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Refined Bohr inequalities for certain classes of functions: analytic, univalent, and convex

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Abstract. In this article, we prove several refined versions of the classical Bohr inequality for the class of analytic self-mappings on the unit disk $\mathbb{D}$, class of analytic functions f defined on $\mathbb{D}$ such that $\operatorname{Re}(f(z)) < 1$, and class of subordination to a function g in $\mathbb{D}$. Consequently, the main results of this article are established as certainly improved versions of several existing results. All the results are proved to be sharp.

1 Introduction

Bohr's remarkable work in the year 1914 on the power series in complex analysis has generated a lot of research activities in complex analysis and related areas. This work is popularly referred to as Bohr phenomenon. The phenomenon has been investigated in various function spaces. Throughout the article, we denote $\mathcal{A}$ be the class of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defined on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. We define two subclasses $\mathcal{B} := \{f \in \mathcal{A} : |f(z)| \leq 1\}$ and $\mathcal{P} := \{f \in \mathcal{A} : \operatorname{Re}(f(z)) < 1\}$.

In 1914, the classical result related to the family $\mathcal{B}$ was discovered by Bohr [18] as follows.

Theorem 1.1 [18] *If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$, then*

$$M_f(r) := \sum_{n=0}^{\infty} |a_n| r^n \leq 1 \ \text{for} \ |z| = r \leq \frac{1}{3}. \tag{1.1}$$

Bohr initially, shows the inequality (1.1) for $|z| = r \leq 1/6$. Subsequently, the inequality (1.1) was improved for $|z| \leq 1/3$ by M. Riesz, I. Schur, and F. Wiener independently and they showed that the constant $1/3$ cannot be improved. The constant $1/3$ and the inequality (1.1) are called, respectively, the Bohr radius and the Bohr inequality for the class $\mathcal{B}$. Moreover, for

$$f_a(z) := \frac{a - z}{1 - az}, \quad a \in [0, 1),$$

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it can be easily shown that $M_{f_a}(r) > 1$ if, and only if, $r > 1/(1+2a)$, and it is easy to see $a \to 1^-$ suggests that the constant $1/3$ is best possible.

Initially, the problem was considered by Harald Bohr while working on the absolute convergence of the Dirichlet series of the form $\sum a_n n^{-s}$, but in recent years, it becomes an active area of research in modern function theory. In fact, this theorem gets much attention as it has been applied to the characterization problem of Banach algebra satisfying the von Neumann inequality [21]. However, several other proofs of this interesting inequality were given in different articles (see [36, 41, 42]). In 1997, Boas and Khavinson [17] first introduced the concept of the n-dimensional Bohr radius $\mathcal{K}_n^\infty$ and established the result: for every $n \in \mathbb{N}$, the n-dimensional Bohr radius $\mathcal{K}_n^\infty$ with $n \geq 2$ satisfies

$$\frac{1}{3\sqrt{n}} < \mathcal{K}_n^\infty < 2\sqrt{\frac{\log n}{n}}.$$

Such study concerning multidimensional Bohr inequality, by Boas and Khavinsion, was an incentive for many mathematicians to link the asymptotic behavior of $\mathcal{K}_n^\infty$ to various problems in analysis, for instance, geometry of Banach spaces, unconditional basic constant for spaces of polynomials, etc.

In the majorant series $\sum_{n=0}^\infty |a_n| r^n$ of the function $f \in \mathcal{B}$, the beginning terms play some significant role in the related discussion about the Bohr inequality. For instance, in the case of $|a_0| = 0$, Tomic [42] has proved the inequality (1.1) for $0 \leq r \leq 1/2$ and if the term $|a_0|$ is replaced by $|a_0|^2$, then the constant $1/3$ can be replaced by $1/2$. In addition, if $|a_0|$ is replaced by $|f(z)|$, then the constant $1/3$ can be replaced by $\sqrt{5}-2$ which is best possible (see [28, 28]).

An extension of the Bohr inequality has established by Paulsen *et al.* [36] and the following sharp inequality for the class $\mathcal{P}$ are obtained.

Theorem 1.2 [36] *Suppose that $f(z) = \sum_{n=0}^\infty a_n z^n \in \mathcal{P}$. Then the following sharp inequality holds:*

$$\sum_{n=0}^\infty |a_n| r^n \leq 1 \ \textit{for} \ \ r \leq \frac{1}{3}.$$

It is generally known that the Bohr radius $1/3$ remains valid in Theorem 1.1 even if the condition $\mathrm{Re}\,(f(z)) < 1$ in $\mathbb{D}$ and $a_0 = f(0) \in [0,1)$ are substituted for the assumption on f. Actually, this condition shows that (see [22, Carathéodory's Lemma, p. 41]) the coefficient bounds as $|a_n| \leq 2(1-a_0)$ for all $n \geq 1$ and with this, the following sharp inequality holds (see [36]):

$$a_0 + \sum_{n=1}^\infty |a_n| r^n \leq a_0 + 2(1-a_0)\frac{r}{1-r} \leq 1 \text{ for } r \leq \frac{1}{3}.$$

For different aspects and some recent developments of the Bohr phenomenon for different classes of functions, we may refer to the articles [4–15, 21, 23, 30, 31, 35] and the references therein.

The exploration of the Bohr radius problem for subordinating families of analytic functions in $\mathbb{D}$ adds another interesting layer to the overall understanding of the Bohr phenomenon. For any two analytic functions f and g in $\mathbb{D}$, the function f is said to be subordinate to g, denoted by $f \prec g$ if there exist an $\omega \in \mathcal{B}$ satisfying $\omega(0) = 0$ and $f(z) = g(\omega(z))$ in $z \in \mathbb{D}$ (see [22]).

