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ON SETS OF PP-GENERATORS OF FINITE GROUPS

JAN KREMPA and AGNIESZKA STOCKA✉

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Abstract

The classes of finite groups with minimal sets of generators of fixed cardinalities, named $\mathcal{B}$-groups, and groups with the basis property, in which every subgroup is a $\mathcal{B}$-group, contain only p-groups and some $\{p, q\}$-groups. Moreover, abelian $\mathcal{B}$-groups are exactly p-groups. If only generators of prime power orders are considered, then an analogue of property $\mathcal{B}$ is denoted by $\mathcal{B}_{pp}$ and an analogue of the basis property is called the pp-basis property. These classes are larger and contain all nilpotent groups and some cyclic q-extensions of p-groups. In this paper we characterise all finite groups with the pp-basis property as products of p-groups and precisely described $\{p, q\}$-groups.

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1. Preliminaries

All groups considered here are finite. For any group G, let $\Phi(G)$ denote the Frattini subgroup of G. An element $g \in G$ will be called a *pp-element* if it is of a prime power order, while by a p-element we mean an element whose order is a power of a prime number p. As in [3], groups containing only pp-elements will be called *CP-groups*. For other notation, terminology and results one can consult, for example, [4, 10].

A subset X of a group G will be called:

- *g-independent* if $\langle Y, \Phi(G)\rangle \neq \langle X, \Phi(G)\rangle$ for every $Y \subset X$;
- a *generating set* if $\langle X\rangle = G$ (or equivalently $\langle X, \Phi(G)\rangle = G$);
- a *g-base* of G if X is a g-independent generating set of G.

In connection with these notions the following invariants are considered (see [2, 5, 8]):

$$m(G) = \sup_X |X| \quad \text{and} \quad d(G) = \inf_X |X|, \tag{1.1}$$

where X runs over all g-bases of G. Then the following properties are defined (see [2, 6, 9]): a group G has *property $\mathcal{B}$ (is a $\mathcal{B}$-group)* if $d(G) = m(G)$ and G has *the basis property* if all its subgroups are $\mathcal{B}$-groups.

Groups with the basis property and $\mathcal{B}$-groups are completely described (see [2, 6, 9]). These classes contain only p-groups and some $\{p, q\}$-groups, are homomorphically closed and soluble. Among direct products, they contain only p-groups.

In characterisations of $\mathcal{B}$-groups and groups with the basis property, as in [2, 6, 9], some cyclic q-extensions of p-groups for $q \neq p$ play an important role. We recall this construction here.

Example 1.1 [5, 6, 9]. Let $p \neq q$ be primes, m be a nonnegative integer and $\mathbb{K} = \mathbb{F}_p[\rho]$ be the field extension of the prime field $\mathbb{F}_p$, where ρ is a primitive q^mth root of $1 \in \mathbb{K}^*$. Let also $Q = \langle x \rangle$ be a cyclic group of order q^m and let V be a vector space over $\mathbb{K}$. Then we can consider an action $\phi : Q \longrightarrow Aut_{\mathbb{K}} V$ via multiplication:

$$x^j\phi : v \longrightarrow v\rho^j \quad \text{for } j = 1, \ldots, q^m.$$

We can also construct the semidirect product $G_\phi = V \rtimes_\phi Q$ with the above-mentioned action. The extension G_ϕ will be invoked here as a *scalar extension*. As in [9], one can check that G_ϕ is a group with the basis property, is a CP-group and $\Phi(G_\phi) = 1$. Moreover,

$$d(V) = [\mathbb{K} : \mathbb{F}_p] \cdot \dim_{\mathbb{K}}(V) \quad \text{and} \quad d(G_\phi) = \dim_{\mathbb{K}}(V) + 1. \tag{1.2}$$

The classes of $\mathcal{B}$-groups and groups with the basis property are rather narrow. Thus, we proposed in [6] a modification of these notions. A subset $X \subseteq G$ is said there to be:

- *pp-independent* if X is a set of pp-elements and is g-independent;
- a *pp-generating set* if X is a set of pp-elements and is a generating set;
- a *pp-base* of G if X is a pp-independent generating set of G.

As in formula (1.1), the following invariants can be considered:

$$m_{\mathrm{pp}}(G) = \sup_X |X| \quad \text{and} \quad d_{\mathrm{pp}}(G) = \inf_X |X|,$$

where X runs over all pp-bases of G. Also, from [6], a group G has *property* $\mathcal{B}_{\mathrm{pp}}$ *(is a* $\mathcal{B}_{pp}$*-group)* if $d_{\mathrm{pp}}(G) = m_{\mathrm{pp}}(G)$ and G has the *pp-basis property* if all its subgroups are $\mathcal{B}_{\mathrm{pp}}$-groups.

