Compositio Math. 140 (2004) 809–817
DOI: 10.1112/S0010437X03000691

On indices of meromorphic 1-forms

W. Ebeling and S. M. Gusein-Zade

Abstract

We discuss meromorphic 1-forms on compact complex manifolds and on complete intersections with isolated singularities. We give a Poincaré–Hopf type formula for them, i.e. we express the Euler characteristic in terms of singularities of a meromorphic 1-form. For that we introduce a suitable notion of an index of a germ of a meromorphic 1-form (with an additional structure) on an isolated complete intersection singularity. For a meromorphic 1-form on a complete intersection V with isolated singularities the indices of the singular points sum up to (plus–minus) the Euler characteristic of a smoothing of V.

Introduction

For an isolated zero of a germ of a holomorphic vector field on a (smooth) complex manifold the notion of its index (or multiplicity) is defined. For a germ of a vector field $X = \sum_{i=1}^{n} X_i(x)\,\partial/\partial x_i$ on $(\mathbb{C}^n, 0)$ it is equal to the dimension of the local algebra $\mathcal{O}_{\mathbb{C}^n,0}/(X_1, \dots, X_n)$. If one has a holomorphic vector field with isolated zeros on a compact complex manifold M^n, then the sum of the indices of all zeros is equal to the Euler characteristic $\chi(M)$ of the manifold, i.e. to the characteristic number $c_n(TM)[M]$ (TM is the tangent bundle of M). Since non-zero holomorphic vector fields (as well as holomorphic 1-forms) on a compact complex manifold rarely exist, it is interesting to consider meromorphic vector fields of which there are a lot (at least for projective manifolds). For a meromorphic vector field defined by a holomorphic section of the vector bundle $TM \otimes L$ (L is a holomorphic line bundle on M), the sum of the indices of its zeros is equal to the corresponding characteristic number $c_n(TM \otimes L)[M]$ of the vector bundle $TM \otimes L$ (see e.g. [BB70]).

For a germ of a holomorphic 1-form $\alpha = \sum_{i=1}^{n} A_i(x)\,dx_i$ on $(\mathbb{C}^n, 0)$ its index $\mathrm{ind}_0\alpha$ is defined in the same way; it is equal to $\dim \mathcal{O}_{\mathbb{C}^n,0}/(A_1, \dots, A_n)$. For a holomorphic 1-form with isolated zeros on a compact manifold M, the sum of the indices of its zeros is equal to the characteristic number $c_n(T^*M)[M] = (-1)^n\chi(M)$, where T^*M is the cotangent bundle (a Poincaré–Hopf type formula).

Let $(V^n, 0) \subset (\mathbb{C}^{n+k}, 0)$ be a germ of an isolated complete intersection singularity (ICIS) defined by equations $f_1 = \dots = f_k = 0$, and let α be a holomorphic 1-form on $(V, 0)$ (i.e., the restriction of a germ of a holomorphic 1-form $\sum_{i=1}^{n+k} A_i(x)\,dx_i$ on $(\mathbb{C}^{n+k}, 0)$) such that α has an isolated singular point at the origin on V. In this situation the index $\mathrm{ind}_{V,0}\alpha$ of the 1-form α was defined in [EG01, EG02]. It was shown that it is equal to the dimension of the factor-algebra of the algebra $\mathcal{O}_{\mathbb{C}^{n+k},0}$ of germs of functions on $(\mathbb{C}^{n+k}, 0)$ factorized by the ideal generated by $f_1, \dots, f_k$ and the $(k+1)\times(k+1)$-

Received 14 March 2002, accepted in final form 24 July 2002.
2000 Mathematics Subject Classification 32Q55, 14B05, 32A20.
Keywords: meromorphic 1-forms, singular points, indices.
Partially supported by the DFG-programme 'Global methods in complex geometry', grants RFBR–01–01–00739, NWO–RFBR–047.008.005, INTAS–00–259, NSh–1272.2003.1.

minors of the matrix

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_{n+k}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_k}{\partial x_1} & \cdots & \dfrac{\partial f_k}{\partial x_{n+k}} \\ A_1 & \cdots & A_{n+k} \end{pmatrix}.$$

Let V^n be a complete intersection in $\mathbb{CP}^{n+k}$ with isolated singularities and let α be a holomorphic 1-form on V (which means that the germ of α at each point is holomorphic). Singular points of α are singular points of the variety V itself and zeros of α on the smooth part of V. Suppose that the singular points of the 1-form α are isolated. Then the sum of their indices is equal to the characteristic number $c_n(T^*\widetilde{V})[\widetilde{V}] = (-1)^n\chi(\widetilde{V})$ of a smoothing $\widetilde{V}$ of the variety V.

