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REGULAR NEIGHBORHOODS OF IMMERSED MANIFOLDS

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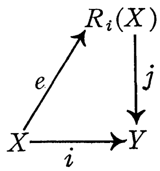
1. Introduction. Let X and Y denote polyhedra, $i : X \to Y$ a PL immersion. A regular neighborhood of X associated with i is a regular neighborhood $(e, R_i(X))$ of X together with an immersion $j : R_i(X) \to Y$ such that the diagram



commutes and for each $x \in X$ there is a neighborhood N of $e(x)$ in $R_i(X)$ such that $j|N$ is an embedding and $j(N)$ is a neighborhood of $f(x)$ in Y. In [**7**], the existence of induced regular neighborhoods is shown. For X and Y PL manifolds this was done in [**1**]. The properties of the regular neighborhood associated with an immersion are also provided in [**7**]. For example, if $i : X \to Y$ is an embedding,

$$X \xrightarrow{\ i\ } R_i(X) \xrightarrow{\ j\ } Y$$

is a regular neighborhood of X associated with i where $R_i(X)$ is a regular neighborhood of $i(X)$ in Y and j is inclusion.

A PL homotopy $f : X \times I \to Y$ between two immersions of a polyhedron X in a polyhedron Y is a regular homotopy if the associated map $F : X \times I \to Y \times I$ defined by $F(x, t) = (f(x, t), t)$ is an immersion. f is a pseudo-regular homotopy if there is some PL immersion $F : X \times I \to Y \times I$ with

$$F^{-1}(Y \times \{i\}) = X \times \{i\}, i = 0, 1$$

and $f = p_1F$ where $p_1 : Y \times I \to Y$ is the natural projection.

The following theorem is taken from [**7**].

THEOREM 1. *Let $i : X \times I \to M$ be an immersion of the polyhedron $X \times I$ in the PL manifold M, with $i^{-1}(\partial M) = X \times \partial I$ and* $\dim M - \dim (X \times I) \geqq 3$.

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Let i_t be the immersion defined by $i_t(x) = i(x, t)$ for all x in X, $t = 0$ or 1, and let

$$X \xrightarrow{e_t} R_{i_t}(X) \xrightarrow{j_t} \partial M$$

be associated regular neighborhoods. Then there is a homeomorphism $h : R_{i_0}(X) \to R_{i_1}(X)$ such that $he_0 = e_1$.

The following is an immediate corollary to Theorem 1.

COROLLARY 2. *Let i_1 and i_2 denote immersions of the polyhedron X into the manifold M with* $\dim M - \dim X \geqq 3$. *Then if i_1 and i_2 are pseudo-regularly homotopic, the associated regular neighborhoods are equivalent.*

In this paper we are concerned with the setting where X is an n-dimensional, compact, connected, orientable PL manifold of dimension $n \geqq 3$ and Y is a $2n$-dimensional PL manifold. It is actually possible, in this setting, to drop the codimesion restriction of Theorem 1 using the fact that a regular neighborhood of a manifold determines a block bundle [**9**]. However, the codimension restriction of Theorem 1 is satisfied in our situation automatically. A superscript is used to denote dimension.

The following is the main theorem:

THEOREM 3. *Suppose M^n is a compact, connected, orientable PL manifold and Q^{2n} is a PL manifold without boundary. Then any two homotopic immersions of M^n into Q^{2n} are pseudo-regularly homotopic provided $n \geqq 3$.*

With same setting as in Theorem 3 we have the following:

COROLLARY 4. *If f and g are homotopic immersions of M^n into Q^{2n}, then $(e, R_f(M))$ is equivalent to $(e, R_g(M))$.*

In [**4**] Hudson shows that the isotopy classes of embeddings of $S^1 \times S^{n-1}$ in R^{2n} are non-trivial for $n \geqq 3$. Corollary 4 shows, however, that their regular neighborhoods are equivalent.

In [**2**] one finds an example to show that Theorem 3 fails when one considers immersions of M^n into Q^q for $q < 2n$.

Finally, in [**1**] one has that regular homotopy classes of immersions of S^n in R^{2n} correspond bijectively with $\pi_n(\tilde{V}_{2n,n}) \simeq \pi_n(V_{2n,n}) \neq 0$ for $n \geqq 4$ where $\tilde{V}_{2n,n}$ and $V_{2n,n}$ are respectively the PL and classical Stiefel varieties [**9**; **5**]. Thus Theorem 3 shows that pseudo-regularly homotopic does not imply regularly homotopic, cf. concordance implies isotopy.

All work is done in the PL category. [**3**] and [**11**] form standard references. Int and ∂ are used to denote interior and boundary respectively, B^n denotes the standard n dimensional simplex and S^{n-1} its boundary, and "$\simeq$" denotes "is PL homeomorphic to".

