BULLETIN **13** (1), 1970

ENCLOSURE THEOREMS FOR EIGENVALUES OF ELLIPTIC OPERATORS

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1. **Introduction.** Let L be the linear, elliptic, self-adjoint partial differential operator given by

$$Lu = -\sum_{i,j=1}^{n} D_i(a_{ij}D_ju)+bu, \quad a_{ij} = a_{ji},$$

where D_j denotes partial differentiation with respect to x_j, $1 \le j \le n$, b is a positive, continuous real-valued function of $x=(x_1,\ldots,x_n)$ in n-dimensional Euclidean space E^n, the a_{ij} are real-valued functions possessing uniformly continuous first partial derivatives in E^n and the matrix $\{a_{ij}\}$ is everywhere positive definite. A solution u of $Lu=0$ is assumed to be of class C^1.

Our purpose is to establish variational formulae for the eigenvalues and eigenfunctions of L for two specific perturbations. These are: the perturbation of E^n to an n-disk D_a of radius a, considered in §2; and the perturbation of the upper half-space H^n of E^n to the upper half of D_a, S_a, discussed in §3. The boundary condition adjoined on the bounding surface of D_a and S_a is

$$Bv = 0, \tag{1-1}$$

where B is the linear boundary operator defined by

$$Bv = \sigma_1 \sum_{i,j=1}^{n} a_{ij}D_jv \cos(\nu,x_j)+\sigma_2 v$$

and ν denotes the outer normal to the bounding surface. It is assumed that the functions σ_1 and σ_2 are defined on the bounding surface, are piecewise continuous and non-negative, and that the sum $\sigma_1+\sigma_2$ has a positive lower bound.

Let H_1, H_2, H_a and H_s denote the Hilbert spaces which are Lebesgue spaces with respective inner products

$$(u, v)_1 = \int_{E^n} u(x)\overline{v(x)}\,dx, \qquad (u, v)_2 = \int_{H^n} u(x)\overline{v(x)}\,dx,$$

$$(u, v)_a = \int_{D_a} u(x)\overline{v(x)}\,dx, \quad \text{and} \quad (u, v)_s = \int_{S_a} u(x)\overline{v(x)}\,dx,$$

and respective norms $\|u\|_1$, $\|u\|_2$, $\|u\|_a$ and $\|u\|_s$. For x in E^n, $|x|$ denotes the usual Euclidean norm. The eigenvalue problem for L on E^n,

$$Lu = \lambda u, \quad u \in H_1, \tag{1-2}$$

Received by the editors May 25, 1969.

(1) This paper represents part of the author's thesis prepared at the University of B.C. under the direction of C. A. Swanson.

https://doi.org/10.4153/CMB-1970-001-8 Published online by Cambridge University Press

will be called the basic problem in §2. Similarly, the eigenvalue problem for L on H^n

$$(1\text{-}3)\qquad \begin{aligned} Lu &= \eta u, \quad u \in H_2, \\ Bu &= 0 \text{ on the } (n-1) \text{ hyperplane } P = \{x \mid x \in E^n,\ x_n = 0\}, \end{aligned}$$

will be called the basic problem in §3. The corresponding perturbed eigenvalue problems will be defined in §§2 and 3.

In general it is not true that the eigenvalues of the perturbed problems tend to limits as $a \to \infty$, even when the spectrum of the basic problem is entirely discrete (see [4] p. 20). The only assumption required in order to obtain the enclosure theorems and representational formulae is that there exists at least one eigenvalue of the basic problem whose corresponding eigenfunctions satisfy some limit property. For example, Theorem I in §2 shows that if the eigenfunctions corresponding to the basic eigenvalue λ of multiplicity m satisfy condition (2-2), then at least m eigenvalues of the perturbed problem (2-1) converge to λ as the radius a of the n-disk tends to infinity.