Proposition 1.2 [6]. *A group G has the basis property if and only if it has the pp-basis property and is a CP-group.*

Theorem 1.3 [6]. *Let G be a group and $H \leq G$ be a normal subgroup.*

(1) *If G is a $\mathcal{B}_{pp}$-group, then G/H is also a $\mathcal{B}_{pp}$-group.*
(2) *If G has the pp-basis property, then G is soluble and G/H has the pp-basis property.*

To exhibit a difference between g-notions and pp-notions explicitly, let us consider a modification of Example 1.1, with some data which will be needed later.

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EXAMPLE 1.4 ([6], §5). Let $p \neq q$ be primes, $m \geq l \geq 0$ and $\mathbb{K}$ a field of characteristic p with a primitive q^lth root ρ of $1 \in \mathbb{K}^*$. Let $Q = \langle x \rangle$, V, the action $\phi : Q \longrightarrow Aut_{\mathbb{K}} V$ and $G_\phi = V \rtimes_\phi Q$ be as in Example 1.1. The centraliser of V in Q is equal to $\langle x^{q^l} \rangle$ and $G_\phi / \langle x^{q^l} \rangle$ is a scalar extension and hence a CP-group. The group G_ϕ will be named here *a generalised scalar extension*. As in [6, 7] one can check that G_ϕ is a group with the pp-basis property, but for $l < m$ it does not have the basis property. Moreover, for $l = 0$, we have $G_\phi = P \times Q$.

2. Main results

In this section we formulate structure theorems for groups with the pp-basis property. For this purpose a group G will be called *(coprimely) indecomposable* if it is not a direct product of nontrivial groups with coprime orders. This class of groups contains CP-groups, generalised scalar extensions with $l > 0$ and all other $\{p, q\}$-groups with a nonnormal Sylow subgroup. It is also easy to check that every group is a direct product of indecomposable groups with coprime orders, and this decomposition is unique up to the order of factors. We also have the following result.

THEOREM 2.1 [6]. *Let G_1 and G_2 be groups with coprime orders.*

(1) *G_1 and G_2 are $\mathcal{B}_{\text{pp}}$-groups if and only if $G_1 \times G_2$ is a $\mathcal{B}_{\text{pp}}$-group;*
(2) *G_1 and G_2 have the pp-basis property if and only if $G_1 \times G_2$ has the pp-basis property.*

COROLLARY 2.2. *Let G be a group. Then G has the pp-basis property if and only if it is a direct product of indecomposable groups with the pp-basis property of coprime orders. This decomposition is unique up to the order of factors.*

We quote, after [6, 7], some properties of $\mathcal{B}_{pp}$-groups and groups with the pp-basis property needed here.

THEOREM 2.3. *Let $G = P \rtimes Q$ be a nontrivial semidirect product, where P is a p-group and Q is a cyclic q-group, for primes $p \neq q$. The following conditions are equivalent:*

(1) *G is a $\mathcal{B}_{\text{pp}}$-group;*
(2) *$G/\Phi(P)$ is a generalised scalar extension;*
(3) *$G/\Phi(G)$ is a scalar extension;*
(4) *G is a $\mathcal{B}$-group.*

THEOREM 2.4. *Let $G = P \rtimes Q$ be a semidirect product, where P is a p-group and Q is a cyclic q-group, for primes $p \neq q$. Then the following conditions are equivalent:*

(1) *G has the pp-basis property;*
(2) *for every subgroup $H \leq G$, either the group $H/\Phi(H)$ is a scalar extension or $H = P_H \times Q_H$, where $P_H = P \cap H$ and Q_H is a Sylow q-subgroup of H.*

Our characterisation of indecomposable groups with the pp-basis property is given by the following results.

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THEOREM 2.5. *Let G be an indecomposable group with the pp-basis property. Then G is either a p-group or a $\{p, q\}$-group.*

THEOREM 2.6. *Let G be an indecomposable $\{p, q\}$-group with the pp-basis property. Then G is either a cyclic q-extension of a p-group or a cyclic p-extension of a q-group.*

Due to the above theorems, we can have various characterisations of indecomposable $\{p, q\}$-groups with the pp-basis property, by applying Theorems 2.3 and 2.4. With the help of the above theorems, the Burnside basis theorem and Corollary 2.2 we obtain a structure theorem for groups with the pp-basis property.

