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SEMI-PRIME RINGS WHOSE HOMOMORPHIC IMAGES ARE SERIAL

LAWRENCE S. LEVY AND PATRICK F. SMITH

A theorem of Eisenbud, Griffith, and Robson states that if R is hereditary and noetherian (on both the left and right) then every proper homomorphic image of R is a generalized unserial ring (see, for example, [**3**, p. 244]). Singh [**11**, p. 883] states a converse: If R is a right bounded, noetherian prime ring, all of whose proper homomorphic images are generalized uniserial rings, then (every divisible right R-module is injective, so) R is right hereditary. (Actually, Singh omitted the clearly necessary "bounded" condition.) Singh's theorem generalizes results of [**9**, Proposition 15], [**2**, Theorem 2.1], and [**8**], about commutative rings.

We will call a semi-prime ring R *essentially right bounded* if each essential right ideal contains a two-sided ideal which is essential as a right ideal. In case R is prime, "essentially right bounded" coincides with "right bounded". The main result of this note is the following generalization of Singh's theorem, which allows R to be semi-prime, and to have non-artinian homomorphic images, even when R is prime.

THEOREM. *Let R be an essentially right bounded, right noetherian, semi-prime ring such that R/I is a serial ring for all right essential two-sided ideals I. Then R is right hereditary.*

Recall that a module is called *serial* provided its submodules are totally ordered by inclusion. A ring is called *right serial* provided it is a direct sum of serial right modules; and is called *serial* provided it is both left and right serial. Serial rings which are both left and right artinian coincide with Nakayama's generalized uniserial rings. The property of serial rings which we will need is [**13**, 1.3, 2.6, 3.4]:

WARFIELD'S THEOREM. *A ring R is (right and left) serial if and only if every finitely presented right R-module is a direct sum of serial modules.*

LEMMA. *Let $M = M_1 \oplus \ldots \oplus M_n$ be a module over some ring, and X a serial submodule of M. Suppose, for each i, that*

(i) *$E(M_1)$, the injective hull of M_i, is a serial module; and*

(ii) *$\pi_i(X) \neq M_i$ (π_i: $M \to M_i$ the projection map).*

Then some serial submodule of M properly contains X.

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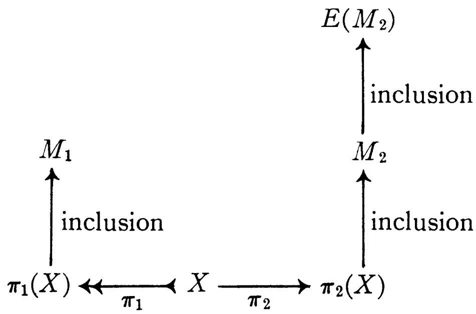
Proof of the lemma. This will be by induction on n, the case $n = 1$ being trivial.

Since $\bigcap_i \ker (X \to M_i) = 0$ and X is serial, we must have some $\ker (X \to M_i) = 0$. Say $\ker (X \to M_1) = 0$, and consider diagram (1) below.

(1)



Composition of the isomorphism $[\pi_1: X \twoheadrightarrow \pi_1(X)]^{-1}$, in (1), with π_2 gives a homomorphism θ of $\pi_1(X)$ into the injective module $E(M_2)$. So θ can be extended to a homomorphism $\theta: M_1 \to E(M_2)$.

Case 1. $\theta(M_1) \subseteq M_2$. Then X is a submodule of

(2) $M' = (1 + \theta)M_1 \oplus M_3 \oplus \ldots \oplus M_n$ and $(1 + \theta)M_1 \cong M_1$.

Moreover, the projection of X in $(1 + \theta)M_1$ is not all of $(1 + \theta)M_1$ because, by hypothesis (ii), $\pi_1(X) \neq M_1$. So, by our induction hypothesis, X is contained in a properly larger serial submodule of M', hence of M.

Case 2. $\theta(M_1) \not\subseteq M_2$. Note that $\theta^{-1}(M_2) \subseteq M_1$, and

(3) $X \subseteq \theta^{-1}(M_2) \oplus M_2 \oplus M_3 \oplus \ldots \oplus M_n$.

Thus replacing M_1 by its submodule $\theta^{-1}(M_2)$ will produce a reduction to Case 1, provided we can show that the projection of X in $\theta^{-1}(M_2)$ is not all of $\theta^{-1}(M_2)$; that is,

(4) $\pi_1(X) \neq \theta^{-1}(M_2)$.

Since $E(M_2)$ is serial, the hypothesis of Case 2 can be rewritten $\theta(M_1) \supset M_2$, so $\theta\theta^{-1}(M_2) = M_2$. On the other hand

$$\theta\pi_1(X) = \pi_2(X) \neq M_2$$

by hypothesis (ii). This establishes (4) and completes the proof of the lemma.

Definitions. A right R-module X is called *divisible* provided $X = Xc$ for every regular element (i.e., non-zero-divisor) c of R. A *torsion element* of X means an element $x \in X$ such that $xc = 0$ for some regular $c \in R$.

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Proof of the theorem. Let R be as in the theorem. First we prove the preliminary result:

(5) The injective hull E of any serial, torsion right R-module T is again a serial, torsion R-module.

To see that E is a torsion module (i.e., all of its elements are torsion elements), take e_0 in E. Since T is an essential submodule of E, the right ideal

$$\{r \in R | \ e_0 r \in T\}$$

is essential in R, and hence contains a regular element c_1 of R [**4**, 3.9]. But then $e_0 c_1 \in T$, a torsion module, so $e_0 c_1 c_2 = 0$ for some regular c_2, as desired.