Throughout the article, we denote $S(g)$, the class of functions f subordinate to a function g. Many authors have studied Bohr phenomenon for functions in the class $S(g)$, where the function g belongs to different class (see [1, 16, 24, 37]).

Our primary interest in this article is to establish sharp refined versions of Bohr-type inequalities for different classes of functions such as $\mathcal{B}$, $\mathcal{P}$, and $\mathcal{S}(g)$, where g is a given function defined in $\mathbb{D}$. The article is organized as follows: In Section 2, we establish a sharp Bohr-type inequalities for the class $\mathcal{B}$ with help of the planar integral S_r. In Section 3, we study refined version of Bohr-type inequality considering certain power of the initial coefficient for functions of the class $\mathcal{P}$. Finally, in Section 4, we prove two results concerning refined versions of the Bohr inequality for the class $S(g)$.

2 Bohr-type inequalities for the class $\mathcal{B}$

Inspired by the notion of Rogosinski's inequality and Rogosinski's radius investigated in [38], Kayumov *et al.* [28] (see also [28]) obtained the following Bohr–Rogosinski inequality and Bohr–Rogosinski radius for the class $\mathcal{B}$.

Theorem 2.1 [28] *Suppose that* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$. *Then*

$$|f(z)| + \sum_{n=N}^{\infty} |a_n| r^n \le 1 \ \textit{for} \ r \le R_N,$$

where R_N *is the unique root of the equation* $2(1+r)r^N - (1-r)^2 = 0$ *in the interval* $(0,1)$. *The radius* R_N *is the best possible. Moreover,*

$$|f(z)|^2 + \sum_{n=N}^{\infty} |a_n| r^n \le 1 \ \textit{for} \ r \le R'_N,$$

where R'_N *is the unique root of the equation* $(1+r)r^N - (1-r)^2 = 0$ *in the interval* $(0,1)$. *The radius* R'_N *is the best possible.*

In a number of articles (see [20, 32, 33] and the references therein) have further refined and sometimes improved Theorem 2.1 for different classes of functions. Our objective in this section is to improve Theorem 2.1. In order to do that, let us recall some well-known formulations. Let f be holomorphic in $\mathbb{D}$, and for $0 < r < 1$, let $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$. Throughout the article, $S_r = S_r(f)$ denotes the planar integral

$$S_r = \int_{\mathbb{D}_r} |f'(z)|^2 dA(z).$$

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, than the quantity S_r has the following series representation:

$$S_r = \pi \sum_{n=1}^{\infty} n |a_n|^2 r^{2n}.$$

In fact, if f is a univalent function, then S_r is the area of $f(\mathbb{D}_r)$. The quantity S_r has a certain significant role in the study of improved versions of Bohr-type inequalities (see, e.g., [7, 25, 27, 32]). For example, Kayumov and Ponnusamy [29] have proved the following improved version of the Bohr inequality in terms of S_r.

Theorem 2.2 [29] *Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$. Then*

$$\sum_{n=0}^{\infty} |a_n| r^n + \frac{16}{9}\left(\frac{S_r}{\pi}\right) \leq 1 \quad \textit{for} \quad r \leq \frac{1}{3}$$

and the numbers 1/3, 16/9 *cannot be improved. Moreover,*

$$|a_0|^2 + \sum_{n=1}^{\infty} |a_n| r^n + \frac{9}{8}\left(\frac{S_r}{\pi}\right) \leq 1 \quad \textit{for} \quad r \leq \frac{1}{2}$$

and the numbers 1/2, 9/8 *cannot be improved.*

In fact, Kayumov and Ponnusamy [28] have proved the following sharp inequality for the function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$:

$$\frac{S_r}{\pi} = \sum_{n=1}^{\infty} n|a_n|^2 r^{2n} \leq r^2 \frac{(1-|a_0|^2)^2}{(1-|a_0|^2 r^2)^2} \text{ for } 0 < r \leq 1/\sqrt{2}. \tag{2.1}$$

Furthermore, Ismagilov *et al.* [26] have observed that

$$1 - \frac{S_r}{\pi} \geq \frac{(1-r^2)(1-r^2|a_0|^4)}{(1-|a_0|^2 r^2)^2}, \tag{2.2}$$

and hence, in view of (2.1) and (2.2), the following inequality can be obtained easily:

$$\frac{S_r}{\pi - S_r} \leq \frac{r^2(1-|a_0|^2)^2}{(1-r^2)(1-r^2|a_0|^4)}. \tag{2.3}$$

In view of the upper bound of the quantity $S_r/(\pi - S_r)$, Theorem 2.2 is investigated further in [26] and obtained the following sharp result replacing the quantity S_r/π by $S_r/(\pi - S_r)$.

Theorem 2.3 [26] *Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$. Then*

$$\sum_{n=0}^{\infty} |a_n| r^n + \frac{16}{9}\left(\frac{S_r}{\pi - S_r}\right) \leq 1 \quad \textit{for} \quad r \leq \frac{1}{3}$$

and the number 16/9 *cannot be improved. Moreover,*

$$|a_0|^2 + \sum_{n=1}^{\infty} |a_n| r^n + \frac{9}{8}\left(\frac{S_r}{\pi - S_r}\right) \leq 1 \quad \textit{for} \quad r \leq \frac{1}{2}$$

and the number 9/8 *cannot be improved.*

For recent development of Bohr-type inequalities with the quantity $S_r/(\pi - S_r)$, we may refer to the article [3]. In fact, there is an ongoing research effort aimed at refining the Bohr inequality for the class $\mathcal{B}$, with the objective of finding sharp results. In this context, recently, Liu *et al.* [32] have studied refined Bohr inequality with various suitable settings and obtained the following result.