THEOREM 2.7. *Let G be a group. Then G has the pp-basis property if and only if it is one of the following groups:*

(1) *a p-group;*
(2) *an indecomposable $\{p, q\}$-group with the pp-basis property;*
(3) *a direct product of groups given in (1) and (2) with pairwise-coprime orders.*

As immediate consequences, we obtain the following results.

COROLLARY 2.8. *Every group with the pp-basis property is nilpotent-by-abelian.*

COROLLARY 2.9. *Let G be a Frattini-free group. Then G is a group with the pp-basis property if and only if G is a direct product of some elementary abelian p-groups and some scalar extensions, with coprime orders.*

3. Proofs

LEMMA 3.1. *Let G be an indecomposable semidirect product of a normal p-subgroup $P \neq 1$ by a q-subgroup $Q \neq 1$. If G has the pp-basis property, then Q is cyclic.*

PROOF. From the assumption, we immediately have $\Phi(P) \lhd G$. Thus, applying Theorem 1.3, we can suppose that P is an elementary abelian p-group. Let C stand for $C_Q(P)$ and $x \in Q \setminus C$. Then $C \lhd G$ and, by assumption, $\langle P, x\rangle$ is a $\mathcal{B}_{pp}$-group. Suppose that $C \cap \langle x\rangle = \langle x^k\rangle$. By Theorem 1.3, $G_x = \langle P, x\rangle/\langle x^k\rangle \simeq \langle PC, x\rangle/C$ is a $\mathcal{B}_{pp}$-group. It follows from Theorem 2.3 that G_x is a scalar extension and so a CP-group. Thus, G/C is also a CP-group and so Q/C acts regularly on P. Hence, by [4, Theorem 5.4.11], Q/C is either cyclic or generalised quaternion. As G/C is a CP-group, then, by Proposition 1.2, G/C has the basis property. Hence, from [9, Proposition 4.2], $Q/C = \langle x_1C\rangle$ for some $x_1 \in Q$.

Suppose that Q is not cyclic. Then there exists $x_2 \in C \setminus \Phi(Q)$. Let a be a nontrivial element of P. Since x_2x_1 acts fixed-point-freely on P, then $o(ax_2x_1) = o(x_2x_1)$ is a power of q. This implies that the sets $\{ax_2x_1, x_1\}$ and $\{a, x_1, x_2\}$ are pp-bases of $\langle a, x_1, x_2\rangle$, contrary to the assumption of the pp-basis property for G. Hence, Q has to be cyclic. □

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Proof of Theorem 2.6. We proceed by induction on $|G|$. Due to the above lemma, we should only take care about existence of a normal Sylow subgroup in G.

Let $G = PQ$, where P is a Sylow p-subgroup and Q is a Sylow q-subgroup of G. If $|G| = pq$, then the result follows easily. Let $|G| > pq$ and let us consider first the case $\Phi(G) \neq 1$. Then, by the induction assumption applied to $G/\Phi(G)$, we obtain that, for example, $P\Phi(G)$ is normal in G. Since P is a Sylow p-subgroup of $P\Phi(G)$, a Frattini argument yields that $N_G(P)\Phi(G) = G$ and hence $G = N_G(P)$ and so P is normal in G.

Now let $\Phi(G) = 1$. If $F(G)$ denotes the Fitting subgroup of G, then we have $F(G) = R \times S$, where R is a maximal normal p-subgroup of G and S is a maximal normal q-subgroup of G. As G is a soluble group, $F(G) \neq 1$. Obviously, $R \leq P$ and $S \leq Q$. If either P or Q is normal in G, then we are done.

Suppose that neither P nor Q is normal in G. Therefore, we may suppose that $1 \neq R \neq P$. Hence, P/R is not normal in G/R and so P/R is cyclic, by the induction assumption. If $Q \lhd QR$, then Q is a characteristic subgroup in $QR \lhd G$. Thus, Q is normal in G, which is a contradiction. Hence, Q is nonnormal in QR. So, Q is cyclic, by the induction assumption.

