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GROUPS WITH A CAYLEY GRAPH ISOMORPHIC TO A HYPERCUBE

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A process is described for enumerating the Cayley graphs isomorphic to a binary d-cube for small values of d. There are 4 Cayley graphs isomorphic to the 3-cube, 14 isomorphic to the 4-cube, 45 isomorphic to the 5-cube and 238 isomorphic to the 6-cube. A similar method may be used for any graph with a prime power number of vertices.

1. INTRODUCTION

Let G be a group and T be a set of generators of G with $T = T^{-1}$ and $1 \notin T$. Then the *Cayley graph* $\mathrm{Cayley}(G,T)$ is the graph with vertex set G and edges (x, xt) $(x \in G,\ t \in T)$. The graph is connected because T generates G, it has no loops because $1 \notin T$, and it is undirected since whenever (x, xt) is an edge so is $\bigl(xt, (xt)t^{-1}\bigr) = (xt, x)$. There is an extensive literature on Cayley graphs, partly because their simple representation and internal symmetry make them useful in applications (see, for example, the recent papers [**7, 8, 10**] and [**12**]).

Two natural questions arise: (1) what graphs can be represented by Cayley graphs; and (2) what different representations does a given graph have? The first of these question seems quite difficult, although some progress has been made; see, for example, [**9**] and the references there. With respect to the second question, most effort seems to have gone into attempts to characterise "Cayley invariant groups" (CI-groups) which were introduced in [**1**] (the definition is given below); see [**4, 5, 6**] and references there.

In the present paper we consider question (2) for d-cubes which are of particular interest for applications. We shall show how to compute a complete set of representations of a d-cube as a Cayley graph for small values of d. It turns out that there are many representations even when $d \leqslant 6$. Our methods apply equally well to any graph with a prime power number of vertices.

One reason that Cayley graphs are interesting is because of their in-built symmetry. If A denotes the automorphism group of $\mathrm{Cayley}(G,T)$ (so A is the group of all

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permutations of the vertex set G which preserve adjacency), then A contains a copy G_0 of G as a regular subgroup acting on the left, namely:

$$G_0 := \{x \mapsto a^{-1}x \mid a \in G\} \cong G.$$

Our constructions below are based on the following converse.

(1.1) Let $\mathcal{G}$ be a connected graph (without loops) whose automorphism group A contains a regular subgroup G (in its action on the vertex set), and let α be a vertex of $\mathcal{G}$. Then $T := \{t \in G \mid \alpha^t \text{ is adjacent to } \alpha \text{ in } \mathcal{G}\}$ satisfies $1 \notin T$ and $T = T^{-1}$, and $\mathcal{G}$ is isomorphic to Cayley(G,T).

PROOF: See [**2**, Lemma 16.3]. ∎

When should two Cayley graphs be considered "the same"? The following is a natural equivalence. Let (G_i, T_i) $(i = 1,2)$ be two pairs where T_i is a generating set for the group G_i such that $1 \notin T_i$ and $T_i = T_i^{-1}$. We say that these pairs are *equivalent* if there exists a group isomorphism ψ from G_1 onto G_2 which maps T_1 onto T_2. Evidently, if (G_1,T_1) is equivalent to (G_2,T_2) under ψ, then ψ induces an isomorphism between Cayley(G_1,T_1) and Cayley(G_2,T_2). In general, pairs which are inequivalent may still have isomorphic Cayley graphs; Babai [**1**] defines G to be a *CI-group* if inequivalent pairs (G,T_1) and (G,T_2) always have nonisomorphic Cayley graphs. The next result shows how we can determine equivalence in the situation of (1.1).

(1.2) Suppose that the automorphism group A of a connected graph $\mathcal{G}$ contains two regular subgroups G_1 and G_2. Let α_1 and α_2 be any two vertices of $\mathcal{G}$ and define $T_i := \{t \in G_i \mid \alpha_i^t \text{ is adjacent to } \alpha_i\}$ for $i = 1,2$. Then (G_1,T_1) is equivalent to (G_2,T_2) if and only if G_1 is conjugate to G_2 in A. In particular, the equivalence type of (G_1,T_1) does not depend on the choice of α_1.