Theorem 2.4 [32] *Suppose that* $f = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$. *Then*

$$|f(z)| + \sum_{n=1}^{\infty} |a_n| r^n + \left(\frac{1}{1+|a_0|} + \frac{r}{1-r} \right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \le 1$$

for $|z| = r \le r_0 = 2/\left(3 + |a_0| + \sqrt{5}(1+|a_0|)\right)$. *The radius* r_0 *is best possible and* $r_0 > \sqrt{5} - 2$. *Moreover,*

$$|f(z)|^2 + \sum_{n=1}^{\infty} |a_n| r^n + \left(\frac{1}{1+|a_0|} + \frac{r}{1-r} \right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \le 1 \tag{2.4}$$

for $|z| = r \le r_0'$, *where* r_0' *is the unique positive root of the equation*

$$(1 - |a_0|^3) r^3 - (1 + 2|a_0|) r^2 - 2r + 1 = 0.$$

The radius r_0' *is best possible and* $1/3 < r_0' < 1/(2 + |a_0|)$.

The above discussion motivates us to pose the following question in order to continue the study on the Bohr phenomenon for further improvement.

Question 2.1 Can we improve the inequality (2.4) in view of incorporating the nonnegative quantities S_r/π and $S_r/(\pi - S_r)$?

We have utilized the proof of techniques discussed in [3, 26, 29] as inspiration to derive the following refined Bohr inequalities to answer the Question 2.1.

Theorem 2.5 *Suppose that* $f = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$. *Then*

$$U_1(z,r) := |f(z)|^2 + \sum_{n=1}^{\infty} |a_n| r^n + \left(\frac{1}{1+|a_0|} + \frac{r}{1-r} \right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} + \lambda \left(\frac{S_r}{\pi} \right) \le 1 \tag{2.5}$$

for $r \le 1/3$, *and the constant* $\lambda = 8/9$ *cannot be improved. Moreover,*

$$U_2(z,r) := |f(z)|^2 + \sum_{n=1}^{\infty} |a_n| r^n + \left(\frac{1}{1+|a_0|} + \frac{r}{1-r} \right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} + \lambda \left(\frac{S_r}{\pi - S_r} \right) \le 1 \tag{2.6}$$

for $r \le 1/3$, *and the constant* $\lambda = 8/9$ *cannot be improved.*

Remark 2.1 The following observations are clear.

(i) If $\lambda = 0$, then the inequalities (2.5) and (2.6) coincide with the inequality (2.4), which shows that Theorem 2.5 improves Theorem 2.4 by improving (2.4).

(ii) The extension of the inequality (2.4) in two distinct cases is achieved by Theorem 2.5, which improves Theorem 2.4 for $\lambda = 8/9$.

The proof of Theorem 2.5 relies heavily on the following two lemmas. The first lemma is established in [32] as a further refinement of Bohr-type inequalities, whereas the second lemma is well-known as "Schwarz–Pick lemma" used for the bounds of analytic function f and their first-order derivatives.

Lemma 2.1 [32] *Suppose that* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$. *Then, for any* $N \in \mathbb{N}$, *the following inequality holds:*

$$\sum_{n=N}^{\infty} |a_n| r^n + sgn(t) \sum_{n=1}^{t} |a_n|^2 \frac{r^N}{1-r} + \left(\frac{1}{1+|a_0|} + \frac{r}{1-r} \right) \sum_{n=t+1}^{\infty} |a_n|^2 r^{2n} \le \frac{(1-|a_0|^2) r^N}{1-r}$$

for $r \in [0,1)$, *where* $t = \lfloor (N-1)/2 \rfloor$.

Lemma 2.2 [40] *(Schwarz–Pick lemma) Suppose that* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$. *Then:*

(i)

$$\frac{|f(z_1) - f(z_2)|}{|1 - \overline{f(z_1)} f(z_2)|} \le \frac{|z_1 - z_2|}{|1 - \overline{z_1} z_2|} \quad \textit{holds for } z_1, z_2 \in \mathbb{D},$$

and the equality sign holds for distinct $z_1, z_2 \in \mathbb{D}$ *if, and only if,* f *is a Möbius transformation.*

(ii)

$$|f'(z)| \le \frac{1 - |f(z)|^2}{1 - |z|^2} \quad \textit{holds for } z \in \mathbb{D},$$

and the equality sign holds for some $z \in \mathbb{D}$ *if, and only if,* f *is a Möbius transformation.*

Proof of Theorem 2.5 We consider the function $U_1(z, r)$ which is given by (2.5). Moreover, by applying the assumption of the result and the Schwarz–Pick lemma to the function f, it is easily obtain

$$|f(z)| \le \frac{r + |a_0|}{1 + r|a_0|} \text{ for } |z| \le r. \tag{2.7}$$

By means of a basic computation involving the inequalities (2.1), (2.7), and Lemma 2.1 (with $N = 1$), it can be shown that

$$U_1(z, r) \le \left(\frac{r + |a_0|}{1 + r|a_0|} \right)^2 + \frac{(1 - |a_0|^2) r}{1 - r} + \frac{8(1 - |a_0|^2)^2 r^2}{9(1 - |a_0|^2 r^2)^2} := U_1^*(r).$$

It is easy to see that $U_1^*(r)$ is a monotonically increasing function of r, and therefore, it suffices to prove the inequality (2.5) for $r = 1/3$. For $r = 1/3$, it can be seen through a simple calculation that

$$U_1^*(1/3) = 1 - \frac{(1 - |a_0|)^2 (1 + |a_0|)(47 - 2|a_0| - |a_0|^2)}{2(9 - |a_0|^2)^2} \le 1,$$

and, therefore, the desired inequality (2.5) holds for $r \le 1/3$ and $|a_0| \in [0, 1)$. Next, we consider the function $U_2(z, r)$ which is given by (2.6). Using the inequalities (2.3) and (2.7), and in view of Lemma 2.1 (with $N = 1$), we obtain