Since $\Phi(G) = 1$, R is an elementary abelian p-group. By [10, 5.2.13], it follows that there exists a subgroup H of G satisfying $G = R \rtimes H$. Thus, $H \simeq G/R$ is metacyclic and there exist elements $a \in Q$ and $b \in P \setminus R$ such that $H = \langle a, b\rangle$. If $H = \langle a\rangle \times \langle b\rangle$, then $P = R \rtimes \langle b\rangle$ is normal in G, which is a contradiction. So, by [10, 10.1.10], $H = \langle a, b \mid a^{q^m} = b^{p^n} = 1, a^b = a^r\rangle$ with $r^{p^n} \equiv 1 \pmod{q^m}$ and $(q^m, r-1) = 1$. Let $1 \neq z \in R \cap Z(P)$. Then we obtain $(za)^{-1}(za)^b = a^{r-1}$. Since $(r-1, q^m) = 1$, we have $\langle a^{r-1}\rangle = \langle a\rangle$. This implies that $\langle za, b\rangle = \langle z, a, b\rangle$ and $o(az) = q^b$. So, the sets $\{za, b\}$ and $\{z, a, b\}$ are pp-bases of the group $\langle z, a, b\rangle$, contrary to our assumption. Thus, either P or Q has to be normal in G. □

Lemma 3.2. *Let G be a group with the pp-basis property. If $|\pi(G)| = 3$, then there exists a Sylow p-subgroup of G which is a direct factor of G.*

Proof. Let $\pi(G) = \{p_1, p_2, p_3\}$. Since G is soluble by Theorem 1.3, there exist Sylow p_i-subgroups P_i of G, for $i = 1, 2, 3$, such that $G = P_1P_2P_3$ and P_iP_j are subgroups of G for all $i, j \in \{1, 2, 3\}$. Furthermore, for all $i \neq j$, either $P_iP_j = P_i \times P_j$ or P_iP_j is indecomposable.

If $P_1, P_2, P_3 \lhd G$, then $G = P_1 \times P_2 \times P_3$. If a Sylow subgroup of G, say P_3, is not normal in G, then either $P_3 \ntriangleleft P_2P_3$ or $P_3 \ntriangleleft P_1P_3$. Hence, by Lemmas 2.6 and 3.1, P_3 is cyclic. Thus, [10, 10.1.10] implies that G has a normal Sylow p_i-subgroup for some $i \in \{1, 2, 3\}$. So, it is enough to consider the following cases:

(1) $P_1 \lhd G$ and $P_2, P_3 \ntriangleleft G$;

(2) $P_1, P_2 \lhd G$ and $P_3 \ntriangleleft G$.

Case 1. In view of Theorem 1.3, by passing to the quotient, we can assume that P_1 is elementary abelian. By arguments as above, P_2 and P_3 are cyclic. Let $P_2 = \langle x\rangle$ and $P_3 = \langle y\rangle$. By Lemma 2.6, one of the Sylow subgroups of P_2P_3 is normal in P_2P_3; we take $P_2 \lhd P_2P_3$. In this case $P_2 \ntriangleleft P_1P_2$.

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Assume that $P_3 \subseteq C_G(P_1)$. Then $P_3 \ntrianglelefteq P_2P_3$ and so y acts fixed-point-freely on P_2. Let $a \in P_1$. Since x acts on P_1 fixed-point-freely (see [6, Proposition 2.4]), we have $o(ax) = o(x)$. Thus, $\langle ax, y\rangle = \langle ax, ax^y, y\rangle = \langle x^{-1}x^y, ax, y\rangle = \langle a, x, y\rangle$ and the sets $\{ax, y\}$, $\{a, x, y\}$ are pp-bases of $\langle a, x, y\rangle$, which is a contradiction.

So, let $P_3 \nsubseteq C_G(P_1)$. Consider the quotient $\bar{G} = G/C_{P_2P_3}(P_1)$. Then every pp-element of $\bar{P}_2\bar{P}_3$ acts on $\bar{P}_1$ fixed-point-freely. From [10, 10.5.5], it follows that $\bar{P}_2\bar{P}_3$ cannot act on $\bar{P}_1$ regularly. Hence, $\bar{P}_2\bar{P}_3$ is not a CP-group. So, there exist elements $\bar{x}_1 \in P_2$ and $\bar{y}_1 \in P_3$ such that $o(\bar{x}_1) = p_2$, $o(\bar{y}_1) = p_3$ and $\bar{x}_1\bar{y}_1 = \bar{y}_1\bar{x}_1$. Let $\bar{a} \in \bar{P}_1$. Since $\bar{x}_1, \bar{y}_1$ act fixed-point-freely on P_1, $o(\bar{a}\bar{x}_1) = o(\bar{x}_1)$ and $o(\bar{a}\bar{y}_1) = o(\bar{y}_1)$. Furthermore, $\langle \bar{a}\bar{x}_1, \bar{a}\bar{y}_1\rangle = \langle \bar{x}_1\bar{y}_1^{-1}, \bar{a}\bar{y}_1\rangle = \langle \bar{x}_1, \bar{y}_1, \bar{a}\rangle$. It follows that the sets $\{\bar{a}\bar{x}_1, \bar{a}\bar{y}_1\}$ and $\{\bar{x}_1, \bar{y}_1, \bar{a}\}$ are pp-bases of $\langle \bar{x}_1, \bar{y}_1, \bar{a}\rangle$, which is a contradiction.

