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A note on energy equality for the fractional Navier-Stokes equations

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This paper proves the energy equality for distributional solutions to fractional Navier-Stokes equations, which gives a new proof and covers the classical result of Galdi [Proc. Amer. Math. Soc. 147 (2019), 785–792].

Keywords: Fractional Navier-Stokes equations; energy equality; distributional solutions

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1. Introduction

In this paper, we consider the energy equality for distributional solutions to the following fractional Navier-Stokes equations

$$\begin{cases} \partial_t u + u\cdot\nabla u + \mu(-\Delta)^\alpha u + \nabla p = 0, & x\in\mathbb{R}^3, t>0\\ \nabla\cdot u = 0, & x\in\mathbb{R}^3, t>0\\ u|_{t=0} = u_0(x), & x\in\mathbb{R}^3 \end{cases} \tag{1.1}$$

with $\mu > 0$ is the kinematic viscosity, for simplicity, we set $\mu = 1$ in the sequel. Here $u = u(x,t) \in \mathbb{R}^3$ and $p = p(x,t) \in \mathbb{R}$ are non-dimensional quantities corresponding to the flow velocity and the total kinetic pressure at the point (x,t), respectively. $u_0(x)$ is the initial velocity field satisfying that $\nabla\cdot u_0 = 0$. We denote the Fourier transform of the function f by $\hat{f}$, then fractional Laplacian is defined by

$$\widehat{(-\Delta)^\alpha} f(\xi) = |\xi|^{2\alpha}\hat{f}(\xi),$$

more details on $(-\Delta)^\alpha$ can be found in [**16**], as a notation, we take Λ as $(-\Delta)^{\frac{1}{2}}$.

The fractional Navier-Stokes equations (1.1) were first considered by Lions [**10**], they showed that if $\alpha \geqslant \frac{5}{4}$, then equations (1.1) has a unique global smooth solution for any smooth initial data (also see [**19**]). In [**19**], Wu showed that when $\alpha > 0$, equations (1.1) with $u_0 \in L^2$ possess a global weak solution and local in time strong solution for given initial value $u_0 \in H^1$. Where, the weak solution means that (u,p) satisfies (1.1) in the distribution sense and satisfies so-called energy inequality. Katz-Pavlović [**7**] showed that if $1 < \alpha < \frac{5}{4}$, the Hausdorff dimension of the singular set at the time of first possible blow-up is at most $(5-4\alpha)$. Recently, Tang-Yu

[17] studied the partial regularity of fractional Navier-Stokes equations (1.1) with $\frac{3}{4} < \alpha < 1$, more precisely, they proved that the suitable weak solution is regular away from a relatively closed singular set whose $(5-4\alpha)$-dimensional Hausdorff measure is zero. Further partial regularity results of equations (1.1), we refer readers to [2, 15]. It is well-know that in case $\alpha < \frac{5}{4}$, it remains unknown whether or not the solution will preserve sufficiently smooth initial regularity. On the other hand, for $\alpha \geqslant \frac{5}{4}$, it is easy to prove that the Leray-Hopf weak solution of the fractional Navier-Stokes equations is energy-conserved by a standard mollifying procedure, and a taking limits argument. From this point of view, there is a rather subtle relationship between energy equality and regularity for weak solutions. Therefore, it is natural to consider the energy conservation problem of the weak solution to the fractional Navier-Stokes equations and expect to have a better understanding for the regularity of the equations (1.1).

When $\alpha = 1$, equations (1.1) reduce to the classical incompressible Navier-Stokes equations:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0, & x \in \mathbb{R}^3, t > 0 \\ \nabla \cdot u = 0, & x \in \mathbb{R}^3, t > 0 \\ u|_{t=0} = u_0(x), & x \in \mathbb{R}^3. \end{cases} \tag{1.2}$$

