SOLUBLE ARTINIAN GROUPS

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Dedicated to H. S. M. Coxeter
on the occasion of his sixtieth birthday

The principal aim of this note is the proof of the following:

MAIN THEOREM. *The following properties of the group G are equivalent*:

I. *G is artinian and soluble.*

II. (a) *To every maximal subgroup S of G there exists a normal subgroup T of G with $G = ST$ and $S \cap T = (S \cap T)^T$.*

(b) *Every minimal normal subgroup M of an epimorphic image of G possesses a Sylow subgroup $S \neq 1$ with a finite class S^M of conjugates in M.*

(c) *$H/\mathfrak{c}_H M$ is finite for every primary minimal normal subgroup M of any epimorphic image H of G.*

(d) *The minimum condition is satisfied by the normal subgroups of G.*

III. (a) *To every maximal subgroup S of any finite epimorphic image F of G there exists a normal subgroup T of F with $F = ST$ and $S \cap T = (S \cap T)^T$.*

(b) *$H/\mathfrak{c}_H M$ is finite for every minimal normal subgroup M of any epimorphic image H of G.*

(c) *The minimum condition is satisfied by the normal subgroups of G.*

IV. (a) *Abelian subgroups of G are artinian.*

(b) *Every epimorphic image $H \neq 1$ of G possesses a normal subgroup $N \neq 1$ whose finitely generated subgroups F meet the following requirements:*

(b′) *The minimum condition is satisfied by the normal subgroups of F.*

(b″) *To every maximal subgroup S of F there exists a normal subgroup T of F with $F = ST$ and $S \cap T = (S \cap T)^T$.*

V. (a) *Abelian subgroups of G are artinian.*

(b) *If F is a finitely generated subgroup of G, then*

(b′) *the minimum condition is satisfied by the normal subgroups of F and*

(b″) *there exists to every maximal subgroup S of F a normal subgroup T of F with $F = ST$ and $S \cap T = (S \cap T)^T$.*

VI. (a) *Abelian subgroups of G are artinian.*

(b) *Every infinite epimorphic image H of G possesses a normal subgroup $N \neq 1$ with finite $H/\mathfrak{c}_H N$.*

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(c) *To every maximal subgroup S of a finite epimorphic image F of G there exists a normal subgroup T of F with $F = ST$ and $S \cap T = (S \cap T)^T$.*

VII. (a) *G is artinian.*

(b) *The maximal subgroups A and B of the subgroup S of G are conjugate in S, if (and only if) A and B contain the same normal subgroups of S.*

(c) *If the simple, finitely generated factor F of G is not primary, then there does not exist a proper subgroup of F which contains for every prime p a p-Sylow subgroup of F.*

(d) *If the simple, finitely generated factor F of G is primary, then the set of maximal subgroups of F is not a partition of F.*

Partly contained in this Main Theorem and partly used in its proof is the following

COROLLARY. *The following properties of the group G are equivalent:*

(i) *G is a finite soluble group.*

(ii) *G is finite and to every maximal subgroup S of G there exists a normal subgroup T of G with $G = ST$ and $S \cap T = (S \cap T)^T$.*

(iii) (a) *G is finitely generated.*

(b) *The minimum condition is satisfied by the normal subgroups of G.*

(c) *If S is a maximal subgroup of the finitely generated characteristic subgroup F of G, then there exists a normal subgroup T of F with $F = ST$ and*

$$S \cap T = (S \cap T)^T.$$

Terminology and notations

G' = commutator subgroup of G.

$G^{(0)} = G, \quad G^{(i+1)} = [G^{(i)}]'$.

The group G is *soluble* if $G^{(i)} = 1$ for almost all i.

The group G is *artinian* if the minimum condition is satisfied by the subgroups of G; this is equivalent to the requirement that every descending chain of subgroups terminates after finitely many steps.

The group G is *almost abelian* if there exists an abelian subgroup A with finite index $[G: A]$; this is equivalent to the requirement that G is an extension of an abelian [normal] subgroup by a finite group.

p-group = group all of whose elements are of order a power of p.

A group is *primary* if it is a p-group for some prime p.

p-Sylow subgroup = maximal p-subgroup.

$x^y = x(x \circ y) = y^{-1}xy$.

X^Y = set of all X^y for y in Y.

$\mathsf{z}G$ = centre of G.

$\mathsf{h}G$ = hypercentre of G = intersection of all normal subgroups X of G with $\mathsf{z}(G/X) = 1$.

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$\mathsf{D}G$ = intersection of all normal subgroups X of G with almost abelian G/X.
$\mathsf{J}G$ = intersection of all subgroups S of G with finite $[G: S]$
= intersection of all normal subgroups X of G with finite G/X.
$\mathsf{c}_G X$ = centralizer of X in G.
$\mathsf{n}_G X$ = normalizer of X in G.
Factor of a group = epimorphic image of a subgroup.
$\{\ldots\}$ = subgroup, generated by the elements (subsets) enclosed in the braces.

Discussion

A. O. Ore (**8**, p. 451, Theorem 9) has shown that a finite group G is soluble if, and only if, every subgroup S of G has the property:

(s*) If the maximal subgroup T of S is not a normal subgroup of S, then there exists a normal subgroup N of S with $S = NT$ and $N \cap T$ a normal subgroup of S.

This condition is much stronger than either of our conditions (ii) and (iii.c) of the Corollary.

B. If G is a symmetric group of degree greater than 4, then G is not soluble. Its alternating subgroup A is its only proper normal subgroup. If M is a maximal subgroup of G, then either $M = A$ or $A \not\subseteq M$ so that $G = MA$. But $M \cap A$ is not a normal subgroup of G. This shows that we cannot weaken the second part of condition (ii) (or (iii.c)) of the Corollary to the requirement:

To every maximal, not normal subgroup M of G there exists a normal subgroup N of G with $N \not\subseteq M \not\subseteq N$.