2. We now proceed to prove Theorem 3.

THEOREM 3. *Suppose M^n is a compact, connected, orientable PL manifold and*

Q^{2n} is a PL manifold without boundary. Then any two homotopic immersions of M^n into Q^{2n} are pseudo-regularly-homotopic provided $n \geqq 3$.

Proof. Let f_0 and f_1 denote homotopic immersions of M into Q. We claim that one can assume, without loss of generality, that dim $S_2(f_i) = 0$, $i = 0, 1$; that is, f_0 and f_1 are general position maps. To see this one can examine the proofs of the general position theorems of Hudson in [**3**] and see that if a map is an immersion, then the shift to general position can be accomplished through a homotopy of immersions. Thus if f_0 and f_1 fail to be general position maps, each is regularly homotopic to such a map. Now let $F : M \times I \rightarrow Q \times I$ denote a map arising from the homotopy between f_0 and f_1 with the property that $F(x, t) = (f_0(x), t)$ for t in $[0, 1/3]$ and $F(x, t) = (f_1(x), t)$ for t in $[2/3, 1]$. Shift F into general position keeping fixed an $Q \times \partial I$. Denote this new map again by F. F is an immersion except for a finite number of branch points all of which lie in int $(M \times I)$. We now proceed to alter the map F on int $(M \times I)$ so as to eliminate the branch points.

Let λ be a PL arc in int $(M \times I)$ which passes through all branch points of $S_2(F)$ but otherwise misses $S_2(F)$. Then $F(\lambda)$ is an arc in int $(Q \times I)$. Let N and P denote regular neighborhoods of λ and $F(\lambda)$ respectively which are chosen so that $F|N$ is a proper map between N and P. Here proper means $F^{-1}(\partial P) = \partial N$. N and P are homeomorphic to B^{n+1} and B^{2n+1} respectively. Since the branch set of F, denoted $\mathrm{Br}(F)$, is in the interior of N, F is an immersion on a neighborhood in N of ∂N. Let N' denote the closure of the complement in N of a collaring of ∂N chosen small enough that $\mathrm{Br}(F) \subset$ int N'. As before $F|\partial N'$ is an immersion and F maps $\partial N'$ into int P. $\partial N' \simeq S^n$ and int $P \simeq R^{2n+1}$. Thus $F|\partial N'$ extends to an immersion $E : N' \rightarrow$ int P as the obstruction to such an extension lies in $\pi_n(\tilde{V}_{2n+1,n}) = 0$. Let $W = (M \times I) -$ int N' and define $F' : M \times I \rightarrow Q \times I$ by

$$F'(x, t) = \begin{cases} F(x, t) & \text{for } (x, t) \in W \\ E(x, t) & \text{for } (x, t) \in N'. \end{cases}$$

At this point we must consider the intersection of $F(W)$ with $E(N')$ as this intersection is the only source of branch points for F'. First we shift $E(N')$ into general position with respect to $F(W)$ keeping $Q \times \partial I$ fixed and also keeping $E(\partial N')$ fixed. This shift can be achieved by an arbitrary small isotopy (see [**3**, Lemma 4.6]). Letting F' again denote the resulting new map, we have $F'|W$ is still an immersion, in fact is unchanged, and $F'|N'$ remains an immersion. $F'(\text{int } W) \cap F'(\text{int } N')$ consist entirely of double points of transversal intersection. Therefore $\mathrm{Br}(F')$ must lie on $\partial N'$. In fact, each point of $\mathrm{Br}(F')$ is a limit point for one or more pairs of rays of $S_2'(F')$. By an alteration described in the proof of Lemma 1 of [**8**] we may assume that each point of $\mathrm{Br}(F')$ is a limit point for precisely one pair of rays of $S_2'(F')$. Thus $S_2(F')$ consists of transversal intersections together with simple closed curves folded by F' at a pair of branch points. Let p_1 and p_2 denote the branch points of

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a simple closed curve which is folded by F'. Let α denote a PL arc in int $(M \times I)$ joining p_1 to p_2 and missing the remaining portion of $S_2(F')$. Then $F'(\alpha)$ is an arc in int $(Q \times I)$. Let A and B denote regular neighborhoods of α and $F'(\alpha)$ respectively such that $F'|A : A \rightarrow B$ is a proper map. Then $A \simeq B^{n+1}$ and $B \simeq B^{2n+1}$. Using, [**8**, Lemma 4] we see that $F'|\partial A$ extends to an immersion of A into P. Repeated application of this procedure to each simple closed curve in $S_2(F')$ containing a pair of branch points leads to an immersion $F'' : M \times I \rightarrow Q \times I$ such that $F''|M \times \{i\} = f_i$, $i = 0, 1$ as was desired.

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