Case 2. If $P_3 \subseteq C_G(P_1)$ or $P_3 \subseteq C_G(P_2)$, then P_1 or respectively P_2 is a direct factor of G. So, assume that $P_3 \nsubseteq C_G(P_1)$ and $P_3 \nsubseteq C_G(P_2)$. Hence, by Lemma 3.1, $P_1 \rtimes P_3$ and $P_2 \rtimes P_3$ are as in Theorem 2.6 and it follows that P_3 is a cyclic group. Let $P_3 = \langle y\rangle$. Analogously to the previous case, we may assume that P_1, P_2 are elementary abelian. So, we can take $x_1 \in P_1$, $x_2 \in P_2$ such that $x_1^y \neq x_1$, $x_2^y \neq x_2$. Thus, x_1y, x_2y are pp-elements and further $\langle x_1y, x_2y\rangle = \langle x_2y, x_1x_2^{-1}\rangle = \langle x_1, x_2, y\rangle$. This implies that $\{x_1y, x_2y\}$ and $\{x_1, x_2, y\}$ are pp-bases of $\langle x_1, x_2, y\rangle$. Hence, G does not have the pp-basis property. □

Proof of Theorem 2.5. By Theorem 1.3, G is a soluble group. From [4, Theorem 6.4.11], there exist Sylow p_i-subgroups P_i, for $i = 1, \ldots, n$, satisfying $G = P_1P_2 \cdot \ldots \cdot P_n$ and P_iP_j is a subgroup of G for $i, j \in \{1, \ldots, n\}$. If $n = 1$, then G is a p-group. If $n = 2$, then G is an indecomposable $\{p, q\}$-group.

Suppose that $n > 2$. By assumption, P_1 is not a direct factor of G. Thus, there exists P_k for some $2 \leq k \leq n$ such that $P_1 \nsubseteq C_G(P_k)$. We can take $k = 2$. Therefore, P_1P_2 is an indecomposable group with the pp-basis property. Lemma 3.2 asserts that $P_1P_2 \subseteq C_G(P_j)$ for every $j = 3, \ldots, n$. Thus, $G = (P_1P_2) \times (P_3 \cdot \ldots \cdot P_n)$, which is a contradiction. □

4. pp-matroid groups

From the Burnside basis theorem we know that, if G is a p-group, then every g-independent (pp-independent) subset of G can be extended to a g-base (pp-base) of G. However, this need not be true in general, even for CP-groups with the basis property (the pp-basis property).

Example 4.1. Let us follow the notation from Example 1.1. In addition, suppose that q does not divide $p - 1$ and let V be the additive group of $\mathbb{K}$. If we take suitable ϕ and $G_\phi = V \rtimes_\phi Q$, then, by formula (1.2), $d(G_\phi) = d_{\mathrm{pp}}(G_\phi) = 2$ and $d(V) = d_{\mathrm{pp}}(V) = [\mathbb{K} : \mathbb{F}_p] \geq 2$. Thus, for $Q \neq 1$, g-bases (pp-bases) of V cannot be extended to g-bases (pp-bases) of G_ϕ.

Recall, as in [11], that G is a *matroid group* if G has property $\mathcal{B}$ and every g-independent subset of G is contained in a g-base of G. Some characterisations of matroid groups can be found in [1, 2, 11].

Analogously, we can give a pp-version of the notion of a matroid group: a group G is a *pp-matroid group* if G has property $\mathcal{B}_{\text{pp}}$ and every pp-independent subset of G is contained in a pp-base of G. We already noted that every p-group is a matroid and a pp-matroid group, but groups from Example 4.1 are neither matroid nor pp-matroid. It is also easy to check that every matroid group is pp-matroid. The converse implication is not true, because every matroid group has to be indecomposable. On the other hand, from Theorem 2.1, one can obtain the following result.