It is well known since the work of Leray [9] and Hopf [6], that for any $u_0 \in L^2_\sigma(\mathbb{R}^3)$ one can construct a global weak solutions to (1.2), namely, a function u that, for each $T > 0$, is in the class

$$u \in L^\infty(0, T; L^2_\sigma(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)) \tag{1.3}$$

and solves (1.1) in a distributional sense. Here, $L^2_\sigma(\mathbb{R}^3)$ is the subspace of $L^2(\mathbb{R}^3)$ of divergence-free vector functions. In addition, such a u satisfies the so-called energy inequality:

$$\|u(t)\|^2_{L^2} + 2\int_0^t \|\nabla u(\tau)\|^2_{L^2}\,\mathrm{d}\tau \leqslant \|u_0\|^2_{L^2}, \quad \forall\, t \geqslant 0. \tag{1.4}$$

Much about the solutions of the Navier-Stokes equation is unknown, including uniqueness and regularity. The main barrier is the fact that the energy equality, which states that for any smooth solution u, it obeys the following basic energy law:

$$\|u(t)\|^2_{L^2} + 2\int_0^t \|\nabla u(\tau)\|^2_{L^2}\,\mathrm{d}\tau = \|u_0\|^2_{L^2}, \quad \forall\, t \geqslant 0. \tag{1.5}$$

In the context of weak solutions even in the class $u \in L^2H^1$, such a manipulation is not feasible due to lack of sufficient regularity to integrate by parts. This leaves room for additional mechanisms of energy dissipation due to the work of the nonlinear term. A natural question that remains open is whether energy equality, which should be expected from a physical point of view, is valid for weak solutions. Thus, an interesting question is: how badly behaved u can keep the energy conservation. Lions [11] and Ladyzhenskaya [8] proved independently that such solutions satisfy the (global) energy equality (1.5) under the additional assumption $u \in L^4L^4$. Shinbrot

[18] generalized the Lions-Ladyzhenskaya condition to

$$u \in L^r(0,T;L^s(\mathbb{R}^d)) \text{ with } \frac{2}{r}+\frac{2}{s} \leqslant 1, s \geqslant 4. \tag{1.6}$$

Yu in [20] given a new proof to the Shinbrot energy conservation criterion.

When considering distributional solutions (see definition 1.1) of 3D incompressible Navier-Stokes equations (1.2), in this case there is not any available regularity on velocity field u, apart the solution being in $L^2_{\text{loc}}(\mathbb{R}^3\times[0,T))$. The interest for distributional solutions dates back to Foias [4], who proved their uniqueness under the solution in Serrin class (i.e., $u\in L^r(0,T;L^s(\Omega))$ with $\frac{2}{r}+\frac{3}{s}=1$, $s>3$). Later, Fabes, Jones and Riviere [3] proved the existence of distributional solutions for the Cauchy problem, while the case of the initial-boundary value problem has been studied mainly starting from the work of Amann [1]. Recently, The possible connection between distributional solutions and the energy equality has been considered by Galdi [5], who proved that if distributional solution in $L^4(0,T;L^4(\mathbb{R}^3))$, and with initial data u_0 in $L^2(\mathbb{R}^3)$, then energy equality (1.5) holds true. The key observation is the use of the duality argument and the above conditions to improve the regularity of the solution (i.e., $L^\infty(0,T;L^2(\mathbb{R}^3))\cap L^2(0,T;H^1(\mathbb{R}^3))$).

Inspired by the above mentioned works on energy conservation of classical Navier-Stokes equations (1.2), the purpose of this note is to prove that, actually, for generalize and extend Galdi's energy conservation result of classical Navier-Stokes equations to fractional Navier-Stokes equations (1.1). More precisely, setting

$$\mathcal{D}_T := \{\varphi \in C_0^\infty(\mathbb{R}^3\times[0,T)) : \ \operatorname{div}\varphi = 0\}.$$

Definition 1.1 Distributional solution. *Let $u_0\in L^2(\mathbb{R}^3)$ with $\nabla\cdot u_0=0$, $T>0$. The function $u\in L^2_{\text{loc}}(\mathbb{R}^3\times[0,T))$ is a distributional solution to the fractional Navier-Stokes equations (1.1) if*

1. *for any $\Phi\in\mathcal{D}_T$, we have*

$$\int_0^T\int_{\mathbb{R}^3} u\cdot\partial_t\Phi - u\cdot\Lambda^{2\alpha}\Phi + u\otimes u : \nabla\otimes\Phi \,\mathrm{d}x\,\mathrm{d}t = -\int_{\mathbb{R}^3} u(x,0)\cdot\Phi(x,0)\,\mathrm{d}x;$$

2. *for any $\varphi\in C_0^\infty(\mathbb{R}^3)$, it holds that*

$$\int_{\mathbb{R}^3} u\cdot\nabla\varphi\,\mathrm{d}x = 0,$$

for a.e. $t\in(0,T)$.