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$$U_2(z,r) \le \left(\frac{r+|a_0|}{1+r|a_0|}\right)^2 + \frac{(1-|a_0|^2)r}{1-r} + \frac{8(1-|a_0|^2)^2r^2}{9(1-r^2)(1-r^2|a_0|^4)} := U_2^*(r).$$

It suffices to show the inequality (2.6) for $r = 1/3$ because $U_2^*(r)$ is an increasing function of r. For $r = 1/3$, by an easy computation, we obtain

$$U_2^*(1/3) = 1 - \frac{(1-|a_0|)^2(1+|a_0|)(45+24|a_0|+10|a_0|^2+8|a_0|^3+|a_0|^4)}{2(9-|a_0|^4)(3+|a_0|)^2} \le 1$$

and, therefore, the desire inequality (2.6) holds for $r \le 1/3$ and $|a_0| \in [0,1)$. In both the cases, the sharpness of the constant $\lambda = 8/9$ can be easily shown with the help of the function $f_a(z) = (a-z)/(1-az)$, $a \in (0,1)$ and hence, we omit the details. ■

3 Bohr-type inequalities for the class $\mathcal{P}$

In geometric function theory, the subclass of the well-known class $\mathcal{S}$ that encompasses convex functions and starlike functions are named as $\mathcal{C}$ and $\mathcal{S}^*$, respectively. Closely related to the classes $\mathcal{C}$ and $\mathcal{S}^*$ is the class $\mathcal{P}$ of all functions φ analytic and having positive real part in $\mathbb{D}$, with $\varphi(0) = 1$ and the function φ can be represented as Poisson–Stieltjes integral

$$\varphi(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t),$$

where $d\mu(t) \ge 0$ and $\int d\mu(t) = 1$.

In 2022, Ponnusamy *et al.* [37] further examined the Bohr inequality by replacing the unimodular boundedness of f by the condition $\operatorname{Re}(f(z)) < 1$ and established the following sharp result.

Theorem 3.1 [37] *Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{P}$. Then*

$$\sum_{n=0}^{\infty} |a_n| r^n + \left(\frac{1}{1+|a_0|} + \frac{r}{1-r}\right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \le 1 \tag{3.1}$$

for $r \le r_*$, where $r_* \approx 0.24683$ is the unique root of the equation $3r^3 - 5r^2 - 3r + 1 = 0$ in the interval $(0,1)$. Moreover, for any $a_0 \in (0,1)$, there exists a uniquely defined $r_0 = r_0(a_0) \in (r_*, 1/3)$ such that

$$\sum_{n=0}^{\infty} |a_n| r^n + \left(\frac{1}{1+|a_0|} + \frac{r}{1-r}\right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \le 1 \quad \text{for} \quad r \in [0, r_0].$$

The radius $r_0 = r_0(a_0)$ can be computed as the solution of the equation

$$4r^3d^2 - (7r^3+3r^2-3r+1)d + 6r^3 - 2r^2 - 6r + 2 = 0, \quad \text{where} \quad d = 1 - a_0.$$

The result is sharp.

Since the initial coefficient $|a_0|$ in the majorant series $M_f(r) = \sum_{n=0}^{\infty} |a_n| r^n$, where $f \in \mathcal{B}$, plays some significant role in the study of Bohr radius, to determine the value corresponding to r_*, we are interested to obtain sharp version of the inequality (3.1)

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by considering general power of $|a_0|$. Henceforth, in order to generalize Theorem 3.1, it is natural to find the answer to the following question.

Question 3.1 Can we prove the inequality (3.1) sharp for any $p \in \mathbb{N}$ if the initial term $|a_0|$ is replaced by $|a_0|^p$?

We obtain the following result answering the Question 3.1.

Theorem 3.2 *Let $f(z)$ be analytic function in $\mathbb{D}$ such that $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{P}$. Then, for any $p \in \mathbb{N}$, the following inequality holds:*

$$V_f(r) := |a_0|^p + \sum_{n=1}^{\infty} |a_n| r^n + \left(\frac{1}{1+|a_0|} + \frac{r}{1-r} \right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \leq 1 \tag{3.2}$$

for $r \leq r_*$, where $r_* \approx 0.24683$ is the unique root of the equation

$$1 - 3r - 5r^2 + 3r^3 = 0$$

in the interval $(0,1)$. The result is sharp.

We have the following remark.

Remark 3.1 In Theorem 3.2, the following observations are clear.

(i) Clearly, if $p = 1$, then Theorem 3.2 reduces to the first part of Theorem 3.1.
(ii) Surprisingly, it is worth noticing that the constant r_*, which is a root of the equation $1 - 3r - 5r^2 + 3r^3 = 0$ is independent of the choice of p.

Proof of Theorem 3.2 To prove this result, we use the following lemma involving subordination.

Lemma 3.1 [37] *Let $f(z)$ and $g(z)$ be two analytic functions in $\mathbb{D}$ with the Taylor series expansions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ for $z \in \mathbb{D}$. If $f(z) \prec g(z)$, then*

$$\sum_{n=0}^{\infty} |a_n| r^n + \left(\frac{1}{1+|a_0|} + \frac{r}{1-r} \right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \leq \sum_{n=0}^{\infty} |b_n| r^n + \left(\frac{1}{1+|b_0|} + \frac{r}{1-r} \right) \sum_{n=1}^{\infty} |b_n|^2 r^{2n}$$

for $r \leq 1/3$. The number $1/3$ cannot be improved.