Let α be a meromorphic 1-form on a compact complex manifold M^n which means that α is a holomorphic 1-form outside of a positive divisor D and in a neighbourhood of each point of M the form α can be written as $\widehat{\alpha}/F$ where $F = 0$ is a local equation of the divisor D and $\widehat{\alpha}$ is a holomorphic 1-form. Let L be the line bundle associated to the divisor D, i.e. L has a holomorphic section s with zeros on D. Then $\omega = s\alpha$ is a holomorphic section of the vector bundle $T^*M\otimes L$. In some constructions (say, as in [BB70]) one defines a meromorphic 1-form on M simply as a holomorphic section of the tensor product $T^*M \otimes L$ for a holomorphic line bundle L. This definition is somewhat different from the one formulated above. For example, in this case only the class of the divisor of poles of a meromorphic 1-form is defined, not the divisor itself. Moreover, in this setting, the value of a meromorphic 1-form on a vector field is not a function, but a section of the line bundle L. In the sequel we use the notation ω for a section of the vector bundle $T^*M \otimes L$, for short calling it a meromorphic 1-form as well. Suppose that the section ω has isolated zeros. Then the sum of their indices is equal to the characteristic number $c_n(T^*M \otimes L)[M]$ and thus depends on L.

For a meromorphic 1-form on a smooth compact complex curve M the characteristic number $c_1(T^*M)[M] = -\chi(M)$ is equal to the number of zeros minus the number of poles counted with multiplicities. Therefore, to express the Euler characteristic of a compact manifold in terms of singularities of a meromorphic 1-form one has to take the divisor D of its poles into account as well. If a meromorphic 1-form on a manifold M^n is defined simply as a section of $T^*M \otimes L$, the divisor of poles is not defined and thus one also cannot define singular points of the form on its pole locus.

The aim of this paper is to give a Poincaré–Hopf type formula for meromorphic 1-forms, i.e. to express the Euler characteristic of a compact manifold or of a smoothing of a complete intersection in terms of singularities of a meromorphic 1-form. For that we introduce a suitable notion of an index of a germ of a meromorphic 1-form (with an additional structure) on an ICIS, so that the indices of the singular points sum up to (plus–minus) the Euler characteristic of a smoothing. There are several descriptions of the index of a meromorphic 1-form, one of them being the alternating sum of the dimensions of certain algebras.

1. Singular points of meromorphic 1-forms

A Poincaré–Hopf type formula can be considered as one describing a localization of an invariant (say, of the Euler characteristic) of a manifold at singular points of, say, a vector field or a 1-form, i.e. a representation of the invariant as the sum of integer invariants ('indices') corresponding to singular points. Therefore one first has to define singular points. Let M^n be a complex manifold, let L be a line bundle on M, and let ω be a holomorphic section of $T^*M \otimes L$. One has zeros of ω on M, but the pole locus of ω is not well defined. In order to discuss singular points of ω on its

https://doi.org/10.1112/S0010437X03000691 Published online by Cambridge University Press

pole locus (which is necessary as we saw in the example when M was a curve), we have to fix this locus. This means that we have to choose a holomorphic section $s = s_1$ of the line bundle L or to choose its zero divisor $D = D_1$. One can say that we have to consider $\alpha = \omega/s$, i.e. to proceed with our initial definition of a meromorphic 1-form.

For the further setting we suppose that the divisor D of poles of the 1-form ω is non-singular (in particular, reduced). It is not very essential for this section but makes the discussion simpler. Since D is a submanifold of M, there is a well defined map $T^*M|_D \to T^*D$ and thus $(T^*M \otimes L)|_D \to T^*D \otimes L|_D$ (the restriction of meromorphic 1-forms to D). (Using the 1-form α, one can also define only a section of the vector bundle $T^*D \otimes L|_D$, but not a meromorphic 1-form on D with precisely defined pole locus.)

Let ω_1 be the restriction of ω to D_1. It is a holomorphic section of the vector bundle $T^*D \otimes L|_D$. Its zeros are well defined and should be considered as singular points of the meromorphic 1-form ω on the pole locus. To discuss its singular points on its pole locus we again have to fix a divisor. Suppose that there exists a (positive) divisor D_2 on M which is the zero locus of another section s_2 of the same line bundle L and which intersects D_1 transversally (in particular, this means that D_2 is non-singular at its intersection points with D_1 and $D_1 \cap D_2$ is non-singular as well). One has the restriction of ω to $D_1 \cap D_2$ and its zeros there.