C. We shall show in the course of the proof that condition IV(b) of the Main Theorem is a fairly simple consequence of V(b), though not conversely. One might say therefore that condition V is at the same time more elegant and stronger than condition IV.

D. The condition III(b) and (c) is identical with condition C of the Proposition below. Naturally we may substitute for these conditions each of the equivalent conditions A–H of the Proposition. Condition II(c) and (d) is only slightly weaker than III(b) and (c).

E. If the maximal subgroup S of G is a normal subgroup of G, then $G = SG$ and $S \cap G = S = (S \cap G)^G$. This shows that the existence of the normal subgroup T in conditions II(a), III(a), IV(b″), V(b″), VI(c), (ii), (iii.c) need only be required for the *non-normal* maximal subgroups S.

F. Cf. also Discussion B of the Proposition.

G. It is an open question whether or not either of the conditions VII(c) or VII(d) is indispensable.

VII(c) imposes upon certain groups F the requirement:

(+) F is a torsion group; and the only subgroup of F which contains a p-Sylow subgroup of F for every prime p is F itself.

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It is a consequence of Sylow's theorems that finite groups meet requirement $(+)$. On the other hand, Wehrfritz has shown:

If K is the absolutely algebraic, algebraically closed field of characteristic 2, then $G = \mathrm{PSL}(2, K)$ is locally finite and contains a proper subgroup which is metabelian and contains a p-Sylow subgroup of G for every prime p.

Condition VII(d) imposes upon certain groups F the requirement:

$(++)$ F is a primary group; and the set of maximal subgroups of F is not a partition of F.

This is certainly the case if F is cyclic; and it is certainly not the case if F is an elementary abelian group of order p^2. If F is a finite p-group, then every maximal subgroup of F is normal of index p. Thus if A and B are different maximal subgroups of F, then $F/(A \cap B)$ is elementary abelian of order p^2; and this shows that the maximal subgroups of F form a partition of F if, and only if, F is elementary abelian of order p^2.

If F happens to be a finitely generated infinite 2-group, and if the maximal subgroups of F form a partition, then consider two different maximal subgroups A and B of F. Clearly A contains an element a of order 2 and B contains an element b of order 2. But $\{a, b\}$ has finite order, since ab is a 2-element. Hence $\{a, b\}$ is part of a maximal subgroup C of F (Maximum Principle of Set Theory). Since $A \cap C$ contains $a \neq 1$, we have $A = C$; and likewise we have $C = B$. This is a contradiction, showing that finitely generated, infinite 2-groups likewise meet requirement $(++)$. (This argument has been used by various people; it goes back to O. Schmidt (or further).)

Artinian soluble groups are almost abelian; see Baer (**3**, p. 18, Lemma 3.3). An artinian group is consequently soluble if, and only if, it is almost abelian and its finite epimorphic images are soluble. Thus it will prove important in our discussion to have criteria at our disposal for a group to be artinian and almost abelian. With this in mind we are going to prove the following characterizations of the class of artinian and almost abelian groups:

PROPOSITION. *The following properties of the group G are equivalent:*

A. *G is artinian and almost abelian.*

B. (1) *The minimum condition is satisfied by the normal subgroups of G.*
(2) *Minimal normal subgroups of epimorphic images of G are finite.*

C. (1) *The minimum condition is satisfied by the normal subgroups of G.*
(2) *$H/\mathbf{c}_H M$ is finite for every minimal normal subgroup M of any epimorphic image H of G.*

D. (1) *The minimum condition is satisfied by the normal subgroups of G.*
(2) *$H/\mathbf{c}_H M$ is almost abelian for every minimal normal subgroup M of any epimorphic image H of G.*

(3) *If the minimal normal subgroup M of an epimorphic image of G is abelian, then M is of finite rank.*

E. (1) *The minimum condition is satisfied by the normal subgroups of G.*

(2) *$H/\mathfrak{c}_H M$ is almost abelian for every minimal normal subgroup M of any epimorphic image H of G.*

(3) *Every primary abelian subnormal subgroup of G is artinian.*

F. (1) *The minimum condition is satisfied by the normal subgroups of G.*

(2) *$H/\mathfrak{c}_H M$ is almost abelian and finitely generated for every minimal normal subgroup M of any epimorphic image H of G.*

G. (1) *If C is a characteristic subgroup of G and G/C is almost abelian and artinian, then the minimum condition is satisfied by the normal subgroups of C.*

(2) *$H/\mathfrak{c}_H M$ is almost abelian for every minimal normal subgroup M of any epimorphic image H of G.*

H. (1) *G is a torsion group.*

(2) *Among the normal subgroups X of G with finite G/X there exists a minimal one.*

(3) *Every abelian subnormal subgroup of G is artinian.*

(4) *Every epimorphic image $H \neq 1$ of G possesses a normal subgroup $N \neq 1$ with almost abelian $H/\mathfrak{c}_H N$.*

Terminological reminder. The (abelian) group A *is of finite rank* (in the sense of Prüfer) if there exists a positive integer n such that every finitely generated subgroup of A may be generated by n (or fewer) elements. S *is a subnormal subgroup of* G if there exists a finite chain of subgroups $S(i)$ of G such that $S = S(0)$, $S(i)$ is a normal subgroup of $S(i + 1)$, $S(n) = G$.

Discussion

A. See (**5**, p. 128, Theorem 4.2) for further criteria for a group to be artinian and almost abelian.

B. There exist groups G with the following properties:

G' is an infinite elementary abelian p-group;

G/G' is abelian of Prüfer's type q^∞;

G' is the one and only one minimal normal subgroup of G and the minimum condition is satisfied by the normal subgroups of G;

$H/\mathfrak{c}_H M$ is a primary abelian torsion group of rank 1 for every minimal normal subgroup M of any epimorphic image H of G.