The principal difficulty in these estimation problems is in establishing a reasonable condition on the basic eigenfunctions so that the norm of the function $f = R_a u - \alpha u$, constructed in Lemma 1, becomes small for large a. For example, in §2 this condition is characterized in terms of the "L-measure" (2-3) which is independent of the basic eigenfunctions. The method employed for the treatment of these problems is an extension of that used by C. A. Swanson [4] for the special case $\sigma_1 = 0$ and the results established here properly contain those in [4].

2. **The perturbation of E^n to an n-disk.** The perturbed eigenvalue problem is

$$(2\text{-}1)\qquad Lv = \mu v \text{ in } D_a = \{x \mid |x| < a, a > 0\}, \quad v \in \mathscr{D}_a,$$

where the perturbed domain $\mathscr{D}_a$ is defined to be the set of all complex-valued functions v such that:

(i) v is twice continuously differentiable on D_a;
(ii) v and $D_j v$, $j = 1, \ldots, n$, are continuous at those points of $\partial D_a = \{x \mid |x| = a\}$ at which σ_1 and σ_2 are continuous and v satisfies (1-1) on ∂D_a.

The only assumption to be made here is that there exists at least one eigenvalue λ of the basic problem (1-2) whose corresponding eigenfunctions u satisfy

$$(2\text{-}2)\qquad \{\max_{\partial D_a} |Bu|\} \|g\|_a / \|u\|_a = o(1) \quad \text{as } a \to \infty,$$

where g is the "L-measure"

$$(2\text{-}3)\qquad \begin{aligned} Lg &= 0 \text{ in } D_a, \\ Bg &= 1 \text{ on } \partial D_a. \end{aligned}$$

It is known [2, 3] that for (2-1) there exists a denumerable sequence of eigenvalues μ_i, $0 < \mu_1 \le \mu_2 \le \cdots$, and a complete orthonormal sequence of eigenfunc-

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tions $\{v_i\}$, such that for some Robin function $R_a(x, y)$ (Green's function of the third kind)

$$v_i(y) = \mu_i R_a v_i(y) = \mu_i \int_{D_a} R_a(x, y) v_i(x)\, dx$$

and any basic eigenfunction u satisfies $LR_a u = u$ in D_a. R_a is constructed in the usual way as the sum of a fixed fundamental solution and the solution of a particular Robin problem. The following notation will be used:

$$\psi_a(u) = \{\max_{\partial D_a} |Bu|\} \|g\|_a / \|u\|_a \quad (u \neq 0),$$

$$\psi_a = \sup_{u \in \Lambda_\lambda} \psi_a(u),$$

$$\rho_a = \lambda\psi_a/(1-\psi_a),$$

where Λ_λ is the eigenspace associated with the basic eigenvalue λ of (1-2). Since every $u \in \Lambda_\lambda$ has the representation $u = \sum_{k=1}^{m} \alpha_i u_i$ in terms of an orthonormal basis $\{u_i\}$, where the basic eigenvalue λ is of multiplicity m,

$$\psi_a(u) \leq m \max_{1 \leq i \leq m} \psi_a(u_i)$$

and it follows from condition (2-2) that $\psi_a = o(1)$ and $\rho_a = o(1)$ as $a \to \infty$.

LEMMA 1. *For* $\alpha = 1/\lambda$

(2-4) $$\|R_a u - \alpha u\|_a \leq \alpha\psi_a \|u\|_a \quad \textit{for every } u \in \Lambda_\lambda.$$

Proof. The eigenvalues of (1-2) are positive. In fact, if $\lambda \leq 0$ for some λ, the maximum principle ([1] p. 326) implies that the eigenfunction u corresponding to λ approaches its maximum as $|x| \to \infty$. This contradicts $u \in L^2(E^n)$ and $u \neq 0$. Let $\alpha = 1/\lambda$ and $f = R_a u - \alpha u$. Then for every $u \in \Lambda_\lambda$, f is a solution of

$$Lf = 0 \text{ in } D_a,$$

$$Bf = -\alpha Bu \text{ on } \partial D_a,$$

and

$$f(y) = -\alpha \int_{\partial D_a} R_a(x, y) Bu\, dS_x \quad \text{for every } y \in D_a.$$