Going on in this way we arrive at the situation when we have fixed n divisors $D_1, \dots, D_n$ (zeros of sections $s_1, \dots, s_n$ of the line bundle L) so that, for each $i = 1, \dots, n$, $D_1 \cap \dots \cap D_i$ is non-singular. The set of singular points of the 1-form ω is the union of the zeros of ω itself and of the restrictions of ω to $D_1 \cap \dots \cap D_i$ for all $i = 1, \dots, n$. (For $i = n$ the intersection $D_1 \cap \dots \cap D_n$ is zero-dimensional and all its points should be considered as zeros.) One can say that we have to consider a collection of meromorphic 1-forms $\omega/s_1, \dots, \omega/s_n$ on M proportional to each other.

2. A Poincaré–Hopf type formula for meromorphic 1-forms

Let M^n be a compact complex manifold, let ω be a meromorphic 1-form on M, that is, a holomorphic section of the bundle $T^*M \otimes L$ where L is a holomorphic line bundle with non-zero holomorphic sections. Suppose that $D_1 = D, D_2, \dots, D_n$ are zero divisors of holomorphic sections of the line bundle L such that, for each $i = 1, \dots, n$, $D_1 \cap \dots \cap D_i$ is non-singular. Suppose that the form ω itself and its restrictions to the submanifolds $D_1 \cap \dots \cap D_i$, $i = 1, \dots, n$, have only isolated zeros. Let m_0 (respectively m_i, $i = 1, \dots, n$) be the number of zeros of the form ω (respectively, of the restriction of ω to the intersection $D_1 \cap \dots \cap D_i$) counted with multiplicities. In particular m_n is the number of points in $D_1 \cap \dots \cap D_n$.

Theorem 1. *We have*

$$c_n(T^*M)[M] = (-1)^n \chi(M^n) = m_0 - m_1 + \dots + (-1)^n m_n.$$

Proof. The proof will use induction on the dimension n. We suppose that

$$c_{n-1}(T^*D)[D] = (-1)^{n-1} \chi(D) = m_1 - m_2 + \dots + (-1)^{n-1} m_n.$$

One has $m_0 = c_n(T^*M \otimes L)[M]$. For the characteristic class $c_n(T^*M \otimes L)$ one has

$$c_n(T^*M \otimes L) = c_n(T^*M) + c_1(L) c_{n-1}(T^*M) + \dots + c_1(L)^n.$$

However, the normal bundle to $D = D_1$ in M is isomorphic to the restriction $L|_D$ and therefore $c(T^*M|_D) = c(T^*D)(1 - c_1(L|_D))$. Let $j : D \hookrightarrow M$ be the inclusion. Then

$$c(T^*D) = j^*(c(T^*M)(1 - c_1(L))^{-1}),$$
$$c_{n-1}(T^*D) = j^*(c_{n-1}(T^*M) + c_1(L) c_{n-2}(T^*M) + \dots + c_1(L)^{n-1}).$$

https://doi.org/10.1112/S0010437X03000691 Published online by Cambridge University Press

Therefore

$$\begin{aligned} & c_{n-1}(T^*D)[D] \\ & \quad = j^*(c_{n-1}(T^*M) + c_1(L)c_{n-2}(T^*M) + \cdots + c_1(L)^{n-1})[D] \\ & \quad = (c_1(L)(c_{n-1}(T^*M) + c_1(L)c_{n-2}(T^*M) + \cdots + c_1(L)^{n-1}))[M] \\ & \quad = (c_n(T^*M \otimes L) - c_n(T^*M))[M] \\ & \quad = m_0 - c_n(T^*M)[M]. \end{aligned}$$

So

$$c_n(T^*M)[M] = m_0 - c_{n-1}(T^*D)[D] = m_0 - m_1 + \cdots + (-1)^n m_n. \qquad \square$$

Remark 1. From the proof of Theorem 1 it follows that

$$m_0 = (-1)^n(\chi(M) - \chi(D)) = (-1)^n\chi(M \setminus D).$$

Suppose that all zeros of ω are outside of D (one can consider this situation as the generic one). Let s be the holomorphic section of the line bundle L with zeros on D. Then $\alpha = \omega/s$ is a holomorphic 1-form on $M \setminus D$ with simple poles along D. So in this case the number m_0 of zeros of the holomorphic 1-form α on $M \setminus D$ coincides with $(-1)^n\chi(M \setminus D)$. This is the relation which holds for holomorphic 1-forms on compact manifolds ($M \setminus D$ is not compact).

Remark 2. In [BB70] there were considered meromorphic vector fields on complex manifolds. To have a version of Theorem 1 for vector fields it is necessary to define zeros of a vector field on its pole divisor. However (in contrast to a 1-form), a restriction of a vector field to a submanifold is not defined (as a vector field on the submanifold). One can use the projection of a vector field to the submanifold for that. However, this operation is not complex analytic and thus after projecting one has to consider zeros of non-meromorphic vector fields. These zeros should have signs and should be counted with them.