See (**5**, pp. 130–131, Example 5.2).

This shows the indispensability of conditions D(3), E(3), and of the strengthened form G(1) of the minimum condition for normal subgroups.

C. It is well known and easily verified that artinian almost abelian groups are of finite rank. Thus we may substitute for D(3) the global, and stronger, requirement that G be of finite rank.

D. The following three properties of the elementary abelian, minimal normal subgroup M of the group H are equivalent:

(i) The rank of M is finite.

(ii) M is finite.

(iii) $H/c_H M$ is finite.

The equivalence of (ii) and (iii) is contained in (1.3) below. The equivalence of (i) and (ii) is immediate, once we notice that an elementary abelian, minimal normal subgroup is always primary and that a primary elementary abelian group is of finite rank if, and only if, it is finite.

It is easy to construct examples of minimal normal subgroups, isomorphic to the additive group of the rational numbers and hence abelian of rank **1**, showing the indispensability of the hypothesis that M be elementary abelian.

As a consequence of the equivalence of conditions (i)–(iii) we may substitute in condition D(3) of the Proposition for our present requirement (i) either of the properties (ii) and (iii).

A. Preliminary results. We collect in this section a number of results which will be used variously in the following.

(1.1) *If H is an epimorphic image of an artinian, almost abelian group, then*

(a) *every minimal normal subgroup of H is finite;*

(b) *$[H: S]$ is finite for every maximal subgroup S of H;*

(c) *H is locally finite;*

(d) *if S is a maximal subgroup of H and H is soluble, then the finite index $[H: S]$ is a power of a prime p and H is the product $H = SP$ of S and any p-Sylow subgroup P of H;*

(e) *any two p-Sylow subgroups of H are conjugate in H.*

Proof. It is an immediate consequence of our hypothesis that H is artinian and almost abelian. Application of Baer (**2**, pp. 7–8, Satz 2.1) shows the validity of (b), (c), and of

(a*) simple factors of H are finite.

Every minimal normal subgroup M of H is artinian, almost abelian and characteristic simple. Apply (**9**, p. 274, Satz 29) to show that M is the direct product of finitely many, isomorphic, simple groups which are, by (a*), finite. Hence M is finite, proving (a).

Since the artinian group H is by (c) locally finite, we may apply (**3**, p. 37, Lemma A.2) to show the validity of (e).

Suppose now that H is soluble and that S is a maximal subgroup of H. Then $[H: S]$ is finite by (b). Hence there exists a normal subgroup T of H with finite H/T and $T \subseteq S$. Since H is soluble, so is H/T. But the maximal subgroup S/T of the finite soluble group H/T is well knownto have prime power index $[H/T: S/T] = [H: S]$. If this index is a power of the prime p, then H/T is the product of S/T and of any p-Sylow subgroup of H/T. Recall finally that every p-Sylow subgroup of H/T is the product of T and of a p-Sylow subgroup of H; and (d) is verified.

(1.2) *If the minimum condition is satisfied by the normal subgroups of G, and if $H/\mathsf{c}_H M$ is almost abelian for every minimal normal subgroup M of any epimorphic image H of G, then*

(+) $\mathsf{D}G$ *is a hypercentral torsion group and only finitely many primes are orders of elements in* $\mathsf{D}G$;

(++) $G/\mathsf{D}G$ *is artinian and almost abelian;*

(+++) *infinite minimal normal subgroups of epimorphic images of G are primary and elementary abelian.*

Terminological reminder. A group is *hypercentral* if it is equal to its hypercentre, and this is equivalent to requiring that its non-trivial epimorphic images have non-trivial centres. $\mathsf{D}G$ is the intersection of all normal subgroups X of G with almost abelian G/X.

Proof. Since the minimum condition is satisfied by the normal subgroups of G, there exists among the normal subgroups X of G with almost abelian G/X a minimal one, D. If X is a normal subgroup of G with almost abelian G/X, then $G/(D \cap X)$ is isomorphic to a subgroup of the direct product

$$(G/D) \otimes (G/X).$$

But a direct product of two almost abelian groups is an almost abelian group; and so are their subgroups. Hence $G/(D \cap X)$ is an almost abelian group. Because of the minimality of D we have $D = D \cap X \subseteq X$ so that $D = \mathsf{D}G$ and $G/\mathsf{D}G = G/D$ is an almost abelian group. Naturally the minimum condition is satisfied by the normal subgroups of $G/\mathsf{D}G$. Application of Baer (**5**, p. 127, Proposition 4.1) shows that $G/\mathsf{D}G$ is artinian. This proves (++).

Form the hypercentre $\mathsf{h}\mathsf{D}G$ of $\mathsf{D}G$. This is the intersection of all normal subgroups X of $\mathsf{D}G$ with trivial centre $\mathsf{z}[\mathsf{D}G/X]$. The hypercentre is always hypercentral and $\mathsf{z}[\mathsf{D}G/\mathsf{h}\mathsf{D}G] = 1$. The set T of torsion elements in the hypercentral group $\mathsf{h}\mathsf{D}G$ is a characteristic subgroup of the characteristic subgroup $\mathsf{h}\mathsf{D}G$ of G and is hence itself a characteristic torsion subgroup of G; see Baer (**1**, p. 207, Corollary). If $\mathsf{D}G$ were not a hypercentral torsion group, then $T \subset \mathsf{D}G$. Since the minimum condition is satisfied by the normal subgroups of G, it is satisfied by the normal subgroups of $H = G/T$. Consequently there exists a minimal normal subgroup M of H with $M \subseteq \mathsf{D}G/T$—note that this latter group is a normal subgroup, not 1, of H. By hypothesis $H/\mathsf{c}_H M$ is almost abelian. Hence

$$M \subseteq \mathsf{D}G/T = \mathsf{D}H \subseteq \mathsf{c}_H M.$$

In particular, M is abelian and $M \subseteq \mathsf{z}[\mathsf{D}G/T]$.