PROOF: See [**1**]. ∎

Finally, the following criterion for equivalence is useful for computations. Its simple proof is omitted.

(1.3) Suppose that (G_i,T_i) $(i = 1,2)$ are two pairs where T_i is a generating set for the group G_i, and that G_1 is a finite group of order g. Then there is an isomorphism ψ of G_1 onto G_2 such that $\psi(T_1) = T_2$ if and only there is a bijection $\lambda : T_1 \to T_2$ such that $\{(t,\lambda(t)) \mid t \in T_1\}$ generates a subgroup of order g in $G_1 \times G_2$ (and in the latter case λ extends to an isomorphism ψ of G_1 onto G_2).

REMARK. In a computation to decide whether two pairs (G_1,T_1) and (G_2,T_2) are equivalent, we work with partial (injective) functions $\lambda : T_1 \to T_2$. The criteria in (1.3) is used to check whether λ can be extended to an isomorphism on the subgroup generated by its domain. If it is, we extend λ so that its domain contains one further element of T_1; otherwise we backtrack by deleting an element from the domain.

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EXAMPLE 1. There are exactly four inequivalent pairs (G,T) for which Cayley(G,T) is isomorphic to the 3-cube. (This is not obvious but see Example 2 below.) These are:

(i) G is elementary Abelian of order 2^3 with $T=\{a,b,c\}$: $a^2=b^2=c^2=1$, $ab=ba$, $ac=ca$ and $bc=cb$ (the "standard example");

(ii) G is a direct product of cyclic subgroups of order 2 and 4 with $T=\{a,b,b^{-1}\}$: $a^2=b^4=1$ and $ab=ba$;

(iii) G is dihedral of order 8 with $T=\{a,b,c\}$: $a^2=b^2=c^2=1$, $ab=ba$, $ac=ca$ and $bc=cba$;

(iv) G is dihedral of order 8 with $T=\{a,b,b^{-1}\}$: $a^2=b^4=1$ and $ab=b^3a$.

The pairs (G,T) in (iii) and (iv) are obviously inequivalent even though the underlying group is the same.

2. AUTOMORPHISMS OF THE d-CUBE

We shall define the (binary) d-cube ($d \geqslant 2$) as the graph whose vertex set is the vector space $V := (\mathbb{F}_2)^d$ of d-tuples over the field of two elements with two vertices adjacent if and only if they differ in exactly one entry. The automorphism group A of the d-cube clearly contains the elementary Abelian 2-group B of order 2^d consisting of all translations of V by elements of V, and also the group $S \cong \mathrm{Sym}(d)$ consisting of the permutations of V induced by permuting the coordinates of V. Since B acts regularly on V, and S is the stabiliser of $(0,0,\dots,0)$ in A, we have $A=SB$ with $S\cap B=1$. In its natural representation, A is a subgroup of $\mathrm{Sym}(V)$ of order $2^d d!$ and degree 2^d. Alternatively, A may be represented as the reflection group consisting of all $d\times d$ monomial matrices with nonzero entries ± 1 (as a Coxeter group A is denoted by A_{d-1}).

The next result follows immediately from (1.1) and (1.2).

(2.1) Let G be a group of order 2^d. Then there exists a generating set T of G with $1\notin T$ and $T=T^{-1}$ such that Cayley(G,T) is isomorphic to the d-cube if and only if G is isomorphic to a regular subgroup R of the group A defined above. The equivalence types (G,T) correspond one-to-one with the A-conjugacy classes of regular subgroups of A isomorphic to G.

For computational purposes we prefer a more compact representation of A. Since A is isomorphic to the wreath product $C_2 \operatorname{wr} \mathrm{Sym}(d)$ it has a faithful permutation representation of degree $2d$ as well as its natural representation of degree 2^d (see, for example, [**3**, Chapter 2]). Specifically, let Π be the partition of $\{1,2,\dots,2d\}$ whose parts are the 2-subsets $\{1,d+1\},\{2,d+2\},\dots,\{d,2d\}$, and let A_0 be the subgroup of $\mathrm{Sym}(2d)$ consisting of all permutations which leave Π invariant. Then A is isomorphic to A_0 with B corresponding to $B_0 := \langle (1,d+1),\dots,(d,2d)\rangle$, and S corresponding to $S_0 := \langle (1,2)(d+1,d+2),(1,2,\dots,d)(d+1,d+2,\dots,2d)\rangle \cong \mathrm{Sym}(d)$. It is easily