Then

$$|f(y)| \leq \alpha\{\max_{\partial D_a} |Bu|\} \int_{\partial D_a} R_a(x, y)\, dS_x = \alpha\{\max_{\partial D_a} |Bu|\} g(y)$$

for every $y \in D_a$, where g is the solution of (2-3). Hence

$$\|f\|_a = \|R_a u - \alpha u\|_a \leq \alpha\{\max_{\partial D_a} |Bu|\}(\|g\|_a/\|u\|_a)\|u\|_a$$

for every $u \in \Lambda_\lambda$, and this proves the lemma.

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THEOREM 1. *Let λ be an m-fold degenerate eigenvalue of (1-2) whose corresponding eigenfunctions satisfy (2-2). Then there exists a positive number a_0 such that at least m perturbed eigenvalues $\mu_i(a)$ of (2-1) are enclosed in the interval* $[\lambda, \lambda+\rho_a]$ *whenever $a \geq a_0$ and converge to λ as $a \to \infty$.*

Proof. Since $\rho_a = o(1)$ as $a \to \infty$, there exists an a_0 such that $\rho_a < 1$ for every $a \geq a_0$. Let $F_{a\epsilon}$ be the subspace of H_a generated by all the eigenfunctions of R_a whose eigenvalues $\beta_i = 1/\mu_i$ lie in the interval $|\beta - \alpha| < \epsilon$. Let P($\epsilon$) be the projection of H_a onto $F_{a\epsilon}$. Then $\|u - P(\epsilon)u\|_a \leq \epsilon^{-1}\|R_a u - \alpha u\|_a$ for every $u \in \Lambda_\lambda$ by ([5] p. 33), since the integral operator R_a is a self-adjoint linear transformation on H_a. By Lemma 1,

$$\|u - P(\epsilon)u\|_a \leq \alpha\psi_a \epsilon^{-1}\|u\|_a$$

for every $u \in \Lambda_\lambda$. Thus, by ([5] p. 35), there are at least m eigenvalues β_i contained in the interval $|\beta_i - \alpha| \leq \alpha\psi_a$, $i = 1, 2, \ldots$, or, more precisely, the interval $|\mu_i - \lambda| \leq \mu_i \psi_a$. Since $D_a \subset E^n$, it follows by the minimax principle for eigenvalues [1] that $\mu_i \geq \lambda$ for every i. Hence, $\lambda \leq \mu_i \leq \lambda + \mu_i\psi_a$, or

$$\lambda \leq \mu_i \leq \lambda/(1-\psi_a) = \lambda + \rho_a, \quad a \geq a_0,$$

for every $i = 1, 2, \ldots$, and the thorem is proved.

The following result is a simple consequence of Theorem 1 and the minimax principle for eigenvalues.

THEOREM 2. *Let λ be as in Theorem 1. If there exists a basic eigenvalue exceeding λ, then there is a positive number $a_1 \geq a_0$ such that exactly m perturbed eigenvalues μ_i are enclosed in the interval* $[\lambda, \lambda+\rho_a]$ *whenever $a \geq a_1$.*

Let

$$k_a(x) = \left(\int_{D_a} |x-y|^{2p} R_a^2(x, y)\, dy\right)^{1/2},$$

where $p = p(n)$ is a positive number with $p(2) = 0$, $p(3) = 0$ and $0 < n - 2p < 4$. In order to establish uniform estimates for the perturbed eigenfunctions, it is assumed that

$$\psi_a^q k_a(x) = o(1) \quad \text{as } a \to \infty \quad (q = (n-2p)/n)$$

uniformly for all $x \in D_a$.

THEOREM 3. *Let u_i be the orthonormal eigenfunctions corresponding to the m-fold degenerate eigenvalue λ of Theorem 2, and v_i those corresponding to the m perturbed eigenvalues μ_i, $i = 1, \ldots, m$. Then*

$$v_i(x) = u_i(x) - f_i(x) + 0(\psi_a^q)k_a(x), \quad i = 1, \ldots, m, \quad x \in D_a, \quad a \geq a_1,$$

where f_i is the solution of $Lf_i = 0$ in D_a, $Bf_i = Bu_i$ on ∂D_a.