Since M is a minimal normal subgroup of H, the group $H/\mathsf{c}_H M$ of automorphisms, induced in M by H, is irreducible. It is an epimorphic image of $H/\mathsf{D}H \cong G/\mathsf{D}G$ and because of (++) it is artinian and almost abelian. Apply

(**6**, p. 393, Theorem) to see that M is a primary elementary abelian group. Let $M = N/T$. Then N is a normal subgroup of G with $T \subset N \subseteq \mathsf{D}G$. Since T and M are torsion groups, so is N. From $M \subseteq \mathsf{z}[\mathsf{D}G/T]$ and $T \subseteq \mathsf{h}\mathsf{D}G$, we deduce that $N \subseteq \mathsf{h}\mathsf{D}G$ so that $N \subseteq T \subset N$, a contradiction. Thus $\mathsf{D}G$ is a hypercentral torsion group. As such it is the direct product of its primary components; see Baer (**1**, p. 198, Corollary 1). Primary components are characteristic subgroups of $\mathsf{D}G$ and hence of G. Apply the minimum condition for normal subgroups to see that almost all of them are trivial. This completes the proof of $(+)$.

Consider now normal subgroups A, B of G such that $A \subset B$ and B/A is a minimal normal subgroup of G/A. If, first, $B \subseteq A\mathsf{D}G$, then $B = A[B \cap \mathsf{D}G]$ by Dedekind's Modular Law so that

$$B/A \cong [B \cap \mathsf{D}G]/[A \cap \mathsf{D}G] \subseteq \mathsf{D}G/[A \cap \mathsf{D}G]$$

is a hypercentral torsion group, since $\mathsf{D}G$ is by $(+)$ a hypercentral torsion group. But as a minimal normal subgroup B/A is characteristic simple. It follows that B/A is primary elementary abelian. If, secondly, $B \not\subseteq A\mathsf{D}G$, then

$$A \subseteq A[B \cap \mathsf{D}G] \subset B;$$

and this implies that

$$A = A[B \cap \mathsf{D}G] = B \cap A\mathsf{D}G,$$

since B/A is a minimal normal subgroup of G/A. It follows that

$$B\mathsf{D}G/A\mathsf{D}G \cong B/[B \cap A\mathsf{D}G] = B/A,$$

and that $B\mathsf{D}G/A\mathsf{D}G$ is therefore a minimal normal subgroup of the epimorphic image $G/A\mathsf{D}G$ of $G/\mathsf{D}G$. The latter group is by $(++)$ artinian and almost abelian; and we deduce from (1.1) the finiteness of $B\mathsf{D}G/A\mathsf{D}G \cong B/A$. Thus we have shown that a minimal normal subgroup of an epimorphic image of G is either finite or else a primary elementary abelian group; and this proves $(+++)$.

(1.3) *The minimal normal subgroup M of the group H is finite if, and only if, $H/\mathsf{c}_H M$ is finite.*

Proof. If, first, M is finite, so is the group of automorphisms, induced by H in M. But this group of automorphisms is essentially the same as $H/\mathsf{c}_H M$, proving the necessity of our condition. If, conversely, $H/\mathsf{c}_H M$ is finite, then we distinguish two cases. If $M \cap \mathsf{c}_H M = 1$, then

$$M = M/[M \cap \mathsf{c}_H M] \cong M\mathsf{c}_H M/\mathsf{c}_H M \subseteq H/\mathsf{c}_H M$$

is clearly finite. If $M \cap \mathsf{c}_H M \neq 1$, then we deduce that

$$M = M \cap \mathsf{c}_H M \subseteq \mathsf{c}_H M$$

from the minimality of M. Hence M is an abelian group. If $t \neq 1$ is an element in M, then $\mathsf{c}_H M \subseteq \mathsf{c}_H t$ so that $[H: \mathsf{c}_H t]$ is finite, implying the finiteness of the

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class t^H of elements, conjugate to t in H. Since M is a minimal normal subgroup of H, we have $M = \{t^H\}$ so that M is a finitely generated abelian group. As a minimal normal subgroup M is free of proper characteristic subgroups; and the only finitely generated abelian groups which are characteristic simple are clearly the finite elementary abelian p-groups. This completes the proof of (1.3).

B. Proof of the Proposition. That B is a consequence of A is contained in (1.1); and (1.3) implies the equivalence of B and C. In the presence of the equivalent conditions B and C, every epimorphic image $H \neq 1$ of G possesses by B(1) a minimal normal subgroup M which is finite by B(2) and which admits obviously a finite group of automorphisms only: $H/\mathsf{c}_H\, M$ is likewise finite. Thus condition (ii) of Baer (**5**, p. 128, Theorem 4.2) is satisfied by G. Hence G is artinian and almost abelian, showing the equivalence of conditions A–C.

If the equivalent conditions A–C are satisfied by G, then we deduce D(1) from C(1), D(2) from C(2) and D(3) from B(2).

If condition D is satisfied by G, then we deduce from D(1) and (2) and (1.2) that every infinite minimal normal subgroup M of an epimorphic image of G is primary, elementary abelian. Thus M is, by D(3), of finite rank. But primary, elementary abelian groups of finite rank are clearly finite. Thus every minimal normal subgroup of an epimorphic image of G is finite. We have deduced B from D, showing the equivalence of conditions A–D.

If the equivalent conditions A–D are satisfied by G, then we note that E(1), (2) is identical with D(1), (2) and E(3) is a consequence of the fact that G and hence all its subgroups are artinian.