Since $\operatorname{Re}(f(z)) < 1$, we may write the given condition as

$$f(z) \prec g(z), \quad g(z) = a_0 - 2(1-a_0)\frac{z}{1-z} = a_0 - 2(1-a_0) \sum_{n=1}^{\infty} z^n.$$

Here, $g(z)$ is a univalent function in $\mathbb{D}$ onto the half-plane $\{w : \operatorname{Re}(w) < 1\}$. According to the Lemma 3.1, if $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then it is sufficient to show that

$$V_g(r) = |b_0|^p + \sum_{n=1}^{\infty} |b_n| r^n + \left(\frac{1}{1+|b_0|} + \frac{r}{1-r} \right) \sum_{n=1}^{\infty} |b_n|^2 r^{2n} \leq 1 \ \text{ for } \ r \leq r_*,$$

where r_* is as in the statement. For simplicity, let $1 - a_0 = \mu$ so that $a_0 = 1 - \mu$ and $b_n = -2\mu$ for $n \geq 1$. This gives for $\mu \in [0,1]$ and $r \in (0,1)$ that

$$\begin{aligned} V_g(r) &= (1-\mu)^p + 2\mu \sum_{n=1}^{\infty} r^n + \left(\frac{1}{2-\mu} + \frac{r}{1-r} \right) 4\mu^2 \sum_{n=1}^{\infty} r^{2n} \\ &= 1 - \left(1 - (1-\mu)^p - 2\mu \frac{r}{1-r} - \frac{(1+r-r\mu)4\mu^2 r^2}{(2-\mu)(1-r)(1-r^2)} \right) \\ &= 1 - \mu \left(1 + (1-\mu) + (1-\mu)^2 + (1-\mu)^3 + \cdots + (1-\mu)^{p-1} - 2\frac{r}{1-r} \right) \\ &\quad + \frac{(1+r-r\mu)4\mu^2 r^2}{(2-\mu)(1-r)(1-r^2)} \\ &= 1 - \frac{\mu \Psi(\mu,r)}{(2-\mu)(1-r)(1-r^2)}, \end{aligned}$$

where

$$\begin{aligned} \Psi(\mu,r) := {} & (1-r)(1-r^2)(2-\mu)\left(1 + (1-\mu) + (1-\mu)^2 + (1-\mu)^3 + \cdots + (1-\mu)^{p-1}\right) \\ & - 2r(1-r^2)(2-\mu) - 4\mu(1+r-r\mu)r^2. \end{aligned}$$

We claim that $\Psi(\mu,r) \geq 0$ for every $r \leq r_*$ and $\mu \in (0,1]$. For simplicity, set $R(r) = (1-r)(1-r^2)$ and by a straightforward calculations, we obtain

$$\begin{aligned} \frac{\partial \Psi(\mu,r)}{\partial \mu} = {} & -\left(1 + (1-\mu) + (1-\mu)^2 + (1-\mu)^3 + (1-\mu)^4 + \cdots + (1-\mu)^{p-1}\right) R(r) \\ & + (2-\mu)R(r)\left(-1 - 2(1-\mu) - 3(1-\mu)^2 - \cdots - (p-1)(1-\mu)^{p-2}\right) \\ & + 2r(1-r^2) - 4r^2(1+r) + 8r^3\mu, \\ \frac{\partial^2 \Psi(\mu,r)}{\partial \mu^2} = {} & 2\left(1 + 2(1-\mu) + 3(1-\mu)^2 + \cdots + (p-1)(1-\mu)^{p-2}\right) R(r) \\ & + (2-\mu)R(r)\left(2 + 6(1-\mu) + 12(1-\mu)^2 \cdots + (p-1)(p-2)(1-\mu)^{p-3}\right) \\ & + 8r^3, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^3 \Psi(\mu,r)}{\partial \mu^3} = {} & -3R(r)\left(2 + 6(1-\mu) + \cdots + (p-1)(p-2)(1-\mu)^{p-3}\right) \\ & - R(r)(2-\mu)\left(6 + 24(1-\mu) + \ldots (p-1)(p-2)(p-3)(1-\mu)^{p-4}\right) \\ \leq {} & 0 \quad \text{for all} \quad \mu \in (0,1]. \end{aligned}$$

Thus, it is easy to see that $\dfrac{\partial^2 \Psi(\mu,r)}{\partial \mu^2}$ is a decreasing function of μ in $(0,1]$, and hence,

$$\frac{\partial^2 \Psi(\mu,r)}{\partial \mu^2} \geq \frac{\partial^2 \Psi(1,r)}{\partial \mu^2} = 4R(r) + 8r^3 \geq 0 \quad \text{for all} \quad r \in [0,1).$$

This shows that $\dfrac{\partial \Psi(\mu, r)}{\partial \mu}$ is an increasing function of μ in $(0,1]$. Therefore, we obtain

$$\frac{\partial \Psi(\mu, r)}{\partial \mu} \leq \frac{\partial \Psi(1, r)}{\partial \mu} = -2 + 4r - 2r^2 \leq 0 \quad \text{for all} \quad r \in [0,1).$$

Clearly,

$$\Psi(\mu, r) \geq \Psi(1, r) = 1 - 3r - 5r^2 + 3r^3 \geq 0 \quad \text{for all} \quad r \leq r_*,$$

where r_* is given as in the statement of the theorem, and hence, the proof is complete. ■

4 Bohr phenomenon for the class of subordinations

Studying the Bohr inequality for the class of subordination is an interesting and difficult exercise. In recent years, researchers are studying the Bohr phenomenon for such classes of functions and till date, the study continues. In [1], Abu-Muhanna showed the following sharp Bohr phenomenon for the subordinate class $S(g)$.

Theorem 4.1 [1] *Suppose g is a univalent function in $\mathbb{D}$ and $f \in S(g)$ such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then*

$$\sum_{n=1}^{\infty} |a_n| r^n \leq d \quad \text{for} \quad |z| = r \leq r_0 = 3 - \sqrt{8}.$$

The radius r_0 is sharp for the Koebe function $f(z) = z/(1-z)^2$.