To see that E is serial, take nonzero elements u and v of E. Then $uR + vR$ has an essential, serial submodule (its intersection with T) and is therefore indecomposable. Since E is a torsion module, $ucR = 0$ and $vc'R = 0$ for regular elements c and c' of R. Therefore by our "right bounded" hypothesis, there is a two-sided, right essential ideal I such that $uI = 0 = vI$. Hence $(uR + vR)I = 0$.

But then $uR + vR$ is a module over the serial ring R/I, and is finitely presented since R/I is right noetherian. So, by Warfield's theorem, above, applied to the ring R/I, the indecomposable R/I-module $uR + vR$ must be serial. So $uR \subseteq vR$ or vice versa, showing that E is serial.

Now we proceed to the theorem itself. To show that R is right hereditary, we quote [**7**, 3.5]: If R has a right quotient ring and every divisible right R-module is injective, then R is right hereditary. Of course, R has a semi-simple artinian right quotient ring by [**4**].

So let D be any divisible right R-module.

Special case (the crux of our argument): D is a torsion module. We can suppose that $D \neq 0$. The idea here is to show that D is a direct sum of generalized $Z_{p\infty}$ groups. (This is the same idea that Singh used, but our modules need not be artinian.)

Let xR be any cyclic serial submodule of D. We wish to conclude, from the lemma, that, for some y in D,

(6) $xR \subset yR$ (proper inclusion; yR serial).

Since x is a torsion element, the "right bounded" hypothesis yields a two-sided, right essential ideal I such that $xI = 0$. Since I is essential, as a right ideal, it contains a regular element c. Since D is divisible, we can find $d \neq 0$ in D such that

(7) $dc = x$ (Note: $(xR)I = 0$).

Note. If $x = 0$, then we are using the "torsion" rather than "divisible" hypothesis to find $d \neq 0$. The element d is likewise annihilated by a two-sided, right essential ideal J. So dR is a finitely presented module over the right noetherian serial ring R/J. Thus, by Warfield's Theorem, above,

(8) $dR = M_1 \oplus \ldots \oplus M_n$ (each $M_i \neq 0$)

with each M_i serial. In fact, each injective hull $E(M_i)$ is serial, by (5) above. To apply the lemma to (8), with $X = xR$, we have to show that every projection $\pi_i(xR)$ in (8) satisfies $\pi_i(xR) \neq M_i$.

If $\pi_i(xR) = M_i$, then by (7),

$$0 = \pi_i(xRI) = \pi_i(xR)I = M_iI = \pi_i(dI).$$

But $x = dc \in dI$ then shows $\pi_i(x) = 0$, so $M_i = \pi_i(xR) = 0$, a contradiction.

Thus we can apply the lemma, getting (6).

We now use (6) repeatedly, beginning with $x_1 = 0$, to get an infinite sequence of proper inclusions

(9) $x_1R \subset x_2R \subset x_3R \subset \ldots$

where each x_iR is a serial submodule of D.

We claim that $X = \bigcup_i x_iR$ is injective. To see this, let E be its injective hull. Since X is serial and torsion, so is E, by (5). Hence, if $E \neq X$, and $e \in E - X$, then $eR \supset X$. But eR is noetherian, and X is not, by (9). This contradiction shows that X is injective.

We conclude: D has an injective, nonzero submodule.

Zorn's lemma immediately yields a family $\{Y_\alpha\}$ of injective submodules of D, with $\{Y_\alpha\}$ maximal with respect to the property that $Y = \sum Y_\alpha$ is a direct sum. Since R is right noetherian, Y is injective, hence $D = Y \oplus D'$ for some submodule D' which is divisible and torsion; properties preserved by homomorphic images. But then $D' = 0$, for otherwise D' would have an injective, nonzero submodule, contradicting maximality of $\{Y_\alpha\}$.

We conclude: Every divisible, torsion right R-module is injective.

General case. Now let D be an arbitrary divisible right R-module, and let D_T be its set of torsion elements. Since R has a right quotient ring, D_T is a submodule of D [**7**, 1.4]. Hence D_T is injective. So

$$D = D_0 \oplus D_T$$

where D_0 is divisible and torsion-free. But, over a ring whose right quotient ring is semi-simple artinian, every divisible, torsion-free module is injective [**7**, 3.3].

Thus every divisible right R-module D is injective, and hence R is right hereditary.

Remark. Zaks, in [**14**], proved the following related result: If R is a right bounded, right noetherian prime ring whose proper homomorphic images are right artinian principal right ideal rings, then R is right hereditary. Then Smith [**12**] showed that the hypothesis "right artinian" can be removed, provided one requires (perhaps unnecessarily) that proper prime ideals of R be maximal. The proof in [**12**] is quite different from the present one. And we were unable to find a common generalization of these closely related results.

Warfield [**13**, 6.6] proves the related-looking result: If R is a module-finite algebra over a commutative, noetherian ring, and R/I is serial whenever it is artinian, then R is a direct sum of (prime) hereditary orders over Dedekind domains and an Artinian PIR. Again the method of proof is quite different.

Non-noetherian, but commutative, rings whose proper homomorphic images are self injective were considered in [**6**]. Again the prime ones turn out to have all finitely generated ideals projective, and the homomorphic images are serial rings.

For additional theorems of this type, see the references in [**12**].

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University of Wisconsin,
Madison, Wisconsin;
University of Glasgow,
Glasgow, Scotland