Assume conversely the validity of E. Then we deduce from E(1), (2) and (1.2) that $\mathsf{D}G$ is a hypercentral torsion group and $G/\mathsf{D}G$ is artinian and almost abelian. Every abelian subnormal subgroup of $\mathsf{D}G$ is a subnormal subgroup of G and hence, by E(3), artinian. Thus condition (3) of (**4**, pp. 359–360, Hauptsatz 8.15, **A**) is satisfied by $\mathsf{D}G$. Hence $\mathsf{D}G$ is artinian and soluble; and this implies in particular that $\mathsf{D}G$ is almost abelian; see (**3**, p. 18, Lemma 3.3). As an extension of the artinian and almost abelian group $\mathsf{D}G$ by the artinian and almost abelian group $G/\mathsf{D}G$ the group G itself is artinian and almost abelian, proving the equivalence of conditions A–E.

It is obvious that F is a consequence of C. If F is satisfied by G, and if M is a minimal normal subgroup of the epimorphic image H of G, then

$$J = H/\mathsf{c}_H\, M$$

is almost abelian and finitely generated (by F(2)); and we deduce from F(1) that the minimum condition is satisfied by the normal subgroups of J. It is a consequence of (**5**, p. 127, Proposition 4.1) that almost abelian groups with minimum condition for normal subgroups are artinian. Hence J is artinian, almost abelian, and finitely generated; and it is a consequence of (1.1, c) that $J = H/\mathsf{c}_H\, M$ is finite. Hence C is a consequence of F, proving the equivalence of conditions A–F.

It is almost obvious that G is a consequence of the equivalent conditions A–F. We assume, conversely, the validity of G. Then we deduce from G(1) that the minimum condition is satisfied by the normal subgroups of G. Combine this property with G(2) and apply (1.2). It follows that $\mathsf{D}G$ is a hypercentral torsion group and that $G/\mathsf{D}G$ is artinian and almost abelian. By G(1), consequently, the minimum condition is satisfied by the normal subgroups of $\mathsf{D}G$. Thus C(1) is satisfied by $\mathsf{D}G$. Since $\mathsf{D}G$ is hypercentral, every minimal normal subgroup M of an epimorphic image H of $\mathsf{D}G$ is contained in the centre $\mathsf{z}H$ of H. Thus C(2) is also satisfied by $\mathsf{D}G$. Since we have already shown the equivalence of A and C, it follows that $\mathsf{D}G$ is artinian and almost abelian. Consequently, G is, as an extension of the artinian and almost abelian group $\mathsf{D}G$ by the artinian and almost abelian group $G/\mathsf{D}G$, itself artinian and almost abelian. Hence A is a consequence of G, proving the equivalence of A–G.

If G meets the equivalent requirements A–G, then we deduce conditions H(1)–H(3) from the minimum condition. If $H \neq 1$ is an epimorphic image of G, then there exists a minimal normal subgroup M of H and $H/\mathsf{c}_H M$ is, by G(2), almost abelian, proving H(4).

Assume, conversely, the validity of H. Then there exists by H(2) a normal subgroup D of G which is minimal among the normal subgroups X of G with finite G/X. If F is some normal subgroup of G with finite G/F, then we deduce from Poincaré's theorem that $G/(D \cap F)$ is likewise finite. Apply the minimality of D to see that $D = D \cap F \subseteq F$. Thus D is the intersection of all normal subgroups X of G with finite G/X. Since D is a characteristic subgroup of G, so is its commutator subgroup $T = D'$.

We consider the hypercentre $\mathsf{h}T$ of T and recall that $\mathsf{h}T$ is the intersection of all the normal subgroups X of T with trivial centre $\mathsf{z}(T/X) = 1$, that $\mathsf{h}T$ is hypercentral, and that the centre $\mathsf{z}(T/\mathsf{h}T) = 1$. Naturally $\mathsf{h}T$ is a characteristic subgroup of T and G. If T were not hypercentral, then $\mathsf{h}T \subset T$. Among the normal subgroups X of G with $T \cap X = \mathsf{h}T$ there exists a maximal one, say W (Maximum Principle of Set Theory). From

$$W \cap T = \mathsf{h}T \subset T$$

we deduce that $W \subset G$. Then $H = G/W \neq 1$. Hence there exists by G(4) a normal subgroup $N \neq 1$ of H with almost abelian $H/\mathsf{c}_H N$. Denote by A and B the uniquely determined normal subgroups of G with $W \subseteq A \cap B$ and $N = A/W$, $\mathsf{c}_H N = B/W$. Note that $1 \neq N$ implies $W \subset A$.

From $H/\mathsf{c}_H N = (G/W)/(B/W) \cong G/B$ we deduce that G/B is almost abelian. Consequently there exists a normal subgroup C of G with $C' \subseteq B \subseteq C$ and finite G/C. Since D has been shown to be the intersection of all normal subgroups X of G with finite G/X, we have $D \subseteq C$ and this implies

$$T = D' \subseteq C' \subseteq B.$$

From

$$1 = N \circ \mathsf{c}_H N = W(A \circ B)/W$$

we deduce $A \circ B \subseteq W$.

From $W \subset A$ and the maximality of W we deduce that

$$\mathfrak{h}T = W \cap T \subset A \cap T.$$

Furthermore, we have

$$(A \cap T) \circ T \subseteq (A \cap T) \circ B \subseteq T \cap (A \circ B) \subseteq T \cap W = \mathfrak{h}T.$$

Hence

$$1 \subset (A \cap T)/\mathfrak{h}T \subseteq \mathfrak{z}[T/\mathfrak{h}T] = 1,$$

a contradiction proving the hypercentrality of T.