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verified that S_0 is the set of all permutations in $\mathrm{Sym}(2d)$ which leave Π invariant and map $\{1,2,\dots,d\}$ into itself, and that a subgroup R of A is regular (in the natural representation) if and only if its image R_0 in A_0 has the properties: (i) $|R_0| = 2^d$; and (ii) $R_0 \cap S_0 = 1$. Now (2.1) gives the following.

(2.2) Let G be a group of order 2^d. Then there exists at least one set T of generators of G with $1 \notin T$ and $T = T^{-1}$ such that $\mathrm{Cayley}(G,T)$ is isomorphic to the d-cube if and only if G is isomorphic to a subgroup R_0 of $\mathrm{Sym}(2d)$ such that:

(i) R_0 leaves the partition Π described above invariant; and

(ii) no nontrivial element of R_0 maps $\{1,2,\dots,d\}$ into itself.

REMARK. In particular, if $\mathrm{Cayley}(G,T)$ is isomorphic to the d-cube, then G has a faithful permutation representation of degree $2d$. This shows, for example, that if k is the integer such that $d < 2^k \leqslant 2d$ then the orders of the elements of G are bounded by 2^k and the solvable length of G is bounded by k.

3. Enumerating conjugacy classes of regular subgroups

We now turn to the question of how to enumerate efficiently the regular subgroups of the automorphism group A of the d-cube. By (1.2) we really want to enumerate the A-conjugacy classes of such groups. It turns out to be easier to enumerate first the P-conjugacy classes of regular subgroups of a Sylow 2-subgroup P of A, and then to remove extraneous groups which are A-conjugate but not P-conjugate. Since the process we describe works equally well for a p-group for any prime p we shall describe it in that more general setting.

PROBLEM. Let p be a prime. Given a finite p-group P and a subgroup P_0 of index p^d, find representatives of the P-conjugacy classes of subgroups R of P satisfying:

$$(*) \qquad |R| = p^d \text{ and } R \cap P_0 = 1.$$

REMARK. In our original problem P is a transitive permutation group of degree 2^d, P_0 is a point stabiliser in P, and the subgroups R with property (*) are the regular subgroups. The more general formulation is helpful because we want to work with the representation of degree $2d$ described in (2.2).

Since P is a p-group, every subgroup is subnormal, so we can find a normal series:

$$P_0 \triangleleft P_1 \triangleleft \dots \triangleleft P_d = P$$

where P_k/P_{k-1} is of order p and generated by $P_{k-1}z_k$, say, for $k = 1,2,\dots,d$. Suppose that R is a subgroup of P with properties (*) and put $R_k := R \cap P_k$. Then

$$1 = R_0 \triangleleft R_1 \triangleleft \dots \triangleleft R_d = R$$

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is a normal series for R whose factors are each of order dividing p. Since $|R| = p^d$, each factor in this series has, in fact, order equal to p, so we have a composition series for R. Thus $|R_k| = p^k$ and $P_k = P_0R_k$. In particular, $z_k \in P_0R_k$ and so $u_kz_k \in R_k$ for some $u_k \in P_0$. Since $u_kz_k \notin P_{k-1}$, we conclude that $R_{k-1}u_kz_k$ generates R_k/R_{k-1}. Note that u_k is completely determined by R since $R \cap P_0 = 1$. Hence we can associate with each subgroup R satisfying (*) a unique d-tuple $(u_1, \dots, u_d)$ of elements from P_0 such that

$$R_k = \langle u_1z_1, \dots, u_kz_k\rangle \text{ for } k = 0, 1, \dots, d.$$

Let $\mathcal{U}$ denote the set of all such d-tuples as R ranges over the subgroups of P satisfying (*).