In [2], Abu-Muhanna *et al.* established the results determining the Bohr radius for subordinating families of analytic functions and bounded harmonic mappings. In [34], Lie *et al.* studied two refined versions of the Bohr inequality and determine the Bohr radius for the derivatives of analytic functions associated with quasi-subordination. Bohr's phenomenon for analytic functions subordinate to starlike or convex function is investigated by Hamada in [24]. Recently, Ponnusamy *et al.* [37] obtained the following refined versions of the Bohr-type inequalities.

Theorem 4.2 [37] *Suppose that g is a univalent function in $\mathbb{D}$ and $f \in S(g)$ such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then*

$$\sum_{n=1}^{\infty} |a_n| r^n + \left(\frac{1}{2-d} + \frac{r}{1-r}\right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \leq d \tag{4.1}$$

for $|z| = r < r'$, where $r' \approx 0.128445$ is the unique root of the equation

$$(1 - 6r + r^2)(1-r)^2(1+r)^3 - 16r^2(1+r^2) = 0$$

in the interval $(0,1)$, where $d = \operatorname{dist}(g(0), \partial g(\mathbb{D})) < 1$. The sharpness of r' is shown by the Koebe function $f(z) = z/(1-z)^2$.

Theorem 4.3 [37] *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and g be analytic in $\mathbb{D}$ such that g is univalent and convex in $\mathbb{D}$. Assume that $f \in S(g)$ and $d = \operatorname{dist}(g(0), \partial g(\mathbb{D})) \leq 1$. Then*

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the inequality (4.1) holds for all $r \leq r_*$, where $r_* \approx 0.24683$ is the unique root of the equation $3r^3 - 5r^2 - 3r + 1 = 0$ in the interval $(0, 1)$. Moreover, for any $d \in (0, 1)$, there exists a uniquely defined $r_0(a_0) \in (r_*, 1/3)$ such that the inequality (4.1) holds for all $r \in [0, r_0(a_0)]$. The radius $r_0(a_0)$ can be calculated as the solution of the equation $4r^3 d^2 - (7r^3 + 3r^2 - 3r + 1)d + 6r^3 - 2r^2 - 6r + 2 = 0$.

In the study of refined Bohr-type inequalities, finding different versions of refined Bohr-type inequalities and their sharpness is interesting in geometric function theory. Our aim in this section is to determine refined versions of Bohr-type inequalities for functions in the class $S(g)$. We now mention here refined versions of Bohr-type inequalities obtained recently by Liu *et al.* [32] for functions in the class $\mathcal{B}$.

Theorem 4.4 [32] *Suppose that $f = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$. Then*

$$\sum_{n=0}^{\infty} |a_n| r^n + \left(\frac{1}{1 + |a_0|} + \frac{r}{1 - r} \right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} + |f(z) - a_0| \leq 1 \ \text{for} \ |z| = r \leq \frac{1}{5}$$

and the number $1/5$ cannot be improved. Moreover,

$$\sum_{n=0}^{\infty} |a_n| r^n + \left(\frac{1}{1 + |a_0|} + \frac{r}{1 - r} \right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} + |f(z) - a_0|^2 \leq 1 \ \text{for} \ |z| = r \leq \frac{1}{3}$$

if, and only if, $0 \leq |a_0| \leq 4\sqrt{2} - 5 \approx 0.656854$.

To continue the research, our objective in this section is to establish the inequalities of Theorem 4.4 for the subordination class $S(g)$, where g may be univalent or convex univalent in $\mathbb{D}$. Therefore, it is natural raise the following questions.

Question 4.1 Can we derive a sharp version of Theorem 4.4 for the class $S(g)$ when g is a univalent function in $\mathbb{D}$?

Question 4.2 Can we derive a sharp version of Theorem 4.4 for the class $S(g)$ when g is the univalent and convex function in $\mathbb{D}$?

We obtain the following two results answering Questions 4.1 and 4.2, respectively.

Theorem 4.5 *Let g be analytic and univalent function in $\mathbb{D}$ and $f \in S(g)$ so that $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then the inequality*

$$X_{1,f}(r) := \sum_{n=1}^{\infty} |a_n| r^n + \left(\frac{1}{2 - d} + \frac{r}{1 - r} \right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} + |f(z) - a_0| \leq d \tag{4.2}$$

for $r \leq r_1$, where $r_1 \approx 0.0888988$ is the unique root of the equation $1 - 9r - 27r^2 + 19r^3 + 3r^4 - 11r^5 - 9r^6 + r^7 = 0$ in the interval $(0, 1)$. Moreover,

$$X_{2,f}(r) := \sum_{n=1}^{\infty} |a_n| r^n + \left(\frac{1}{2 - d} + \frac{r}{1 - r} \right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n} + |f(z) - a_0|^2 \leq d \tag{4.3}$$

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for $r \leq r_2$, *where* $r_2 \approx 0.10469$ *is the unique root of the equation*

$$1 - 5r - 39r^2 - 37r^3 - 53r^4 - 23r^5 - 5r^6 + r^7 = 0$$

in the interval $(0,1)$, *where* $d = \operatorname{dist}(g(0), \partial g(\mathbb{D})) < 1$. *The sharpness of* r_1 *and* r_2 *are shown by the Koebe function* $f(z) = z/(1-z)^2$.

Theorem 4.6 *Let* g *be analytic in* $\mathbb{D}$ *such that* g *is univalent and convex in* $\mathbb{D}$ *and* $f \in S(g)$ *so that* $f(z) = \sum_{n=0}^{\infty} a_n z^n$. *Then the inequality (4.2) holds for* $r \leq r_3$, *where* $r_3 \approx 0.174789$ *is the unique root of the equation* $1 - 5r - 5r^2 + 5r^3 = 0$ *in the interval* $(0,1)$. *Moreover, the inequality (4.3) holds for* $r \leq r_4$, *where* $r_4 \approx 0.20473$ *is the unique root of the equation*

$$1 - 3r - 9r^2 - r^3 = 0$$

in the interval $(0,1)$, *where* $d = \operatorname{dist}(g(0), \partial g(\mathbb{D})) \leq 1$.