Every abelian subnormal subgroup of D is an abelian subnormal subgroup of G and hence by H(3) artinian. Since $D/T = D/D'$ is abelian and T is hypercentral, one verifies readily that non-trivial epimorphic images of D possess non-trivial, abelian, normal subgroups: D is hyperabelian. D is by H(1) a torsion group. Hence condition (3) of (**4**, pp. 359–360, Hauptsatz 8.15, **A**) is satisfied by D so that D is artinian and soluble and consequently almost abelian; see (**3**, p. 18, Lemma 3.3). Since G/D is finite, G itself is almost abelian and artinian, proving the equivalence of A–H.

C. The $\mathfrak{s}$-groups. It will be convenient to introduce the following condition:

$\mathfrak{s}$: *If M is a maximal subgroup of G, then there exists a normal subgroup N of G with $G = MN$ and $M \cap N = (M \cap N)^N$.*

This is a weak solubility requirement, as will be substantiated by Lemma 3 below. Furthermore, condition $\mathfrak{s}$ is, mutatis mutandis, identical with the conditions II(a), III(a), IV(b″), V(b″), VI(c) of the Main Theorem and with conditions (ii), (iii.c) of its Corollary. On the other hand, it is clear that groups without maximal subgroups are always $\mathfrak{s}$-groups. Note furthermore that since N in property $\mathfrak{s}$ is normal, $M \cap N$ is normalized by M and hence by $MN = G$ so that $\mathfrak{s}$ requires $M \cap N$ to be a normal subgroup of G.

LEMMA 1. *The following properties of the group G are equivalent:*

(i) *G is an $\mathfrak{s}$-group.*

(ii) *If M is a maximal subgroup of G, then there exists a normal subgroup N of G with $G = MN$ and $M_G = M \cap N$.*

(iii) *If M is a maximal subgroup of G, then there exists a normal subgroup N of G with $G = MN$ and $M \cap N \subseteq M_G$.*

Terminological reminder. $M_G = \bigcap_{x \in G} M^x =$ product of all normal subgroups of G contained in M; this is the most comprehensive normal subgroup of G contained in M, the core of M in G.

Proof. It is clear that (ii) implies (i) and we have noted before that (i) implies (iii). Assume therefore the existence of a normal subgroup N of G with

$G = MN$ and $M \cap N \subseteq M_G$. The product $N^* = NM_G$ is likewise a normal subgroup of G; and it is obvious that $G = MN^*$. Furthermore, we deduce from Dedekind's Modular Law that

$$M \cap N^* = M \cap M_G N = M_G(M \cap N) = M_G.$$

Hence (ii) is a consequence of (iii).

Remark. Actually we have shown slightly more than indicated in the Lemma, to wit: If M is a subgroup, then the existence of a normal subgroup N of G, meeting one of the requirements (i)–(iii), implies the existence of a normal subgroup N with property (ii).

LEMMA 2. *Epimorphic images of* $\mathfrak{s}$*-groups are* $\mathfrak{s}$*-groups.*

Proof. Suppose that M is a maximal subgroup of the epimorphic image H of the $\mathfrak{s}$-group G; and denote by σ some epimorphism of G upon H. The inverse image $J = M^{\sigma^{-1}}$ is clearly a maximal subgroup of G which satisfies $J^\sigma = M$. Since G is an $\mathfrak{s}$-group, there exists a normal subgroup N of G with $G = NJ$ and $N \cap J = J_G$. Then $H = G^\sigma = N^\sigma J^\sigma = LM$ where $L = N^\sigma$ is a normal subgroup of H. If x belongs to $L \cap M$, then there exists an element y in N with $y^\sigma = x$. Since x is in M, the element y belongs to $M^{\sigma^{-1}} = J$. Hence y belongs to $N \cap J$ and $y^\sigma = x$ belongs to $(J \cap N)^\sigma$. It follows that

$$L \cap M \subseteq (J \cap N)^\sigma \subseteq J^\sigma \cap N^\sigma = M \cap L.$$

Hence

$$L \cap M = (J \cap N)^\sigma = (J_G)^\sigma$$

is a normal subgroup of H so that $L \cap M \subseteq M_H$. Thus we have verified Lemma 1 (iii), proving that H too is an $\mathfrak{s}$-group.

LEMMA 3. *Hyperabelian groups are* $\mathfrak{s}$*-groups.*

Terminological reminder. A group is termed *hyperabelian* if every epimorphic image, not 1, possesses an abelian normal subgroup, not 1.

Proof. Assume that M is a maximal subgroup of the hyperabelian group G. Then $M_G \subseteq M \subset G$ so that G/M_G is an epimorphic image, not 1, of G. Consequently, there exists an abelian normal subgroup, not 1, of G/M_G; and this implies the existence of a normal subgroup N of G with $N' \subseteq M_G \subset N$. Since the normal subgroup N of G is not part of the core of M, it is not part of the maximal subgroup M of G either. Hence $G = MN$. Furthermore,

$$(M \cap N) \circ N \subseteq N' \subseteq M_G \subseteq M \cap N$$

so that $M \cap N$ is normalized by N. Thus G meets requirement $\mathfrak{s}$, as we wanted to show.

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Remark. It is well known that every factor of a hyperabelian group is likewise hyperabelian. Hence every factor of a hyperabelian group is an $\mathfrak{s}$-group. It may be remarked, furthermore, that the proof of Lemma 3 could easily be extended to show considerably more than just the property $\mathfrak{s}$.

D. Groups with conjugate maximal subgroups. For our applications only a subclass of this class of groups will be needed.

LEMMA 4. *Assume that G is a finitely generated, simple group whose maximal subgroups are artinian and soluble and whose maximal subgroups are conjugate in G.*

A. *If G is not primary, then the p-Sylow subgroups of G are conjugate in G; and every maximal subgroup of G contains for every prime p a p-Sylow subgroup of G.*

B. *If G is primary, but not cyclic, then a partition of G is formed by the maximal subgroups of G.*

Terminological reminder. The set λ of subgroups of G is *a partition of G* if every element $g \neq 1$ in G belongs to one and only one subgroup in λ.