The main result of this paper is

Theorem 1.2. *Suppose that $1\leqslant\alpha<\infty$ and $u\in L^2_{\text{loc}}(\mathbb{R}^3\times[0,T))$ be a distributional solution in the sense of definition 1.1 to system (1.1). If*

$$u\in L^{\frac{4\alpha}{2\alpha-1}}\left(0,T;L^4(\mathbb{R}^3)\right),$$

then

$$\int_{\mathbb{R}^3} |u(t,x)|^2 \, \mathrm{d}x + 2\int_0^t \int_{\mathbb{R}^3} |\Lambda^\alpha u(x,\tau)|^2 \, \mathrm{d}x \, \mathrm{d}\tau = \int_{\mathbb{R}^3} |u_0|^2 \, \mathrm{d}x$$

for any $t \in [0,T]$.

Remark 1.3. This result extends the well-known Galdi's energy conservation criterion to fractional Navier-Stokes equations (1.1).

Remark 1.4. The non-uniqueness of weak solutions to the 3D Navier-Stokes equations with fractional hyperviscosity $(-\Delta)^\alpha$ was showed in [**13**], where $[1, \frac{5}{4})$. However, theorem 1.2 reveals that this non-uniqueness mechanism is inhibited in the class of $L^{\frac{4\alpha}{2\alpha-1}}(0,T;L^4(\mathbb{R}^3))$. In other words, this non-uniqueness property destroys the energy conservation of weak solutions. Here, the uniqueness property of weak solutions refers to the identically vanishing solution.

2. Proof of theorem 1.2

This section is devoted to proof of theorem 1.2. For the sake of simplicity, we will proceed as if the solution is differentiable in time. The extra arguments needed to mollify in time are straightforward.

Let $\eta : \mathbb{R}^3 \to \mathbb{R}$ be a standard mollifier, i.e., $\eta(x) = C\mathrm{e}^{\frac{1}{|x|^2-1}}$ for $|x|<1$ and $\eta(x)=0$ for $|x| \geqslant 1$, where constant $C>0$ selected such that $\int_{\mathbb{R}^3} \eta(x)\,\mathrm{d}x = 1$. For any $\varepsilon>0$, we define the rescaled mollifier $\eta_\varepsilon(x) = \varepsilon^{-3}\eta(\frac{x}{\varepsilon})$. For any function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, its mollified version is defined as

$$f^\varepsilon(x) = (f * \eta_\varepsilon)(x) = \int_{\mathbb{R}^3} \eta_\varepsilon(x-y) f(y) \, \mathrm{d}y.$$

If $f \in W^{1,p}(\mathbb{R}^3)$, the following local approximation is well known

$$f^\varepsilon(x) \to f \quad \text{in} \quad W^{1,p}_{loc}(\mathbb{R}^3) \quad \forall p \in [1,\infty).$$

The key ingredient to prove theorem 1.2 is the following several important lemmas.

Lemma 2.1 [**12**]. *Let ∂ be a partial derivative in one direction. Let $f, \partial f \in L^p(\mathbb{R}^+ \times \mathbb{R}^3)$, $g \in L^q(\mathbb{R}^+ \times \mathbb{R}^d)$ with $1 \leqslant p,q \leqslant \infty$, and $\frac{1}{p}+\frac{1}{q} \leqslant 1$. Then, we have*

$$\|\partial(fg) * \eta_\varepsilon - \partial(f(g*\eta_\varepsilon))\|_{L^r(\mathbb{R}^+\times\mathbb{R}^3)} \leqslant C\|\partial f\|_{L^p(\mathbb{R}^+\times\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^+\times\mathbb{R}^3)}$$

for some constant $C>0$ independent of ε, f and g, and with $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. In addition,

$$\partial(fg) * \eta_\varepsilon - \partial(f(g*\eta_\varepsilon)) \to 0 \quad \text{in } L^r\left(\mathbb{R}^+\times\mathbb{R}^3\right)$$

as $\varepsilon \to 0$, if $r<\infty$.