(3.1) A d-tuple $(u_1, \dots, u_d)$ of elements from P_0 lies in $\mathcal{U}$ if and only if, for $i = 1, \dots, d$, we have:

(i) u_iz_i normalises $\langle u_1z_1, \dots, u_{i-1}z_{i-1}\rangle$; and

(ii) $(u_iz_i)^p \in \langle u_1z_1, \dots, u_{i-1}z_{i-1}\rangle$.

PROOF: It is easy to check that the two conditions are necessary, so assume that (i) and (ii) both hold; we shall show that $(u_1, \dots, u_d) \in \mathcal{U}$.

Put $R_i := \langle u_1z_1, \dots, u_iz_i\rangle$. Then $1 = R_0 \leqslant R_1 \leqslant \dots \leqslant R_d$ is a normal series by (i), and the factors have orders dividing p by (ii), so $|R_d| \leqslant p^d$. Since $P_0R_d = \langle P_0, z_1, \dots, z_d\rangle = P$, R_d is of order at least $|P : P_0| = p^d$. Hence $|R_d| = p^d$, and now $P_0R_d = P$ implies $P_0 \cap R_d = 1$. Thus R_d satisfies (*) as required. ∎

Now consider the condition of P-conjugacy. Suppose that $(u_1, \dots, u_d)$ and $(v_1, \dots, v_d)$ lie in $\mathcal{U}$ and define subgroups R and Q, respectively, which satisfy conditions analogous to (*). Since $P = P_0R$, R is conjugate to Q under P if and only if R is conjugate to Q under P_0. Suppose $w \in P_0$ such that $w^{-1}Rw = Q$. Then w normalises P_k so $w^{-1}R_kw = Q_k$ for each k. Thus to find the P-conjugacy classes of subgroups satisfying (*) in a backtrack program, it is enough to choose exactly one P_0-conjugacy class of subgroups R_k at each stage.

Assume that $k \geqslant 1$. Suppose that $(u_1, \dots, u_{k-1})$ is a $(k-1)$-tuple of elements from P_0 which extends to two k-tuples of elements from P_0, say $(u_1, \dots, u_k)$ and $(u_1, \dots, u_{k-1}, v)$, both of which satisfy conditions of type (i) and (ii) of (3.1) up to this stage. As before, put $R_i := \langle u_1z_1, \dots, u_iz_i\rangle$ for $i = 0, \dots, k$, and $Q_k := \langle u_1z_1, \dots, u_{k-1}z_{k-1}, vz_k\rangle$.

(3.2) There exists $w \in P_0$ such that $w^{-1}R_kw = Q_k$ if and only if:

(i) w normalises R_{k-1}; and

(ii) $R_{k-1}vz_k = R_{k-1}w^{-1}u_kz_k\mathrm{w}$.

PROOF: The conditions are clearly sufficient for R_k and Q_k to be conjugate under w. Conversely, assume $w^{-1}R_kw = Q_k$ for some $w \in P_0$. Then (i) holds because

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$R_{k-1} = R_k \cap P_{k-1} = Q_k \cap P_{k-1}$. Moreover, $vz_k \in Q_k \cap P_k$, $u_k z_k \in R_k \cap P_k$ and $w \in P_{k-1}$, therefore $wvz_k w^{-1} \in R_k \cap P_{k-1}u_k z_k = (R_k \cap P_{k-1})u_k z_k = R_{k-1}u_k z_k$. Condition (ii) now follows because w normalises R_{k-1}. □

The conditions of (3.1) and (3.2) can be used in a backtrack program to generate a complete set of representatives of P-conjugacy classes (= P_0-conjugacy classes) of subgroups of P satisfying (*). The d-tuples in $\mathcal{U}$ are generated one element at a time, discarding branches which will lead to P_0-conjugate subgroups. Suppose that $k \leqslant d$ and that $(u_1, \dots, u_{k-1})$ is a $(k-1)$-tuple of elements of P_0 such that conditions (i) and (ii) of (3.1) hold for $i = 1, \dots, k-1$. Put $R_{k-1} := \langle u_1 z_1, \dots, u_{k-1} z_{k-1}\rangle$. Then in the next step we must construct a subset $W \subseteq P_0$ such that

(i) $(u_1, \dots, u_{k-1}, u_k)$ satisfies (i) and (ii) of (3.1) for $i = k$ for each $u_k \in W$; and

(ii) $R_k := \langle u_1 z_1, \dots, u_k z_k\rangle$ $(u_k \in W)$ is a complete set of representatives of the P_0-conjugacy classes of subgroups obtained by extending R_{k-1}.