Proof. Every maximal subgroup of G is by hypothesis artinian and soluble. Application of Baer (**3**, p. 18, Lemma 3.3) shows that

(1) maximal subgroups of G are artinian, soluble, almost abelian, locally finite.

If S is a maximal subgroup of G, then the intersection $\mathsf{J}S$ of all the subgroups of finite index in S is abelian, since S is almost abelian; and $S/\mathsf{J}S$ is finite, since S is artinian. It follows that $\mathsf{J}S$ is an artinian, abelian group without proper finite epimorphic images. Since $\mathsf{J}S$ is a characteristic subgroup of S, it is normalized by the maximal subgroup S of G. Hence $\mathfrak{n}_G\,\mathsf{J}S$ is either S or G; and the latter implies that $\mathsf{J}S = 1$ because of the simplicity of G.

Consider now maximal subgroups A and B of G with $\mathsf{J}A \cap \mathsf{J}B \neq 1$. Then $1 \subset \mathsf{J}A \cap \mathsf{J}B \subset G$ so that $\mathsf{J}A \cap \mathsf{J}B$ is, because of the simplicity of G, not a normal subgroup of G. Noting that $\mathsf{J}A$ and $\mathsf{J}B$ are both abelian, it follows that

$$\{\mathsf{J}A, \mathsf{J}B\} \subseteq \mathfrak{n}_G(\mathsf{J}A \cap \mathsf{J}B) \subset G.$$

Since G is finitely generated, application of the Maximum Principle of Set Theory shows the existence of a maximal subgroup C of G with

$$\mathfrak{n}_G(\mathsf{J}A \cap \mathsf{J}B) \subseteq C.$$

Since $C/\mathsf{J}C$ is finite and $\mathsf{J}A$, $\mathsf{J}B$ are free of proper finite epimorphic images, we have $\{\mathsf{J}A, \mathsf{J}B\} \subseteq \mathsf{J}C$. Thus A and the abelian group $\mathsf{J}C$ are part of the normalizer of $\mathsf{J}A$. But $\mathsf{J}A \neq 1$ is a consequence of $\mathsf{J}A \cap \mathsf{J}B \neq 1$ so that

$$\{A, \mathsf{J}C\} \subseteq \mathfrak{n}_G\,\mathsf{J}A \subset G.$$

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This implies that $\mathsf{J}C \subseteq A$ because of the maximality of A; and since $\mathsf{J}C$ is free of proper finite epimorphic images, we may conclude that $\mathsf{J}C \subseteq \mathsf{J}A$. But we have shown before that $\mathsf{J}A \subseteq \mathsf{J}C$. Hence $\mathsf{J}A = \mathsf{J}C$; and we see similarly that $\mathsf{J}C = \mathsf{J}B$. It follows that

$$A = \mathsf{n}_G \mathsf{J}A = \mathsf{n}_G \mathsf{J}B = B.$$

Thus we have shown that

(2) $\mathsf{J}A \cap \mathsf{J}B = 1$, if A and B are different maximal subgroups of G.

By hypothesis any two maximal subgroups of G are conjugate in G and as such they are in particular isomorphic. Hence

(3) there exists a positive integer j such that $[S: \mathsf{J}S] = j$ for every maximal subgroup S of G.

Consider now two different maximal subgroups S and T of G. Then

$$(S \cap T)/(T \cap \mathsf{J}S) \cong \mathsf{J}S(S \cap T)/\mathsf{J}S \subseteq S/\mathsf{J}S,$$
$$T \cap \mathsf{J}S = (T \cap \mathsf{J}S)/(\mathsf{J}T \cap \mathsf{J}S) \cong \mathsf{J}T(T \cap \mathsf{J}S)/\mathsf{J}T \subseteq T/\mathsf{J}T$$

by (2); and it is a consequence of (3) that

$$[S \cap T: 1] = [S \cap T: T \cap \mathsf{J}S][T \cap \mathsf{J}S: 1] \leqslant j^2.$$

Thus we have shown:

(4) If S and T are two different maximal subgroups of G, then $S \cap T$ is a finite group of an order not exceeding j^2.

We assume next that G is not cyclic and that the normalizer condition is satisfied by every maximal subgroup of G. Thus if T is a proper subgroup of the maximal subgroup S of G, then $T \subset \mathsf{n}_S T$. In other words: the maximal subgroups of G are the only normalizer equal proper subgroups of G. If g is any element of G, then $\{g\} \subset G$. Since G is finitely generated, application of the Maximum Principle of Set Theory shows the existence of a maximal subgroup of G, containing g. Assume now by way of contradiction the existence of a pair of different maximal subgroups of G with non-trivial intersection. Because of (4) there exists amongst these pairs one A, B with maximal (non-trivial) intersection $A \cap B$. Since A and B are distinct maximal subgroups of G, we have

$$A \cap B \subset A \text{ and } A \cap B \subset B;$$

and this implies, because of the normalizer condition,

$$A \cap B \subset \mathsf{n}_A(A \cap B) \quad \text{and} \quad A \cap B \subset \mathsf{n}_B(A \cap B).$$

Since G is simple and $1 \subset A \cap B \subset G$, we conclude that $\mathsf{n}_G(A \cap B) \subset G$. Since G is finitely generated, application of the Maximum Principle of Set