LEMMA 2.2 Gagliardo-Nirenberg inequality. *[14] Let $0 \leqslant m, \alpha \leqslant l$, then we have*

$$\|\Lambda^\alpha f\|_{L^p(\mathbb{R}^3)} \leqslant C \|\Lambda^m f\|^{1-\theta}_{L^q(\mathbb{R}^3)} \left\|\Lambda^l f\right\|^{\theta}_{L^r(\mathbb{R}^3)},$$

where $\theta \in [0,1]$ and α satisfies

$$\frac{\alpha}{3} - \frac{1}{p} = \left(\frac{m}{3} - \frac{1}{q}\right)(1-\theta) + \left(\frac{l}{3} - \frac{1}{r}\right)\theta.$$

Here, when $p = \infty$, we require that $0 < \theta < 1$.

LEMMA 2.3. *Let $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ and let u be a distributional solution in the sense of definition 1.1 to system (1.1) and satisfies*

$$u \in L^{\frac{4\alpha}{2\alpha-1}}\left(0,T; L^4(\mathbb{R}^3)\right),$$

then we have

$$\sup_{t\geqslant 0} \|u^\varepsilon(\cdot,t)\|^2_{L^2} + \int_0^t \int_{\mathbb{R}^3} |\Lambda^\alpha u^\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}\tau \leqslant \tilde{C}, \quad \forall\, t \in [0,T],$$

where $\tilde{C}$ is a constant depending only on $\|u_0\|_{L^2}$ and $\int_0^T \|u\|^{\frac{4\alpha}{2\alpha-1}}_{L^4}\,\mathrm{d}t$.

REMARK 2.4. Lemma 2.3 shows that u can be identified with $u \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^\alpha(\mathbb{R}^3))$. This proves that u falls into the class of Leray-Hopf weak solutions, we know that $\alpha \geqslant \frac{5}{4}$ implies energy equality to fractional Navier-Stokes equations. Therefore, a natural question which arises is whether fractional Navier-Stokes equations (1.1) with $\alpha \geqslant \frac{5}{4}$ is also satisfy energy equality for distributional solutions.

Proof of lemma 2.3. By the definition of distributional solutions to (1.1), we obtain that following identity

$$\int_{\mathbb{R}^3} u \cdot \partial_t \Phi^\varepsilon - u \cdot \Lambda^{2\alpha}\Phi^\varepsilon + u \otimes u : \nabla \otimes \Phi^\varepsilon \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} u(x,t) \cdot \Phi^\varepsilon(x,t)\,\mathrm{d}x,$$

for all $\Phi^\varepsilon \in \mathcal{D}_T$. Which in turn gives

$$\int_{\mathbb{R}^3} u^\varepsilon \cdot \partial_t \Phi - u^\varepsilon \cdot \Lambda^{2\alpha}\Phi + (u \otimes u)^\varepsilon : \nabla \otimes \Phi \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} u^\varepsilon(x,t) \cdot \Phi(x,t)\,\mathrm{d}x.$$

Now, choosing $\Phi = u^\varepsilon$ in above identity, integrate by parts to find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} |u^\varepsilon|^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} |\Lambda^\alpha u^\varepsilon|^2 \,\mathrm{d}x = \int_{\mathbb{R}^3} (u \otimes u)^\varepsilon \cdot \nabla u^\varepsilon \,\mathrm{d}x. \tag{2.1}$$