Example. Let

$$\alpha = \frac{x\,dy - y\,dx + dz}{x^2 + 4y^2 + z^2 + 1}$$

be a meromorphic 1-form on the projective space $\mathbb{CP}^3$ (x, y, z are affine coordinates). One can see that the zeros of the corresponding ω on $\mathbb{CP}^3$ and also the zeros of $\omega|_D$, $D = D_1 = \{x^2+4y^2+z^2+1 = 0\}$, are isolated and $m_0 = 0$, $m_1 = 4$. To define other singular points one has to choose D_2 and D_3 (e.g., $D_2 = \{x^2 + y^2 + 4z^2 = 0\}$, $D_3 = \{x^2 + y^2 + z^2 = 0\}$). One has $m_2 = 8$, $m_3 = 8$, $0 - 4 + 8 - 8 = -4 = (-1)^3\chi(\mathbb{CP}^3)$. Pay attention that as a meromorphic 1-form on $\mathbb{CP}^3$ with poles on the hypersurface $P_d(x, y, z) = 0$, $\deg P_d = d$, it is natural to take

$$\frac{A_{d-2}\,dx + B_{d-2}\,dy + C_{d-2}\,dz}{P_d},$$

where A_{d-2}, B_{d-2}, and C_{d-2} are polynomials in x, y, z of degree $d-2$ (α is not of this form). However, one can show that such a form has non-isolated zeros on $\mathbb{CP}^3$ (at infinity).

3. Meromorphic 1-forms on ICIS

Let V^n be a compact subvariety of a complex manifold M^{n+k} such that in a neighbourhood of each point V is a complete intersection in M with only isolated singularities (M is not supposed to be compact). Let ω be a meromorphic 1-form on M, i.e. a holomorphic section of the bundle $T^*M \otimes L$, where L is a line bundle on M with non-zero holomorphic sections. By a smoothing $\widetilde{V}$ of V we understand a smooth (C^∞) manifold which is obtained from V by smoothing its singular points (which are ICIS) in the usual way (as complex ICIS). (If V is a complete intersection in the projective space $\mathbb{CP}^{n+k}$, a smoothing of V can be obtained as an analytic submanifold of $\mathbb{CP}^{n+k}$ as well.)

https://doi.org/10.1112/S0010437X03000691 Published online by Cambridge University Press

We want to formulate a Poincaré–Hopf type formula for the Euler characteristic of V and of its smoothing $\widetilde{V}$ in terms of singular points of the 1-form ω on V. As above we suppose that there exist n holomorphic sections $s_1,\dots,s_n$ of the line bundle L with zero divisors $D_1,\dots,D_n$ such that, for each $i=1,\dots,n$, the intersection $V\cap D_1\cap\dots\cap D_i$ is of dimension $n-i$ and has only isolated singularities (and thus is reduced for $i<n$). We also suppose that the restriction of ω to the non-singular part of V and its restriction to the non-singular part of $V\cap D_1\cap\dots\cap D_i$ $(i=1,\dots,n)$ have only isolated zeros. As singular points of the 1-form ω on V we consider all zeros of its restrictions to the non-singular parts of V and of the intersections $V\cap D_1\cap\dots\cap D_i$ $(i=1,\dots,n)$ and all singular points of V and of $V\cap D_1\cap\dots\cap D_i$ as well.

Let P be a singular point of the 1-form ω on V. Suppose that the divisors $D_1,\dots,D_\ell$ $(0\leqslant\ell\leqslant n)$ pass through the point P and (if $\ell<n$) $D_{\ell+1}$ does not. In a neighbourhood of the point P in some local coordinates on M centred at the point P we have the following situation. The variety (ICIS) V (possibly non-singular) is defined by k equations $f_1=\dots=f_k=0$. Let divisors $D_1,\dots,D_\ell$ be defined by equations $f_{k+1}=0,\dots,f_{k+\ell}=0$. Let us recall that $\{f_1=\dots=f_k=f_{k+1}=\dots=f_{k+i}=0\}$ is an ICIS for each $i=0,1,\dots,\ell$. After choosing a local trivialization of the line bundle L, the 1-form ω can be written as $\sum_{i=1}^{n+k}A_i(x)\,dx_i$ where $A_i(x)$ are holomorphic. One can say that we consider a collection of meromorphic 1-forms $\omega/f_{k+1},\dots,\omega/f_{k+\ell}$ on V proportional to each other. Let us denote the set of local data $(V,\omega,f_{k+1},\dots,f_{k+\ell})$ by Ω.

Let $\operatorname{ind}^{(0)}\Omega$ (respectively $\operatorname{ind}^{(i)}\Omega$, $i=1,\dots,\ell$) be the index of the holomorphic 1-form ω on V (respectively on the ICIS $V\cap D_1\cap\dots\cap D_i$) defined in [EG01, EG02].