It follows from (3.1) and (3.2) that W has these properties provided the elements $R_{k-1}uz_k$ $(u \in W)$ run over a complete set of representatives of the $N_{P_0}(R_{k-1})$-orbits (under conjugation) of the set of elements of order p of the form $R_{k-1}vz_k$ $(v \in P_0)$ in $N_{P_k}(R_{k-1})/R_{k-1}$.

It may happen that no extension of the $(k-1)$-tuple is possible ($W = \emptyset$). On the other hand, suppose $W \neq \emptyset$ and choose $u_k \in W$.

Put $R_k := \langle u_1 z_1, \dots, u_k z_k\rangle$, and $N_i := N_{P_0}(R_i)$ for $i = 0, \dots, k$. Then $N_k \leqslant N_{k-1} \leqslant \dots \leqslant N_0 = P_0$ because $R_{i-1} = R_i \cap P_{k-1}$. Indeed, since $u_k z_k$ normalises R_{k-1}, the elements of the form $R_{k-1}vz_k$ $(v \in P_0)$ in $N_{P_k}(R_{k-1})/R_{k-1}$ are precisely those of the form $R_{k-1}uu_k z_k$ with $u \in N_{k-1}$. Hence we can choose W as any set with the property that vz_k $(v \in W)$ is a set of representatives of the N_{k-1}-conjugacy classes of elements in $\{uu_k z_k \mid u \in N_{k-1} \text{ and } (uu_k z_k)^p \in R_{k-1}\}$. Note that $R_{k-1}N_{k-1}u_k z_k$ is invariant under N_{k-1}. Indeed, if $w \in N_{k-1}$, then $w^{-1}u_k z_k w$ lies in $P_{k-1}u_k z_k$ (since $P_{k-1} \triangleleft P_k$) as well as in $N_P(R_{k-1})$. Thus $w^{-1}u_k z_k w \in R_{k-1}N_{k-1}u_k z_k$ as required because $N_P(R_{k-1}) \cap P_{k-1} = R_{k-1}N_{k-1}$.

EXAMPLE 2. We explain how to enumerate the Cayley graphs isomorphic to the 3-cube. In this case we can take $P := \langle (1,2,4,5), (1,4), (3,6)\rangle$ of order 16 as a Sylow 2-subgroup of Sym(6) which leaves the partition $\Pi := \{\{1,4\},\{2,5\},\{3,6\}\}$ invariant. The subgroup $P_0 := \langle w\rangle \leqslant P$ with $w := (1,2)(4,5)$ has index 8, and corresponds to a point stabiliser in the representation of degree 2^3 of P (see (2.2)). If we take $T := \{(1,4),(2,5),(3,6)\}$, then $Q := \langle T\rangle$ is an elementary Abelian normal subgroup of order 8 in P with $Q \cap P_0 = 1$. Indeed Cayley(Q,T) gives the standard representation of the 3-cube. We next find suitable elements, $z_1 := (3,6)$, $z_2 := (1,4)(2,5)$ and

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$z_3 := (2,5)$, so that $1 \lhd \langle z_1\rangle \lhd \langle z_1, z_2\rangle \lhd \langle z_1, z_2, z_3\rangle = T$ is a P_0-composition series for T. We must now find the 3-tuples (u_1, u_2, u_3) with $u_i \in P_0$ for which $\langle u_1 z_1, u_2 z_2, u_3 z_3\rangle$ has order 8. Since z_1, z_2, z_3 and w all commute except for the relation $z_3^{-1} w z_3 = w z_2$, we find using (3.1) that the only 3-tuples with the required property are (I,I,I), (I,I,w), (w,I,I) and (w,I,w). It can be checked using (3.2) that the corresponding groups, say G_i $(i = 1,\dots,4)$, are not P_0-conjugate. In order to choose a set T_i of generators for G_i so that $\mathrm{Cayley}(G_i, T_i)$ is isomorphic to the 3-cube $(i = 1,\dots,4)$ we refer back to T. We must choose $T_i \subseteq G_i$ to consist of three elements such that right multiplication by the jth element of T_i maps P_0 onto $P_0 t_j$ where t_j is the jth element in T. This gives $T_1 = \{z_2 z_3, z_3, z_1\}$, $T_2 = \{w z_3 z_2, w z_3, z_1\}$, $T_3 = \{z_2 z_3, z_3, w z_1\}$ and $T_4 = \{w z_3 z_2, w z_3, w z_1\}$ and so we obtain the four Cayley graphs listed in Example 1. Since the pairs (G_i, T_i) are obviously mutually inequivalent, we are finished.

