assume that the restriction of ψ to $(\mathbb{A}_F^f)^\times \cap U_p$ is locally algebraic.

Let $\sigma := \sigma_{\mathrm{alg}} \otimes \sigma_{\mathrm{sm}}$ be an irreducible locally algebraic representation of $U_{\mathfrak{p}} = \mathcal{O}_{D_{\mathfrak{p}}}^\times$ with central character ζ, where $\sigma_{\mathrm{alg}} = \mathrm{Sym}^b L^2 \otimes \det^a$ and σ_{sm} is smooth. To σ_{sm} one can attach an inertial type $\tau : I_{\mathbb{Q}_p} \to \mathrm{GL}_2(L)$ such that τ extends to a representation of the Weil group $W_{\mathbb{Q}_p}$, the kernel of τ is an open subgroup of $I_{\mathbb{Q}_p}$ and the following holds: if π' is a smooth irreducible representation of $D_{\mathfrak{p}}^\times$, π is a smooth irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ corresponding to π' via the classical Jacquet–Langlands correspondence, and $\mathrm{rec}_p(\pi)$ is the Weil–Deligne representation corresponding to π via the classical local Langlands correspondence, then $\mathrm{Hom}_{U_{\mathfrak{p}}}(\sigma_{\mathrm{sm}}, \pi') \neq 0$ if and only if $\mathrm{rec}_p(\pi)|_{I_{\mathbb{Q}_p}} \cong \tau$ (see [22, Theorem 3.3]). Note that in that case, either π is supercuspidal, τ extends to an irreducible representation of $W_{\mathbb{Q}_p}$ and the monodromy operator of $\mathrm{rec}_p(\pi)$ is zero, or π is special series and the monodromy operator of $\mathrm{rec}_p(\pi)$ is nonzero. Moreover, the inertial type τ determines σ_{sm} up to conjugation by a uniformiser ϖ_D of D.

Let $\mathbf{w} = (1-a, -a-b)$ and let $R_{\bar{r}}^{\square}(\mathbf{w},\tau,\psi)$ be the framed deformation ring of $\bar{r}$ parameterising potentially semistable lifts of $\bar{r}$ which have Hodge–Tate weights equal to $\mathbf{w}$, inertial type τ and determinant $\psi\varepsilon^{-1}$. Let $R_{\bar{r}}^{\square}(\mathbf{w},\tau,\psi)^{\mathrm{ds}}$ be the closure of closed points in the generic fibre which do not correspond to crystabelline representations. In the supercuspidal case, $R_{\bar{r}}^{\square}(\mathbf{w},\tau,\psi)^{\mathrm{ds}}$ and $R_{\bar{r}}^{\square}(\mathbf{w},\tau,\psi)$ coincide. In the special series case, $R_{\bar{r}}^{\square}(\mathbf{w},\tau,\psi)^{\mathrm{ds}}$ is the union of irreducible components of $R_{\bar{r}}^{\square}(\mathbf{w},\tau,\psi)$ of discrete series type, and the potentially crystalline locus has codimension 1. Mapping a deformation to its trace induces a natural map $R_{\mathrm{tr}\,\bar{r}}^{\mathrm{ps}} \to R_{\bar{r}}^{\square}(\mathbf{w},\tau,\psi)^{\mathrm{ds}}$, and we let $R_{\mathrm{tr}\,\bar{r}}^{\mathrm{ps}}(\mathbf{w},\tau,\psi)^{\mathrm{ds}}$ be its image.