Definition. The alternating sum

$$\operatorname{ind}_P\Omega=\operatorname{ind}^{(0)}\Omega-\operatorname{ind}^{(1)}\Omega+\dots+(-1)^\ell\operatorname{ind}^{(\ell)}\Omega$$

is called the index of the 1-form ω at the point P with respect to the divisors $D_1,\dots,D_n$.

According to [EG01, EG02] one has the following three equivalent descriptions of the index $\operatorname{ind}_P\Omega$.

1) Let B_δ be the ball of sufficiently small radius δ centred at the origin in $\mathbb{C}^{n+k}$ and let S_δ be its boundary. Let $\varepsilon=(\varepsilon_1,\dots,\varepsilon_k,\dots,\varepsilon_{k+\ell})\in(\mathbb{C}^{k+\ell},0)$ be small enough and such that $\varepsilon^{(k+i)}=(\varepsilon_1,\dots,\varepsilon_k,\dots,\varepsilon_{k+i})$ is not a critical value of the map $F_{k+i}=(f_1,\dots,f_k,\dots,f_{k+i}):(\mathbb{C}^{n+k},0)\to(\mathbb{C}^{k+i},0)$ for each $i=0,1,\dots,\ell$. The restriction of the 1-form ω to the smooth manifold $V_i=F_{k+i}^{-1}(\varepsilon^{(k+i)})$ has isolated zeros in B_δ. Let m_i be their number counted with multiplicities. Then $\operatorname{ind}^{(i)}\Omega=m_i$, $\operatorname{ind}_P\Omega=m_0-m_1+\dots+(-1)^\ell m_\ell$.
2) Let K_i $(i=0,1,\dots,\ell)$ be the link of the ICIS $V\cap D_1\cap\dots\cap D_i$, i.e. the intersection $V\cap D_1\cap\dots\cap D_i\cap S_\delta$. Let d_i be the degree of the map
$$(\omega,df_1,\dots,df_k,\dots,df_{k+i}):K_i\to W_{k+i+1}(\mathbb{C}^{n+k})$$
($W_{k+i+1}(\mathbb{C}^{n+k})$ is the Stiefel manifold of $(k+i+1)$-frames in the dual $\mathbb{C}^{n+k}$). Then $\operatorname{ind}^{(i)}\Omega=d_i$, $\operatorname{ind}_P\Omega=d_0-d_1+\dots+(-1)^\ell d_\ell$.
3) Let J_i $(i=0,1,\dots,\ell)$ be the ideal of the ring $\mathcal{O}_{\mathbb{C}^{n+k},0}$ of germs of holomorphic functions of $n+k$ variables at the origin generated by $f_1,\dots,f_k,\dots,f_{k+i}$ and the $(k+i+1)\times(k+i+1)$-minors of the matrix
$$\begin{pmatrix}\dfrac{\partial f_1}{\partial x_1}&\cdots&\dfrac{\partial f_1}{\partial x_{n+k}}\\ \vdots&\ddots&\vdots\\ \dfrac{\partial f_{k+i}}{\partial x_1}&\cdots&\dfrac{\partial f_{k+i}}{\partial x_{n+k}}\\ A_1&\cdots&A_{n+k}\end{pmatrix}.$$

https://doi.org/10.1112/S0010437X03000691 Published online by Cambridge University Press

Let $\nu_i = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{n+k},0}/J_i$. Then $\mathrm{ind}^{(i)}\Omega = \nu_i$, $\mathrm{ind}_P\omega = \nu_0 - \nu_1 + \cdots + (-1)^\ell \nu_\ell$.

Remark 3. In [GSV91] and [SS96] there is defined the index of a holomorphic vector field on an ICIS. One advantage of using 1-forms instead of vector fields is that a description of type 3) does not exist for the index of a vector field.

Let us return to the global situation described at the beginning of the section.

Theorem 2. *We have*

$$\sum_P \mathrm{ind}_P\Omega = (-1)^n \chi(\widetilde{V}),$$

where $\widetilde{V}$ is a smoothing of the variety V.

Proof. If there exist an analytic smoothing $\widetilde{V}$ of V inside M^{n+k} and analytic smoothings of the divisors $D_1, \ldots, D_n$ (i.e. sections $\widetilde{s}_1, \ldots, \widetilde{s}_n$ of the line bundle L close to $s_1, \ldots, s_n$ for which their zero divisors $\widetilde{D}_1, \ldots, \widetilde{D}_n$ are such that, for each $i = 1, \ldots, n$, the intersection $\widetilde{V} \cap \widetilde{D}_1 \cap \cdots \cap \widetilde{D}_i$ is smooth), then the statement follows directly from Theorem 1 since the index of a singular point counts the alternating sum of numbers of zeros on a smoothing vanishing at the point (see the description 1) of the index above).