4. Cayley graphs isomorphic to the d-cube

Example 2 shows how to enumerate the Cayley graphs isomorphic to the 3-cube. For $d > 3$ the procedure is similar. It can be described as follows.

We start with a Sylow 2-subgroup P of $\mathrm{Sym}(2d)$ leaving the partition $\Pi := \{\{1, d+1\}, \{2, 2+d\}, \dots, \{d, 2d\}\}$ invariant, and the subgroup P_0 of P consisting of the permutations which map $\{1,\dots,d\}$ into itself. Then $Q := \langle (1, d+1), (2, 2+d), \dots (d, 2d)\rangle$ is an elementary Abelian normal subgroup of P of order 2^d such that $P = P_0 Q$ and $Q \cap P_0 = 1$. We can therefore choose z_i $(i = 1,\dots,d)$ in Q such that $1 \lhd \langle z_1\rangle \lhd \dots \lhd \langle z_1, z_2, \dots, z_d\rangle = Q$ is a P_0-composition series for Q. With these values of z_i we carry out the backtrack program sketched in Section 3 to find a complete set of representatives for the P-conjugacy classes of subgroups R of P which satisfy the condition (*). For each of these subgroups we find a set T_R of generators such that $\mathrm{Cayley}(R, T_R)$ is isomorphic to the d-cube. Finally extraneous pairs (R, T_R) which are equivalent to other pairs in our list are eliminated, first using a crude criterion based on conjugacy classes to distinguish groups which are obviously not isomorphic, and then using (1.3) to decide equivalence in the remaining undetermined cases.

All calculations were done using GAP 3.1 [**11**].

4.1 The Cayley graphs isomorphic to the 4-cube.

For $d = 4$, we obtained 22 P-conjugacy classes of subgroups R satisfying (*). These 22 classes yielded 14 equivalence classes of Cayley graphs isomorphic to the 4-cube. Generators for representatives of these classes are given by:

$[a4, c4, d4, e4]$, $[b4, f4, g4, h4]$, $[c4, d4, e4, f4]$, $[a4, b4, g4, h4]$, $[i4, j4, k4, l4]$,
$[e4, f4, i4, k4]$, $[a4, h4, m4, n4]$, $[c4, d4, j4, l4]$, $[b4, g4, o4, p4]$, $[e4, f4, m4, n4]$,
$[a4, h4, i4, k4]$, $[c4, d4, o4, p4]$, $[q4, r4, s4, t4]$, $[e4, n4, u4, v4]$

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where

$a4 := (3,7),\ b4 := (4,8),\ c4 := (2,4,6,8),\ d4 := c4^{-1},\ e4 := (1,5)(2,4)(6,8),$
$f4 := (2,4)(3,7)(6,8),\ g4 := (2,6),\ h4 := (1,5),\ i4 := (1,3)(2,4,6,8)(5,7),$
$j4 := (1,3,5,7)(2,4)(6,8),\ k4 := i4^{-1},\ l4 := j4^{-1},\ m4 := (1,3)(4,8)(5,7),$
$n4 := (1,3)(2,6)(5,7),\ o4 := (1,3,5,7),\ p4 := o4^{-1},\ q4 := (1,2,3,4,5,6,7,8),$
$r4 := (1,2,7,8,5,6,3,4),\ s4 := r4^{-1},\ t4 := q4^{-1},\ u4 := (1,2,3,8,5,6,7,4),\ v4 := u4^{-1}.$

Many of the groups involved are isomorphic. For example, the first group is isomorphic to the second via the mapping $a4 \mapsto h4, c4 \mapsto a4c4$ and $e4 \mapsto b4$.