Applying the Gagliardo-Nirenberg inequality and the Hölder inequality, one has

$$\begin{aligned} \left|\int_{\mathbb{R}^3} (u \otimes u)^\varepsilon \cdot \nabla u^\varepsilon dx\right| &\leqslant C\|(u \otimes u)^\varepsilon\|_{L^2} \|\nabla u^\varepsilon\|_{L^2} \\ &\leqslant C\|(u \otimes u)^\varepsilon\|_{L^2} \|u^\varepsilon\|^{1-\frac{1}{\alpha}}_{L^2} \|\Lambda^\alpha u^\varepsilon\|^{\frac{1}{\alpha}}_{L^2} \\ &\leqslant C\|(u \otimes u)^\varepsilon\|^{\frac{2\alpha}{2\alpha-1}}_{L^2} \|u^\varepsilon\|^{\frac{\alpha-1}{2\alpha-1}}_{L^2} + \epsilon \|\Lambda^\alpha u^\varepsilon\|^2_{L^2}. \end{aligned} \tag{2.2}$$

Putting the above estimates (2.2) into (2.1), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|u^\varepsilon|^2\,\mathrm{d}x+\frac{3}{2}\int_{\mathbb{R}^3}|\Lambda^\alpha u^\varepsilon|^2\,\mathrm{d}x\leqslant C\|(u\otimes u)^\varepsilon\|_{L^2}^{\frac{2\alpha}{2\alpha-1}}\|u^\varepsilon\|_{L^2}^{\frac{\alpha-1}{2\alpha-1}}$$
$$\leqslant C\|(u\otimes u)^\varepsilon\|_{L^2}^{\frac{2\alpha}{2\alpha-1}}\left(\|u^\varepsilon\|_{L^2}^2+1\right). \tag{2.3}$$

Next we apply Gronwall's inequality to conclude that

$$\sup_{t\geqslant 0}\|u^\varepsilon(\cdot,t)\|_{L^2}^2+\int_0^t\int_{\mathbb{R}^3}|\Lambda^\alpha u^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\tau\leqslant\|u_0\|_{L^2}^2\exp C\int_0^t\|(u\otimes u)^\varepsilon\|_{L^2}^{\frac{2\alpha}{2\alpha-1}}\,\mathrm{d}\tau$$
$$\leqslant C\exp C\int_0^t\|u\|_{L^4}^{\frac{4\alpha}{2\alpha-1}}\,\mathrm{d}\tau$$
$$\leqslant\tilde{C}. \tag{2.4}$$

for all $t\in[0,T]$, where $\tilde{C}$ is a constant depending only on viscosity u_0 and $\int_0^T\|u\|_{L^4}^{\frac{4\alpha}{2\alpha-1}}\,\mathrm{d}t$. Let $\varepsilon\to 0$ in (2.4), one has

$$\sup_{t\geqslant 0}\|u(\cdot,t)\|_{L^2}^2+\int_0^t\int_{\mathbb{R}^3}|\Lambda^\alpha u|^2\,\mathrm{d}x\,\mathrm{d}\tau\leqslant C, \tag{2.5}$$

and this completes the proof of lemma 2.3. □

Proof of theorem 1.2. With lemmas 2.1 and 2.3 in hand, we are ready to prove our main result. First, we define the function $\Xi=u^\varepsilon$ and note that $\operatorname{div}u^\varepsilon=0$. Using Ξ^ε to test the first equation of system (1.1), one has

$$\int_{\mathbb{R}^3}\Xi^\varepsilon\left(\partial_t u+u\cdot\nabla u+(-\Delta)^\alpha u+\nabla p\right)\,\mathrm{d}x=0, \tag{2.6}$$

thus we have

$$\int_{\mathbb{R}^3}u^\varepsilon\left(\partial_t u+u\cdot\nabla u+(-\Delta)^\alpha u+\nabla p\right)^\varepsilon\,\mathrm{d}x=0. \tag{2.7}$$

This yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|u^\varepsilon|^2\,\mathrm{d}x+\int_{\mathbb{R}^3}|\Lambda^\alpha u^\varepsilon|^2\,\mathrm{d}x=-\int_{\mathbb{R}^3}\operatorname{div}(u\otimes u)^\varepsilon\cdot u^\varepsilon\,\mathrm{d}x. \tag{2.8}$$