If such smoothings do not exist, then the statement is derived from the following arguments. One can easily see that the statement of Theorem 1 and its proof hold in the situation when the manifold M is almost complex (i.e. has a complex structure in the tangent bundle) and the manifolds $D_1 \cap \cdots \cap D_i$, $i = 1, \ldots, n$ (D_i is the zero locus of a section s_i of the complex line bundle L), are almost complex submanifolds of M (i.e. their tangent spaces are complex subspaces of the tangent spaces to the manifold M). The only difference is that in this case the multiplicity of a zero of a complex 1-form ω (a section of the complex bundle $T^*M \otimes L$) or of its restriction to the intersection $D_1 \cap \cdots \cap D_i$ is in general not positive. However, if the manifold M, the intersections $D_1 \cap \cdots \cap D_i$ and the 1-form ω are in fact complex analytic in neighbourhoods of all the singular points of ω, then all the multiplicities are in fact positive.

One can construct almost complex smoothings $\widetilde{V}$ and $\widetilde{V} \cap \widetilde{D}_1 \cap \cdots \cap \widetilde{D}_i$ of the varieties V and $V \cap D_1 \cap \cdots \cap D_i$ $(i = 1, \ldots, n)$ and a complex 1-form $\widetilde{\omega}$ approximating ω in the following way. Let P be a singular point of V or of an intersection $V \cap D_1 \cap \cdots \cap D_i$. Suppose that $P \in V \cap D_1 \cap \cdots \cap D_\ell$, $P \notin V \cap D_1 \cap \cdots \cap D_{\ell+1}$ (here ℓ may be equal to zero if $P \notin D_1$). Choose a small ball B of radius r_0 in a coordinate neighbourhood of the point P ($P = 0$) so that inside this ball V is given by equations $f_1 = \cdots = f_k = 0$, D_i is given by an equation $g_i = 0$ $(i = 1, \ldots, \ell)$, and there are no other singular points of the 1-form ω inside the ball B. Let B' be the ball of radius $r_0/2$ and let $\theta(r)$ be a smooth monotonous function such that $0 \leqslant \theta(r) \leqslant 1$, $\theta(r) = 1$ in a neighbourhood of $r_0/2$, and $\theta(r) = 0$ in a neighbourhood of r_0. Let $\varepsilon_0 \in \mathbb{C}^k$, $\varepsilon_1, \ldots, \varepsilon_\ell \in \mathbb{C}$ be small enough and such that the collection $\{f = \varepsilon_0\}$, $\{f = \varepsilon_0\} \cap \{g_1 = \varepsilon_1\} \cap \cdots \cap \{g_i = \varepsilon_i\}$ $(i = 1, \ldots, \ell)$ is a complex analytic smoothing of the varieties V, $V \cap D_1 \cap \cdots \cap D_i$ $(i = 1, \ldots, \ell)$. Let $\widetilde{V}$ and $\widetilde{D}_i$, $i = 1, \ldots, \ell$, be submanifolds of M which coincide with V and D_i respectively outside of the ball B and which are given by the equations $f(z) = \theta(r(z))\varepsilon_0$ and $g_i(z) = \theta(r(z))\varepsilon_i$ respectively, where $r(z)$ is the length of z in the coordinates in B.

The tangent space $T_{z_0}\widetilde{V}$ to the manifold $\widetilde{V}$ at a point $z_0 \in \widetilde{V} \cap \overline{B \setminus B'}$ is close to the tangent space $T_{z_0}V_{z_0}$ to the complex submanifold $V_{z_0} = \{f(z) = \theta(r(z_0))\varepsilon_0\}$ at the same point. If $z_0 \in \widetilde{V} \cap \widetilde{D}_1 \cap \cdots \cap \widetilde{D}_i$, then also the tangent space $T_{z_0}(\widetilde{V} \cap \widetilde{D}_1 \cap \cdots \cap \widetilde{D}_i)$ is close to the tangent space $T_{z_0}(V_{z_0} \cap D_{1,z_0} \cap \cdots \cap D_{i,z_0})$ where D_{j,z_0} is the complex analytic submanifold $\{g_j(z) = \theta(r(z_0))\varepsilon_j\}$. The spaces $T_{z_0}V_{z_0}$ and $T_{z_0}(V_{z_0} \cap D_{1,z_0} \cap \cdots \cap D_{i,z_0})$ are complex vector subspaces of the tangent space $T_{z_0}M$. To define a complex structure on the tangent space $T_{z_0}(\widetilde{V})$ one can construct a (real linear) map H_{z_0} from the tangent space $T_{z_0}M$ to itself close to the identity (and thus non-degenerate)