4.2 THE CAYLEY GRAPHS ISOMORPHIC TO THE 5-CUBE.

For $d = 5$, we obtained 68 P-conjugacy classes of subgroups R satisfying (*). These 68 classes yielded 45 equivalence classes of Cayley graphs isomorphic to the 5-cube. Generators for representatives of these classes are given by:

$[a5, c5, b5, d5, e5],\ [f5, g5, h5, d5, i5],\ [d5, j5, l5, k5, m5],\ [n5, o5, p5, d5, q5],$
$[c5, d5, j5, k5, e5],\ [g5, r5, d5, s5, i5],\ [a5, b5, d5, l5, m5],\ [f5, h5, t5, d5, u5],$
$[c5, r5, d5, s5, e5],\ [g5, d5, j5, k5, i5],\ [a5, b5, t5, d5, u5],\ [v5, x5, d5, y5, w5],$
$[c5, z5, r5, s5, e5],\ [z5, aa5, bb5, cc5, dd5],\ [a5, b5, z5, t5, u5], [z5, v5, x5, y5, w5],$
$[z5, aa5, s5, bb5, e5],\ [ee5, a5, c5, b5, e5],\ [f5, g5, ee5, h5, i5],\ [ee5, j5, l5, k5, m5],$
$[ee5, r5, t5, s5, u5],\ [f5, ee5, c5, h5, e5],\ [g5, ee5, a5, b5, i5],\ [ee5, r5, l5, s5, m5],$
$[ee5, t5, j5, k5, u5],\ [ee5, c5, j5, k5, e5],\ [g5, ee5, r5, s5, i5],\ [ee5, a5, b5, l5, m5],$
$[f5, ee5, h5, t5, u5],\ [ee5, c5, r5, s5, e5],\ [g5, ee5, j5, k5, i5],\ [f5, ee5, h5, l5, m5],$
$[ee5, a5, b5, t5, u5],\ [g5, ee5, a5, b5, e5],\ [f5, ee5, c5, h5, i5],\ [ff5, f5, g5, h5, i5],$
$[ff5, j5, l5, k5, m5],\ [ff5, c5, j5, k5, e5],\ [ff5, a5, b5, l5, m5],\ [ff5, f5, h5, t5, u5],$
$[ff5, aa5, s5, bb5, e5]$

where

$a5 := (2,4,7,9),\ b5 := a5^{-1},\ c5 := (2,4)(3,8)(7,9),\ d5 := (1,3)(2,4)(5,10)(6,8)(7,9),$
$e5 := (1,6)(2,4)(7,9),\ f5 := (4,9),\ g5 := (3,8),\ h5 := (2,7),\ i5 := (1,6),$
$j5 := (1,3)(2,4,7,9)(6,8),\ k5 := j5^{-1},\ l5 := (1,3,6,8)(2,4)(7,9),\ m5 := l5^{-1},$
$n5 := (1,2)(3,4,8,9)(6,7),\ o5 := n5^{-1}\ p5 := (1,2,6,7)(3,4)(8,9),\ q5 := p5^{-1},$
$r5 := (1,3)(4,9)(6,8),\ s5 := (1,3)(2,7)(6,8),\ t5 := (1,3,6,8),\ u5 := t5^{-1},$
$v5 := (1,2,3,4,6,7,8,8),\ w5 := v5^{-1},\ x5 := (1,2,8,9,6,7,3,4),\ y5 := x5^{-1},$
$z5 := (1,2)(3,4)(5,10)(6,7)(8,9),\ aa5 := (1,2,3,9,6,7,8,4),\ bb5 := aa5^{-1},$
$cc5 := (1,4,3,2,6,9,8,7),\ dd5 := cc5^{-1},\ ee5 := (2.4)(5,10)(7,9),\ ff5 := (5,10).$

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4.3 The Cayley graphs isomorphic to the 6-cube.

For $d = 6$, 894 P-conjugacy classes of subgroups R satisfying (*) were obtained. These 894 classes yielded 238 equivalence classes of Cayley graphs isomorphic to the 6-cube.

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