Proof. Let $\epsilon = \alpha - \alpha'$ in (2-4) where $\alpha = 1/\lambda$ and $\alpha' = 1/\lambda'$. It follows from Theorem 2 that $\alpha\psi_a < \epsilon$ for $a \geq a_1$. Then $F_{a\epsilon}$ is m-dimensional by Theorem 2, and $\|u - P(\epsilon)u\|_a < \|u\|_a$ implies that $u = 0$ if $P(\epsilon)u = 0$, $u \in \Lambda_\lambda$. Therefore, m uniquely determined linearly independent eigenfunctions z_i corresponding to α are mapped by $P(\epsilon)$ into the orthonormal functions v_i. By (2-4), $\|z_i - v_i\|_a = 0(\psi_a)$. Let $\{u_i\}$ be the orthonormal sequence constructed by the Schmidt process as linear combinations of the z_i. Then $\|u_i - z_i\|_a = 0(\psi_a)$ and

$$\|u_i - v_i\|_a = 0(\psi_a), \quad i = 1, \dots, m. \tag{2-5}$$

Let u be an element of $\{u_i\}$ and v the corresponding element in $\{v_i\}$. Then by Theorem 2 and (2-5)

$$\|\mu v - \lambda u\|_a \le \mu\|v-u\|_a + (\mu-\lambda)\|u\|_a = 0(\psi_a).$$

Let

$$w_a(x) = \left(\left\{ \int_{D_a - d_\delta} + \int_{d_\delta} \right\} (|x-y|^{-2p} |\mu v(y) - \lambda u(y)|^2)\, dy \right)^{1/2},$$

where d_δ is the n-disk with centre x and radius δ. If we choose $\delta = \psi_a^{2/n}$ we obtain the uniform estimate $w_a(x) = 0(\psi_a^q)$, where $0 < q = (n-2p)/n < 4/n$. It is asserted that $\lambda R_a u(x)$ gives a uniform estimate for $v(x)$, since

$$|v(x) - \lambda R_a u(x)| = |R_a(\mu v(x) - \lambda u(x))| \le k_a(x) w_a(x) = 0(\psi_a^q) k_a(x). \tag{2-6}$$

The function

$$\xi(x) = \lambda R_a u(x) - u(x) + f(x) \tag{2-7}$$

is the solution of $L\xi = 0$ in D_a, $B\xi = 0$ on ∂D_a. By ([3] p. 97), L is positive definite on $\mathscr{D}_a$. Hence $\xi \equiv 0$ in D_a and the theorem follows from (2-6) and (2-7).

Let u and v be as in Theorem 3. By assumption there exists $c_2 > 0$ such that $\sigma_1 + \sigma_2 > c_2$. Let ω_1 be the set of points of ∂D_a on which $\sigma_1 > c_2/2$ and ω_2 the set on which $\sigma_2 > c_2/2$. Then $\partial D_a = \omega_1 \cup \omega_2$ and we can write Green's symmetric identity in the form

$$\begin{aligned} &(Lu, v)_a - (u, Lv)_a \\ &= \int_{D_a - \omega_1} (1/\sigma_2) \sum_{i,j=1}^{n} a_{ij} D_j \bar{v} \cos(\nu, x_i) Bu\, dS - \int_{\omega_1} (\bar{v}/\sigma_1) Bu\, dS = \{uv\}_a. \end{aligned} \tag{2-8}$$