We claim that the action of $R_{\mathrm{tr}\,\bar{r}}^{\mathrm{ps}}$ on $\mathrm{Hom}_{U_{\mathfrak{p}}}(\sigma, \widehat{H}_{\psi}^1(U^p,\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} L)$ via the homomorphism $R_{\mathrm{tr}\,\bar{r}}^{\mathrm{ps}} \to \mathbb{T}(U^p)_{\mathfrak{m}}$ factors through the quotient $R_{\mathrm{tr}\,\bar{r}}^{\mathrm{ps}}(\mathbf{w},\tau,\psi)^{\mathrm{ds}}$. It follows from Proposition 6.4 that as a $U_p^{\mathfrak{p}}$-representation, $\mathrm{Hom}_{U_{\mathfrak{p}}}(\sigma, \widehat{H}_{\psi}^1(U^p,\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} L)$ is isomorphic to a finite direct sum of copies of the Banach space $C_{\psi}(U_p^{\mathfrak{p}})$ of continuous functions $f : U_p^{\mathfrak{p}} \to L$, on which the centre acts by ψ and $U_p^{\mathfrak{p}}$ acts by right translations. Since the action of $R_{\mathrm{tr}\,\bar{r}}^{\mathrm{ps}}$ is continuous, it is enough to show that it factors through $R_{\mathrm{tr}\,\bar{r}}^{\mathrm{ps}}(\mathbf{w},\tau,\psi)^{\mathrm{ds}}$ on a dense subspace. For this, choose an open subgroup $V_p^{\mathfrak{p}}$ of $U_p^{\mathfrak{p}}$ such that the restriction of ψ to the centre of $V_p^{\mathfrak{p}}$ is algebraic, and let λ be an algebraic representation of $V_p^{\mathfrak{p}}$ with central character ψ. The union of λ-isotypic subspaces inside $C_{\psi}(U_p^{\mathfrak{p}})$ as a $K_p^{\mathfrak{p}}$-representation taken over all open subgroups $K_p^{\mathfrak{p}}$ of $V_p^{\mathfrak{p}}$ will be dense, since locally constant functions are dense in the space of continuous functions. Thus it is enough to prove that the action of $R_{\mathrm{tr}\,\bar{r}}^{\mathrm{ps}}$ on $\mathrm{Hom}_{K_p^{\mathfrak{p}} U_{\mathfrak{p}}}(\lambda \otimes \sigma, \widehat{H}_{\psi}^1(U^p,\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} L)$ factors through $R_{\mathrm{tr}\,\bar{r}}^{\mathrm{ps}}(\mathbf{w},\tau,\psi)^{\mathrm{ds}}$ for all open subgroups $K_p^{\mathfrak{p}}$. It follows from Emerton's spectral sequence in [19, Corollary 2.2.8] (see [34, Proposition 5.2]) that we have an isomorphism of $\mathbb{T}(U^p)_{\mathfrak{m}}$-modules:

$$\mathrm{Hom}_{U_{\mathfrak{p}}}(\sigma_{\mathrm{sm}}, H^1(X(K_p^{\mathfrak{p}} U_{\mathfrak{p}}), \mathcal{V}_W)_{\mathfrak{m}}[\psi_{\mathrm{sm}}]) \cong \mathrm{Hom}_{K_p^{\mathfrak{p}} U_{\mathfrak{p}}}(\lambda \otimes \sigma, \widehat{H}_{\psi}^1(U^p,\mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} L),$$

where $\mathcal{V}_W$ is a local system on $X(K_p^{\mathfrak{p}}U_{\mathfrak{p}})$ corresponding to the algebraic representation $W := (\lambda\otimes\sigma)^*$, $\psi_{\mathrm{sm}} = \psi\psi_{\mathrm{alg}}^{-1}$ and ψ_{alg} is the central character of $\lambda\otimes\sigma$. The left-hand side of the equation is a semisimple $\mathbb{T}(U^p)_{\mathfrak{m}}[1/p]$-module, and the eigenvalues correspond to an automorphic form (see [19, §3]) satisfying local conditions imposed by σ_{sm} at $\mathfrak{p}$, W at infinite places. The compatibility of local and global Langlands correspondences implies that if ρ_x is the Galois representation attached to such an automorphic form, then $\operatorname{tr}\rho_x|_{G_{F_{\mathfrak{p}}}}$ gives a point in $\operatorname{Spec} R^{\mathrm{ps}}_{\operatorname{tr}\bar{r}}(\mathbf{w},\tau,\psi)^{\mathrm{ds}}$. This finishes the proof of the claim.

Fix $x\in \text{m-Spec}\,\mathbb{T}(U^p)_{\mathfrak{m}}[1/p]$, let y be its image in $\operatorname{Spec} R^{\mathrm{ps}}_{\operatorname{tr}\bar{r}}$ and assume that

$$\operatorname{Hom}_{U(\mathfrak{sl}_2)}(\operatorname{Sym}^b L^2, (\widehat{H}^1_{\psi,\lambda}(U^{\mathfrak{p}},\mathcal{O})\otimes_{\mathcal{O}} L)[\mathfrak{m}_x]) \neq 0.$$

It follows from Proposition 6.13 that

$$\operatorname{Hom}_{U_{\mathfrak{p}}}(\sigma\otimes\eta\circ\operatorname{Nrd}, (\widehat{H}^1_{\psi,\lambda}(U^{\mathfrak{p}},\mathcal{O})\otimes_{\mathcal{O}} L)[\mathfrak{m}_x]) \neq 0,$$

where σ is as before and η is either trivial or $\sqrt{\operatorname{pr}(\chi_{\mathrm{cyc}})}$. If η is trivial, then the claim implies that y lies in $\operatorname{Spec} R^{\square}_{\bar{r}}(\mathbf{w},\tau,\psi)^{\mathrm{ds}}$. Note that the assumption on $\bar{r}$ implies that any reducible potentially semistable lift r with Hodge–Tate weights $\mathbf{w}$ is crystabelline and its $\mathrm{WD}(r)|_{W_{\mathbb{Q}_p}}$ is a direct sum of distinct characters. Such representations cannot correspond to points on irreducible components of discrete series type. Hence $\rho_x|_{G_{F_{\mathfrak{p}}}}$ is irreducible and thus determined by its trace.