We now discuss the proof of Theorems 4.5 and 4.6.

Proof of Theorem 4.5 Let $f \prec g$, where $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is a univalent mapping on $\mathbb{D}$ onto a simply connected domain $\Omega = g(\mathbb{D})$. Then it is well known from the Koebe estimate and Rogosinski's coefficient estimates for univalent functions that (see [19, 39])

$$\frac{1}{4}|g'(0)| \leq d \leq |g'(0)|, \quad |b_n| \leq n|g'(0)| \ \text{ for } \ n \geq 1,$$

where $d = \operatorname{dist}(g(0), \partial\Omega)$. Also, the above first inequality gives $|b_n| \leq 4nd$ for $n \geq 1$. Since $f \prec g$, [16, Lemma 1] we see that

$$|f(z) - a_0| = \left|\sum_{n=1}^{\infty} a_n z^n\right| \leq \sum_{n=1}^{\infty} |a_n| r^n \leq \sum_{n=1}^{\infty} |b_n| r^n \ \text{ for } \ r \leq \frac{1}{3}. \tag{4.4}$$

By the similar argument used in the proof of Theorem 4.2 and in view of (4.4) and Lemma 3.1, an easy computation shows that

$$\begin{aligned} X_{1,g}(r) &\leq 8d \sum_{n=1}^{\infty} n r^n + \left(\frac{1}{2-d} + \frac{r}{1-r}\right) 16 d^2 \sum_{n=1}^{\infty} n^2 r^{2n} \\ &= d - d\left(\frac{1 - 10r + r^2}{(1-r)^2} - \frac{(1 + r - rd) 16 d r^2 (1 + r^2)}{(1-r)(1-r^2)^3(2-d)}\right) \\ &= d - \frac{d A_1(d,r)}{(1-r)(1-r^2)^3(2-d)}, \end{aligned}$$

where

$$A_1(d,r) := (1 - 10r + r^2)(1-r)^2(1+r)^3(2-d) - 16d(r^2 + r^4)(1 + r - dr).$$

Our aim is to show that $A_1(d,r) \geq 0$ for every $r \leq r_1$ and $d \in (0,1]$. We see that

$$\frac{\partial^2 A_1(d,r)}{\partial^2 d} \geq 0 \ \text{ for } \ d \in (0,1]$$

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and thus, $\dfrac{\partial A_1(d,r)}{\partial d}$ is an increasing function of d. This gives

$$\frac{\partial A_1(d,r)}{\partial d} \le \frac{\partial A_1(1,r)}{\partial d} = (1-r)(-1+8r+3r^2-35r^4-8r^5+r^6) \le 0$$

for $r \le 0.120502$, and hence $A_1(d,r)$ is a decreasing function of d in interval $(0,1]$. Therefore,

$$A_1(d,r) \ge A_1(1,r) = 1-9r-27r^2+19r^3+3r^4-11r^5-9r^6+r^7 \ge 0$$

for all $r \le r_1$, where $r_1 \approx 0.0888988$ is the unique root in the interval $(0,1)$ of the equation

$$1-9r-27r^2+19r^3+3r^4-11r^5-9r^6+r^7 = 0.$$

Moreover, in order to show the inequality (4.3), it is enough to show that

$$X_{2,g} = \sum_{n=1}^{\infty}|b_n|r^n + \left(\frac{1}{2-d}+\frac{r}{1-r}\right)\sum_{n=1}^{\infty}|b_n|^2r^{2n} + \left(\sum_{n=1}^{\infty}|b_n|r^n\right)^2 \le d \ \text{ for } \ r \le r_2.$$

Since $|b_n| \le 4nd$ for $n \ge 1$, an easy computation shows that

$$\begin{aligned} X_{2,g}(r) &\le 4d\sum_{n=1}^{\infty}nr^n + \left(\frac{1}{2-d}+\frac{r}{1-r}\right)16d^2\sum_{n=1}^{\infty}n^2r^{2n} + 16d^2\left(\sum_{n=1}^{\infty}nr^n\right)^2 \\ &= d - d\left(1-\frac{4r}{(1-r)^2}-\frac{(1+r-rd)16dr^2(1+r^2)}{(1-r)(1-r^2)^3(2-d)}-\frac{16dr^2}{(1-r)^4}\right) \\ &= d - \frac{dA_2(d,r)}{(1-r)(1-r^2)^3(2-d)}, \end{aligned}$$

where

$$\begin{aligned} A_2(d,r) := {} & (1-6r+r^2)(1-r)^2(1+r)^3(2-d) - 16d(r^2+r^4)(1+r-dr) \\ & - 16dr^2(1+r)^3(2-d). \end{aligned}$$

We claim that $A_2(d,r) \ge 0$ for every $r \le r_2$ and $d \in (0,1]$. It can be shown that

$$\frac{\partial^2 A_2(d,r)}{\partial^2 d} \ge 0 \ \text{ for } \ d \in (0,1]$$

and hence, $\dfrac{\partial A_2(d,r)}{\partial d}$ is an increasing function of d. Evidently,

$$\frac{\partial A_2(d,r)}{\partial d} \le \frac{\partial A_2(1,r)}{\partial d} = (1-r)(-1+4r-5r^2-27r^4-4r^5+r^6) \le 0$$

for every $r \le 1$, and hence, $A_2(d,r)$ is a decreasing function of d in interval $(0,1]$. Therefore,

$$A_2(d,r) \ge A_2(1,r) = 1-5r-39r^2-37r^3-53r^4-23r^5-5r^6+r^7 \ge 0$$

for all $r \le r_2$, where $r_2 \approx 0.10469$ is the unique root of the equation

$$1 - 5r - 39r^2 - 37r^3 - 53r^4 - 23r^5 - 5r^6 + r^7 = 0$$

in the interval $(0,1)$.