Since u and v are as in Theorem 3, $\{uv\}_a = (\lambda - \mu)(u, v)_a$ and from (2-5) $|(u, v)_a - (v, v)_a| = 0(\psi_a)$. Hence

$$\lambda - \mu = \{uv\}_a (1 + 0(\psi_a)). \tag{2-9}$$

Let f be the solution of $Lf = 0$ in D_a, $Bf = Bu$ on ∂D_a. Application of (2-8) to $Lf = 0$, $Lv = \mu v$ and $Lu = \lambda u$ gives

$$-\mu(f, v)_a = \{fv\}_a = \{uv\}_a \tag{2-10}$$

and

$$-\lambda(f, u)_a = \{fu\}_a. \tag{2-11}$$

By (2-9) and (2-10) and the fact that $\mu=\lambda+0(\psi_a)$,

$$\lambda-\mu = \lambda(f, v)_a(1+0(\psi_a)).$$

Finally, application of Theorem 3 and (2-11) gives the following asymptotic formulae for the perturbed eigenvalues

$$\lambda-\mu = (\{fu\}_a+\lambda(f,f)_a)(1+0(\psi_a))+0(\psi_a^q)(f, k_a)_a.$$

An Example. Let L be given in E^2 by

$$Lu = -\Delta_2 u+(x_1^2+x_2^2+2)u,$$

where $\Delta_2 \equiv D_1^2+D_2^2$, and let $\sigma_1=\sigma_2=1$ in (1-1).

L is an operator of the Schrödinger type with potential function $V=(x_1^2+x_2^2+2)$. By the method of separation of variables we obtain the orthonormal eigenfunfunctions

$$u_{n,m} = (n!\,m!\,2^{n+m})^{-1/2}H_n(x_1)H_m(x_2)\exp(-(x_1+x_2)/2)$$

corresponding to the eigenvalues $\lambda=2(n+m)+4$, $(n, m=0, 1, \ldots)$, where $H_n(x_1)$ denotes the Hermite polynomial of degree n in x_1. After a routine transformation to polar coordinates (r, θ) and separation of variables we obtain

$$g = (a+1)^{-1}\exp((r^2-a^2)/2)$$

for g in (2-3) and it follows easily that (2-2) is satisfied for every $u_{n,m}$.

3. **The perturbation of H^n to S_a.** Let H^n be the upper half-space of E^n, $H^n=\{x \mid x\in E^n,\ x_n>0\}$. $S_a=\{x \mid x\in D_a\cap H^n\}$ and the boundary of ∂S_a, S_a, can be expressed as the union of two disjoint sets A_a and C_a, where $A_a=\{x \mid x\in E^n$, $x_n=0$, $|x|\le a\}$ and $C_a=\{x \mid x$ on $\partial D_a\cap H^n\}$. It may be noted that it is not necessary to restrict the domains to half-spaces and half-disks. It can easily be verified that the results indicated in this section would also apply to any solid n-cone J^n in E^n perturbed to the solid spherical cone $J_a^n=J^n\cap D_a$.

The perturbed eigenvalue problem to be considered here is

$$Lw = \gamma w \text{ in } S_a, \quad w\in\mathscr{D}_s,$$

and the perturbed domain $\mathscr{D}_s$ is defined as $\mathscr{D}_a$ with v replaced by w and D_a by S_a. For $n>1$ in this domain perturbation it is not possible to characterize the condition on the basic eigenfunctions u in terms of the "L-measure" g. Therefore, it is assumed here that there exists at least one eigenvalue η of the basic problem (1-3) whose corresponding eigenfunctions satisfy

$$\|h\|_s/\|u\|_s = o(1) \quad \text{as } a\to\infty,$$

where h is the solution of

$$Lh = 0 \text{ in } S_a,$$
$$Bh = Bu \text{ on } C_a, \qquad Bh = 0 \text{ on } A_a.$$

Then setting

$$\psi_s(u) = \|h\|_s/\|u\|_s \quad (u \neq 0),$$
$$\psi_s = \sup_{u \in \Lambda_\eta} \psi_s(u)$$

and

$$\rho_s = \eta\psi_s/(1-\psi_s),$$

where $\Lambda\eta$ is the eigenspace associated with the basic eigenvalue η, theorems analogous to Theorems 1, 2 and 3 of §2 can be obtained for this perturbation.

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https://doi.org/10.4153/CMB-1970-001-8 Published online by Cambridge University Press