If η is not trivial, then by twisting by its inverse and using the claim again we deduce that $\operatorname{tr}(\rho_x|_{G_{F_{\mathfrak{p}}}}\otimes\eta^{-1})$ gives a point in $\text{m-Spec}\, R^{\square}_{\bar{r}}(\mathbf{w},\tau,\psi\eta^{-2})^{\mathrm{ds}}[1/p]$. Note that the character $\psi\eta^{-2} = \psi\operatorname{pr}(\chi_{\mathrm{cyc}})^{-1}$ is locally algebraic, but the representation $\lambda\otimes\eta^{-1}\circ\operatorname{Nrd}$ is not. That is why we cannot appeal directly to the results of Emerton in this case. Thus, $\rho_x|_{G_{F_{\mathfrak{p}}}}$ determines the integers a and b, the representation σ_{sm} up to its conjugate by the uniformiser ϖ_D of $D_{\mathfrak{p}}^{\times}$ and whether η is trivial or not, by comparing whether the Hodge–Tate weight of $\det\rho_x|_{G_{F_{\mathfrak{p}}}}$ has the same parity as the Hodge–Tate weight of $\psi|_{G_{F_{\mathfrak{p}}}}$. This implies that $\check{\mathcal{S}}^1(\Pi)^{1\text{-alg}}$ is isomorphic as a representation of $U_{\mathfrak{p}}$ to a finite direct sum of copies of $\sigma\otimes\eta\circ\operatorname{Nrd}$ or its conjugate by ϖ_D. In particular, $\check{\mathcal{S}}^1(\Pi)^{1\text{-alg}}$ is finite-dimensional. $\square$

Remark 6.16. If $F=\mathbb{Q}$, then the proof of Proposition 6.15 can be simplified, since after twisting we can directly appeal to the results of Emerton on locally algebraic vectors in completed cohomology.

Theorem 6.17. *Assume the setup of Theorem 6.11 and the notation of Proposition 6.15. The quotient $\check{\mathcal{S}}^1(\Pi)/\check{\mathcal{S}}^1(\Pi)^{1\text{-alg}}$ contains an irreducible closed subrepresentation of δ-dimension 1.*

Proof. Since the δ-dimension of $\check{\mathcal{S}}^1(\Pi)$ is 1 by Proposition 6.8 and the δ-dimension of $\check{\mathcal{S}}^1(\Pi)^{1\text{-alg}}$ is 0 by Proposition 6.15 and Lemma 6.9, the quotient $\check{\mathcal{S}}^1(\Pi)/\check{\mathcal{S}}^1(\Pi)^{1\text{-alg}}$ is nonzero and has δ-dimension 1. Since it is admissible, it will contain an irreducible subrepresentation B. We have $\delta(\mathrm{B})\leq 1$. If $\delta(\mathrm{B})=1$, then we are done; otherwise, $\delta(\mathrm{B})=0$ and so B is a finite-dimensional L-vector space by Lemma 6.9. The extension of $\check{\mathcal{S}}^1(\Pi)^{1\text{-alg}}$ by B inside $\check{\mathcal{S}}^1(\Pi)$ is a finite-dimensional L-vector space, and Proposition 6.13 implies

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that the action of $D_{\mathfrak{p}}^{\times,1}$ on it is locally algebraic. Since $\check{\mathcal{S}}^1(\Pi)^{1\text{-alg}}$ is the maximal subspace of $\check{\mathcal{S}}^1(\Pi)$ with this property, we conclude that $\mathrm{B} = 0$. □

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Addendum Gabriel Dospinescu has pointed out to us that the claim in the proof of Proposition 6.7 that both $\pi_1^\vee$ and $\pi_2^\vee$ occur as subquotients of $k \otimes_A S_{\psi,\lambda}(U^{\mathfrak{p}}, \mathcal{O})^d_{\mathfrak{m}}$ has not been justified. Contrary to the assertion made in the proof of Proposition 6.7 the proof of Proposition 5.17 does not contain the proof of the claim. This can be fixed as follows.

Let $M := S_{\psi,\lambda}(U^{\mathfrak{p}}, \mathcal{O})^d_{\mathfrak{m}}$. If $M \otimes_A k$ contains $\pi_1^\vee$, but not $\pi_2^\vee$, then it follows from [37, Corollary 7.7] that $M \cong (\mathrm{Ind}_B^G \mathrm{Ord}_B(M^\vee))^\vee$. This would contradict the projectivity of M in $\mathrm{Mod}^{\mathrm{pro}}_{K,\zeta}(\mathcal{O})$, for example by looking at the growth of co-invariants by congruence subgroups of K.

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