The sharpness of r_1 and r_2 can be easily shown by the Koebe function $f(z) = z/(1-z)^2$ and hence, we omit the details. ■

Proof of Theorem 4.6 Let $f \prec g$, where $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is a univalent function on $\mathbb{D}$ onto a convex domain $\Omega = g(\mathbb{D})$. Then it is well known from the growth estimate for convex functions and Rogosinski's coefficient estimate (see [19, 39]) that

$$\frac{1}{2}|g'(0)| \le d \le |g'(0)|, \quad \text{and} \quad |b_n| \le |g'(0)| \quad \text{for} \quad n \ge 1,$$

where $d = \operatorname{dist}(g(0), \partial\Omega)$. It follows that $|b_n| \le 2d$ for $n \ge 1$. Combining these inequalities and the inequality (4.4), we see that the desired inequality follows with the help of Lemma 3.1 if we able to show the inequality

$$W_{1,g}(r) := 2\sum_{n=1}^{\infty}|b_n|r^n + \left(\frac{1}{2-d} + \frac{r}{1-r}\right)\sum_{n=1}^{\infty}|b_n|^2 r^{2n} \le d \quad \text{for} \quad r \le r_3.$$

Since $|b_n| \le 2d$ for $n \ge 1$, we have

$$\begin{aligned} W_{1,g}(r) &\le 4d\sum_{n=1}^{\infty} r^n + \left(\frac{1}{2-d} + \frac{r}{1-r}\right)4d^2\sum_{n=1}^{\infty} r^{2n} \\ &= d - d\left(1 - \frac{4r}{1-r} + \frac{(1+r-rd)4dr^2}{(1-r)(2-d)(1-r^2)}\right) \\ &= d - \frac{d\Phi_1(d,r)}{(1-r)(1-r^2)(2-d)}, \end{aligned}$$

where

$$\Phi_1(d,r) := (1-5r)(1-r^2)(2-d) - 4dr^2(1+r-dr).$$

We claim that $\Phi_1(d,r) \ge 0$ for every $r \le r_3$ and $d \in (0,1]$. We see that

$$\frac{\partial^2 \Phi_1(d,r)}{\partial^2 d} \ge 0 \quad \text{for} \quad d \in (0,1]$$

and thus, $\dfrac{\partial \Phi_1(d,r)}{\partial d}$ is an increasing function of d. This gives

$$\frac{\partial \Phi_1(d,r)}{\partial d} \le \frac{\partial \Phi_1(1,r)}{\partial d} = -1 + 5r - 3r^2 - r^3 \le 0$$

for every $r \le 0.236068$, and hence $\Phi_1(d,r)$ is a decreasing function of d in interval $(0,1]$. Therefore

$$\Phi_1(d,r) \ge \Phi_1(1,r) = 1 - 5r - 5r^2 + 5r^3 \ge 0 \quad \text{for all} \quad r \le r_3,$$

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where $r_3 \approx 0.174789$ is the unique root of the equation $1 - 5r - 5r^2 + 5r^3 = 0$ in the interval $(0,1)$. Moreover, for the inequality (4.3), it suffices to show that

$$W_{2,g} := \sum_{n=1}^{\infty} |b_n| r^n + \left(\frac{1}{2-d} + \frac{r}{1-r} \right) \sum_{n=1}^{\infty} |b_n|^2 r^{2n} + \left(\sum_{n=1}^{\infty} |b_n| r^n \right)^2 \le d.$$

Since $|b_n| \le 2d$ for $n \ge 1$, we have

$$\begin{aligned} W_{2,g}(r) &\le 2d \sum_{n=1}^{\infty} r^n + \left(\frac{1}{2-d} + \frac{r}{1-r} \right) 4d^2 \sum_{n=1}^{\infty} r^{2n} + \left(2d \sum_{n=1}^{\infty} r^n \right)^2 \\ &= d - d\left(1 - \frac{2r}{1-r} + \frac{(1+r-rd)4dr^2}{(1-r)(2-d)(1-r^2)} + \frac{4dr^2}{(1-r)^2} \right) \\ &= d - \frac{d\Phi_2(d,r)}{(1-r)^2(1-r^2)(2-d)}, \end{aligned}$$

where

$$\Phi_2(d,r) := (1-3r)(1-r^2)(2-d) - 4r^2(1+r-dr)d - 4dr^2(1+r)(2-d).$$

Our aim is to show that $\Phi_2(d,r) \ge 0$ for every $r \le r_4$ and $d \in (0,1]$. A simple computation shows that

$$\frac{\partial^2 \Phi_2(d,r)}{\partial^2 d} \ge 0 \quad \text{for} \quad d \in (0,1]$$

which implies that $\dfrac{\partial \Phi_2(d,r)}{\partial d}$ is an increasing function of d. This gives

$$\frac{\partial \Phi_2(d,r)}{\partial d} \le \frac{\partial \Phi_2(1,r)}{\partial d} = -1 + 3r - 3r^2 + r^3 \le 0 \quad \text{for} \quad r \le 1,$$

and hence $\Phi_2(d,r)$ is a decreasing function of d in interval $(0,1]$. Therefore,

$$\Phi_2(d,r) \ge \Phi_2(1,r) = 1 - 3r - 9r^2 - r^3 \ge 0 \quad \text{for} \quad r \le r_4,$$

where $r_4 \approx 0.20473$ is the unique root in the interval $(0,1)$ of the equation $1 - 3r - 9r^2 - r^3 = 0$. The sharpness of r_3 and r_4 can be easily shown by the function $f(z) = 1/(1-z)$ and hence, we omit the details. ∎

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