https://doi.org/10.1112/S0010437X03000691 Published online by Cambridge University Press

which sends $T_{z_0}\widetilde{V}$ to $T_{z_0}V_{z_0}$ and (if $z_0 \in \widetilde{V} \cap \widetilde{D}_1 \cap \cdots \cap \widetilde{D}_i$) the tangent space $T_{z_0}(\widetilde{V} \cap \widetilde{D}_1 \cap \cdots \cap \widetilde{D}_i)$ to $T_{z_0}(V_{z_0} \cap D_{1,z_0} \cap \cdots \cap D_{i,z_0})$. Such a map (smoothly dependent on z_0) can easily be constructed using a hermitian scalar product on TM.

The map H_{z_0} induces a complex structure on the tangent space $T_{z_0}(\widetilde{V})$ so that $T_{z_0}(\widetilde{V} \cap \widetilde{D}_1 \cap \cdots \cap \widetilde{D}_i)$ is a complex subspace of it and also a complex 1-form $\widetilde{\omega}$ on $T_{z_0}(\widetilde{V})$. Since H_{z_0} is close to the identity, $\widetilde{\omega}$ has no singular points on $\widetilde{V} \cap \overline{B \setminus B'}$. Therefore the 'almost complex version' of Theorem 1 applied to $\widetilde{V}$, $\widetilde{D}_1, \ldots, \widetilde{D}_n$, $\widetilde{\omega}$ yields the result. □

In order to get the Euler characteristic of the variety V itself one has to correct the formula of Theorem 2 taking the Milnor numbers of the singular points of V into account. For $P \in V$, let μ_P denote the Milnor number of the ICIS V at the point P (see e.g. [Loo84]). Note that, if V is non-singular at P, then $\mu_P = 0$.

Theorem 3. *We have*

$$(-1)^n \chi(V) = \sum_P (\mathrm{ind}_P \Omega - \mu_P).$$

Remark 4. The Milnor number μ_P can also be written as an alternating sum of indices of holomorphic 1-forms on several ICIS. One can say that the term μ_P corresponds to the difference between two possible definitions of the index as in [SS98, Proposition 1.4].

4. Multiple pole divisor

In some cases it is natural to consider the situation when the pole locus of the meromorphic 1-form ω is a multiple of an irreducible divisor. Examples are the following:

1) A polynomial 1-form on $\mathbb{C}^n$ is a meromorphic 1-form on $\mathbb{CP}^n$ the pole locus of which is a multiple of the infinite hyperplane $\mathbb{CP}^{n-1}_\infty$.
2) Let f be a meromorphic function with the pole locus kD where D is an irreducible divisor (i.e. in a neighbourhood of each point f can be written as $\widetilde{f}/F^k$ where $\widetilde{f}$ is holomorphic and $F = 0$ is a local equation of D). Then its differential df is a meromorphic 1-form with the pole locus $(k+1)D$ (i.e. in a neighbourhood of each point df can be written as $\widetilde{\omega}/F^{k+1}$ where $\widetilde{\omega}$ is a holomorphic 1-form). A polynomial function of degree k on $\mathbb{C}^n$ is a meromorphic function on $\mathbb{CP}^n$ with the pole locus $k\mathbb{CP}^{n-1}_\infty$.

Let us give a version of Theorem 1 for this case. The fact that the pole divisor of a 1-form is multiple means that the corresponding line bundle is a power of another one. Let ω be a meromorphic 1-form on a compact complex manifold M^n, that is, a holomorphic section of the bundle $T^*M \otimes L$, where $L = \lambda^k$, $k > 1$. Suppose that $D_1 = D$, $D_2, \ldots, D_n$ are zero divisors of holomorphic sections of the line bundle λ such that, for each $i = 1, \ldots, n$, $D_1 \cap \cdots \cap D_i$ is non-singular and the form ω itself and its restrictions to $D_1 \cap \cdots \cap D_i$ have isolated zeros. Let m_0 (respectively m_i, $i = 1, \ldots, n$) be the number of zeros of the 1-form ω (respectively of its restriction to $D_1 \cap \cdots \cap D_i$) counted with multiplicities.