Clearly,

$$\int_{\mathbb{R}^3}|u^\varepsilon|^2\,\mathrm{d}x-\int_{\mathbb{R}^3}|u_0^\varepsilon|^2\,\mathrm{d}x+2\int_0^t\int_{\mathbb{R}^3}|\Lambda^\alpha u^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\tau$$
$$=-2\int_0^t\int_{\mathbb{R}^3}\operatorname{div}(u\otimes u)^\varepsilon\cdot u^\varepsilon\,\mathrm{d}x\,\mathrm{d}\tau. \tag{2.9}$$

Notice that

$$-2\int_0^t\int_{\mathbb{R}^3}\operatorname{div}(u^\varepsilon\otimes u^\varepsilon)\cdot u^\varepsilon\,\mathrm{d}x\,\mathrm{d}\tau=0,$$

by using Höder's equality and lemma 2.1, one has

$$
\begin{aligned}
&-2\int_0^t\int_{\mathbb{R}^3} \operatorname{div}(u\otimes u)^\varepsilon\cdot u^\varepsilon - \operatorname{div}(u^\varepsilon\otimes u^\varepsilon)\cdot u^\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau\\
&= 2\int_0^t\int_{\mathbb{R}^3} [(u\otimes u)^\varepsilon - (u^\varepsilon\otimes u^\varepsilon)]\cdot\nabla u^\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau\\
&\leqslant 2\int_0^t\int_{\mathbb{R}^3} |(u\otimes u)^\varepsilon - u^\varepsilon\otimes u^\varepsilon|\,|\nabla u^\varepsilon| \,\mathrm{d}x\,\mathrm{d}\tau\\
&\leqslant 2\int_0^t\int_{\mathbb{R}^3} (|(u\otimes u)^\varepsilon - u\otimes u| + |u\otimes u - u\otimes u^\varepsilon| + |u\otimes u^\varepsilon - u^\varepsilon\otimes u^\varepsilon|)\,|\nabla u^\varepsilon| \,\mathrm{d}x\,\mathrm{d}\tau\\
&\leqslant C\,\|(u\otimes u)^\varepsilon - u\otimes u\|_{L^{\frac{2\alpha}{2\alpha-1}}(0,T;L^2(\mathbb{R}^3))}\|\nabla u^\varepsilon\|_{L^{2\alpha}(0,T;L^2(\mathbb{R}^3))}\\
&\quad + C\,\|u-u^\varepsilon\|_{L^{\frac{4\alpha}{2\alpha-1}}(0,T;L^4(\mathbb{R}^3))}\|u\|_{L^{\frac{4\alpha}{2\alpha-1}}(0,T;L^4(\mathbb{R}^3))}\|\nabla u^\varepsilon\|_{L^{2\alpha}(0,T;L^2(\mathbb{R}^3))}\\
&\quad + C\,\|u-u^\varepsilon\|_{L^{\frac{4\alpha}{2\alpha-1}}(0,T;L^4(\mathbb{R}^3))}\|u^\varepsilon\|_{L^{\frac{4\alpha}{2\alpha-1}}(0,T;L^4(\mathbb{R}^3))}\|\nabla u^\varepsilon\|_{L^{2\alpha}(0,T;L^2(\mathbb{R}^3))}\\
&\to 0,\quad as\quad \epsilon\to 0,
\end{aligned}
\tag{2.10}
$$

where we used the facts that

$$\int_0^T \|\nabla u^\varepsilon\|_{L^2}^{2\alpha}\,\mathrm{d}t \leqslant C\int_0^T \|u^\varepsilon\|_{L^2}^{2\alpha-2}\|\Lambda^\alpha u^\varepsilon\|_{L^2}^2\,\mathrm{d}t \leqslant \tilde{C}$$

and

$$u\in L^{\frac{4\alpha}{2\alpha-1}}\left(0,T;L^4(\mathbb{R}^3)\right).$$

Letting ε goes to zero in (2.9), and using the facts (2.10), what we have proved is that in the limit

$$\int_{\mathbb{R}^3}|u(t,x)|^2\,\mathrm{d}x + 2\int_0^t\int_{\mathbb{R}^3}|\Lambda^\alpha u(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau = \int_{\mathbb{R}^3}|u_0|^2\,\mathrm{d}x.$$

This ends our proof of theorem 1.2. □

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Conflict of interest
The author declares that there is no conflict of interest.

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