Theorem 4.2. *Let G_1 and G_2 be groups of coprime orders. Then $G_1 \times G_2$ is a pp-matroid group if and only if both G_1 and G_2 are pp-matroid groups.*

Based on these definitions, some analogues of properties of matroid groups can be proved for pp-matroid groups.

Theorem 4.3. *Let G be a group and $H \leq G$ be a normal subgroup such that $H \leq \Phi(G)$. Then G is a pp-matroid group if and only if G/H is a pp-matroid group.*

Proposition 4.4. *Let G be a Frattini-free pp-matroid group. If H is a proper subgroup of G, then H is a $\mathcal{B}_{pp}$-group and $d_{\text{pp}}(H) < d_{\text{pp}}(G)$.*

Proof. Let X be a pp-base of H. By assumption, $\langle X\rangle \neq G$ and X is a pp-independent subset of G. However, X can be embedded in a pp-base B of G. Hence, we obtain $d_{\text{pp}}(H) < d_{\text{pp}}(G)$. It is easy to check that H is a $\mathcal{B}_{pp}$-group. □

Theorem 4.5. *Let G be a group and let $H = G/\Phi(G)$. The group G is a pp-matroid group if and only if one of the following holds:*

(1) *G is a p-group for some prime p;*
(2) *$H = P \rtimes Q$ is a scalar extension for primes $p \neq q$, where $q|(p-1)$ and Q is cyclic of order q;*
(3) *G is a direct product of groups given in* (1) *and* (2) *with coprime orders.*

Proof. Let G be a pp-matroid group. Then, by Theorem 4.3, H has the pp-basis property. Hence, by Theorem 2.7, H is a direct product of p-groups and indecomposable $\{p, q\}$-groups with the pp-basis property. Hence, in view of Theorem 4.2, we can assume that H is a Frattini-free indecomposable $\{p, q\}$-group with the pp-basis property, which is pp-matroid. Then H is a scalar extension of an elementary abelian p-group P by a cyclic q-group $Q = \langle x\rangle$. Suppose that Q has order greater than q. Then a pp-base of $P \rtimes \langle x^q\rangle$ cannot be extended to a pp-base of H. So, $|Q| = q$.

From (1.2),

$$d(H) = \dim_{\mathbb{K}}(P) + 1 \quad \text{and} \quad d(P) = [\mathbb{K} : \mathbb{F}_p] \cdot \dim_{\mathbb{K}}(P).$$

On the other hand, by Proposition 4.4, $d(P) < d(H)$. Hence, $[\mathbb{K} : \mathbb{F}_p] = 1$ and so $q|(p-1)$.

Conversely, suppose that H is a group as in (2). Since H is a CP-group, by [2, Theorem 5.1] we know that H is a matroid group and so H is pp-matroid. Hence, with the help of Theorem 4.2, the proof can be completed. □

Corollary 4.6. *Let G be a Frattini-free group. Then G is a matroid group if and only if G is an indecomposable pp-matroid group.*

Example 4.7 [6, Example 3.3]. Let $p \neq q$ be primes such that q is odd and $q|(p-1)$. Consider the group

$$P = \langle a, b, c \mid a^p = b^p = c^p = 1 = [a, c] = [b, c], c = [a, b] \rangle.$$

Let $Q = \langle x \rangle$ be the cyclic group of order q. There exists an element $i \in \mathbb{F}_p^*$ of order q. Thus, the group Q acts on P in the following way:

$$a^{x^j} = a^{i^j} \quad \text{and} \quad b^{x^j} = b^{i^j} \quad \text{for } 1 \leq j \leq q.$$

It is easy to observe that G is a CP-group and we have $\Phi(G) = \Phi(P) = \langle c \rangle$. Thus, G is a $\mathcal{B}$-group and a $\mathcal{B}_{pp}$-group. However, if $H = \langle a, c, x \rangle$, then $\Phi(H) = 1$ and H is not a scalar extension and not a $\mathcal{B}_{pp}$-group. Hence, G is a pp-matroid CP-group, but does not satisfy the pp-basis property, because H is not a $\mathcal{B}_{\mathrm{pp}}$-group and is not pp-matroid. Obviously, G is also a matroid group and H is not a matroid group.

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JAN KREMPA, Institute of Mathematics, University of Warsaw,
Banacha 2, 02-097 Warszawa, Poland
e-mail: jkrempa@mimuw.edu.pl

AGNIESZKA STOCKA, Institute of Mathematics,
University of Białystok, Akademicka 2, 15-267 Białystok, Poland
e-mail: stocka@math.uwb.edu.pl