Theorem 4. *We have*

$$c_n(T^*M)[M] = (-1)^n \chi(M) = m_0 - km_1 + \cdots + (-1)^n k m_n.$$

Proof. For $n = 1$, i.e. for curves, the statement is obvious. Suppose that the statement holds for manifolds of dimension $n-1$, in particular,

$$c_{n-1}(T^*D)[D] = m_1 - km_2 + \cdots + (-1)^{n-1} k m_n.$$

https://doi.org/10.1112/S0010437X03000691 Published online by Cambridge University Press

Since $m_0 = c_n(T^*M \otimes L)[M]$, $m_1 = c_{n-1}(T^*D \otimes L)[D]$, to prove the formula it is sufficient to show that the number

$$c_n(T^*M)[M] + c_{n-1}(T^*D)[D] - c_n(T^*M \otimes L)[M] + (k-1)c_{n-1}(T^*D \otimes L)[D] \tag{1}$$

is equal to zero. In order to prove this let us suppose that

$$c(T^*M) = \prod_{i=1}^{n}(1+\xi_i).$$

Let $c_1 := c_1(\lambda)$ and let $j : D \hookrightarrow M$ be the inclusion. Then

$$c(T^*D) = j^*\left(\frac{\prod(1+\xi_i)}{1-c_1}\right),$$
$$c(T^*M \otimes L) = \prod(1+\xi_i+kc_1),$$
$$c(T^*D \otimes L) = j^*\left(\frac{\prod(1+\xi_i+kc_1)}{1+(k-1)c_1}\right)$$

(everywhere the product is taken for $i = 1, \dots, n$). Since $j^*(a)[D] = c_1 a[M]$, the expression (1) is equal to the residue of

$$\frac{1}{t^{n+1}}\left(\prod(1+\xi_i t) + c_1 t\frac{\prod(1+\xi_i t)}{1-c_1 t} - \prod(1+(\xi_i+kc_1)t) + (k-1)c_1 t\frac{\prod(1+(\xi_i+kc_1)t)}{1+(k-1)c_1 t}\right) \tag{2}$$

at $t = 0$. The function (2) has simple poles at $t_1 = 1/c_1$ and $t_2 = -1/(k-1)c_1$. One easily calculates the residues of (2) at t_1, t_2 and ∞. The residue at ∞ turns out to be equal to zero and the residues at t_1 and t_2 cancel. By the residue theorem the residue at the origin is equal to zero. □

Remark 5. The proof of Theorem 4 does not work for $k = 1$ directly (the corresponding expression which differs from (2) by absence of the last summand has non-zero residues at $t = 1/c_1$ and $t = \infty$ which cancel). Moreover, the proof of Theorem 1 is more straightforward. This is the reason why we kept it.

Remark 6. In [HS98, Theorem 3.1] there is given a formula for the sum of the residues corresponding to the singular points of the foliation on the projective plane $\mathbb{CP}^2$ given by $df = 0$, where f is a polynomial function of degree k on $\mathbb{C}^2$ (which defines a meromorphic function on $\mathbb{CP}^2$). In our terms this is the formula for the number m_0 of zeros of the meromorphic 1-form df, the pole locus of which is $k+1$ times the infinite line. Thus Theorem 4 can be considered as a generalization of [HS98, Theorem 3.1] to higher dimensions and to meromorphic 1-forms which are not, in general, differentials of functions.

Acknowledgement

The authors would like to thank the referee for useful comments.

References

BB70 P. F. Baum and R. Bott, *On the zeros of meromorphic vector fields*, in *Essays on topology and related topics (Mémoires dédiés à Georges de Rham)* (Springer, New York, 1970), 29–47.

EG01 W. Ebeling and S. M. Gusein-Zade, *On the index of a holomorphic 1-form on an isolated complete intersection singularity*, Russ. Math. Dokl. **380** (2001), 458–461.

EG02 W. Ebeling and S. M. Gusein-Zade, *Indices of 1-forms on an isolated complete intersection singularity*, Mosc. Math. J. **3** (2003), no. 2, 439–455.

GSV91 X. Gómez-Mont, J. Seade and A. Verjovsky, *The index of a holomorphic flow with an isolated singularity*, Math. Ann. **291** (1991), 737–751.

HS98 T. Honda and T. Suwa, *Residue formulas for meromorphic functions on surfaces*, Ann. Fac. Sci. Toulouse Math. (6) **7** (1998), 443–463.

Loo84 E. J. N. Looijenga, *Isolated singular points on complete intersections*, London Math. Soc. Lecture Note Series, vol. 77 (Cambridge University Press, Cambridge, 1984).

SS96 J. Seade and T. Suwa, *A residue formula for the index of a holomorphic flow*, Math. Ann. **304** (1996), 621–634.

SS98 J. Seade and T. Suwa, *An adjunction formula for local complete intersections*, Int. J. Math. **9** (1998), 759–768.

W. Ebeling ebeling@math.uni-hannover.de
Universität Hannover, Institut für Mathematik, Postfach 6009, D-30060 Hannover, Germany

S. M. Gusein-Zade sabir@mccme.ru
Moscow State University, Faculty of Mechanics and Mathematics, Moscow, 119992, Russia

https://doi.org/10.1112/S0010437X03000691 Published online by Cambridge University Press