Theory shows the existence of a maximal subgroup C of G with $\mathfrak{n}_G(A \cap B) \subseteq C$. It follows that

$$A \cap B \subset \mathfrak{n}_A(A \cap B) = A \cap \mathfrak{n}_G(A \cap B) \subseteq A \cap C;$$

and from the maximality of the intersection $A \cap B$ we deduce that $A = C$. Likewise it follows that $C = B$. Hence $A = B$, a contradiction proving that $X \cap Y = 1$ for every pair of different maximal subgroups X, Y of G. Thus a partition is formed by the maximal subgroups of G; and we have shown:

(5) *If G is not cyclic and the normalizer condition is satisfied by every maximal subgroup of G, then the set of maximal subgroups of G is a partition of G.*

Assume now that G is not cyclic, but a p-group. If S is a maximal subgroup of G, then every finitely generated subgroup of S is by (1) a finite p-group. Application of **(3**, p. 21, Satz 4.1**)** shows the hypercentrality of S. But every hypercentral group meets the normalizer requirement; see **(7**, p. 219**)**. Application of (5) shows that the set of maximal subgroups of G is a partition of G. This proves B.

We assume next that G is not primary. Since G is finitely generated, there exists a maximal subgroup L of G (Maximum Principle of Set Theory). Consider a p-Sylow subgroup P of G. Since G is not primary, $P \subset G$; and since G is finitely generated, application of the Maximum Principle of Set Theory shows the existence of a maximal subgroup S of G with $P \subseteq S$. But any two maximal subgroups of G are by hypothesis conjugate in G. Hence there exists an element a in G with $S^a = L$ so that $P^a \subseteq S^a = L$. Suppose now that Q is another p-Sylow subgroup of G. From what we have shown already there exists an element b in G with $Q^b \subseteq L$. Since P^a and Q^b are p-Sylow subgroups of G, contained in the maximal subgroup L, they are p-Sylow subgroups of the by (1) artinian and almost abelian group L. Apply (1.1, e) to show that P^a and Q^b are conjugate in L. This proves A.

E. Proof of Main Theorem and its Corollary. The proof of our Main Theorem will be effected in several steps. The first part of this proof will be used in the proof of the Corollary; and the Corollary will be needed to complete the proof of the Main Theorem.

Proof of the validity of conditions II–VII *in artinian and soluble groups.* If the group G is artinian and soluble, then so is every subgroup and every epimorphic image. This shows the validity of conditions II(d), III(c), IV(a), IV(b′), V(a), V(b′), VI(a); and an immediate application of Lemma 3 shows the validity of conditions II(a), III(a), IV(b″), and V(b″), VI(c).

If M is a minimal normal subgroup of the epimorphic image H of G, then M is soluble, artinian, and characteristic simple. It follows that M is a finite, elementary abelian, primary group, showing the validity of II(b), and II(c), III(b), VI(b)—the latter three since $H/\mathfrak{c}_H M$ is essentially the same as the group of automorphisms induced in M by H.

It is clear that VII(a) is satisfied by G. Consider next maximal subgroups A and B of the subgroup S of G and suppose that A and B contain the same normal subgroups of S. It is a consequence of (1.1, b) that $[S: A]$ and $[S: B]$ are both finite, implying that $\mathsf{J}S \subseteq A \cap B$. Since S is as a subgroup of G artinian and soluble, $S/\mathsf{J}S$ is finite and soluble. Hence $A/\mathsf{J}S$ and $B/\mathsf{J}S$ are maximal subgroups of the finite soluble group $S/\mathsf{J}S$, containing the same normal subgroups of $S/\mathsf{J}S$. Application of a theorem of Ore (**8**, p. 451, Theorem 11) shows that $A/\mathsf{J}S$ and $B/\mathsf{J}S$ are conjugate in $S/\mathsf{J}S$. Hence A and B are conjugate in S, proving VII(b). Every simple factor of G is finite—see (**3**, pp. 7–8, Satz 2.1)—and soluble, hence cyclic of order a prime; and this shows trivially the validity of VII(c) and (d).

Derivation of I *from* II. Assume that G meets requirement II and consider a minimal normal subgroup M of the epimorphic image H of G. Assume by way of contradiction that M is not primary. By II(b) there exists a p-Sylow subgroup $S \neq 1$ (for some prime p) of M with finite S^M. It follows that all the p-Sylow subgroups of M are conjugate in M; see, e.g., (**7**, p. 163, bottom). Since M is not primary, $1 \subset S \subset M$. Since M is a minimal normal subgroup of H, it follows that S is not a normal subgroup of H and that therefore $\mathsf{n}_H S \subset H$. If x is an element in H, then S^x is a p-Sylow subgroup of the normal subgroup M. But all p-Sylow subgroups of M are conjugate to S in M. Hence there exists an element y in M such that $S^y = S^x$; and it follows that S is normalized by xy^{-1}. Since xy^{-1} belongs to $\mathsf{n}_H S$ and y belongs to M, we conclude that x belongs to $M\mathsf{n}_H S$, proving that $H = M\mathsf{n}_H S$. Since $S^H = S^M$ is finite, $[H: \mathsf{n}_H S]$ is finite and different from 1. This implies the existence of a maximal subgroup L of H containing $\mathsf{n}_H S$. We note that $S \subset \mathsf{n}_H S \subseteq L$ and $1 \subset S \subseteq M \cap L$. It is a consequence of II(a) and Lemma 2 that H is an s-group. Consequently there exists a normal subgroup J of H with $H = JL$ and $J \cap L = L_H$ (by Lemma 1). From $M \cap J = 1$ we would deduce that these two normal subgroups centralize each other. Hence $S \subset M$ would be centralized by J so that

$$J \subseteq \mathsf{c}_H M \subseteq \mathsf{c}_H S \subseteq \mathsf{n}_H S \subseteq L \subset H = JL = L,$$

a contradiction. Hence $M \cap J \neq 1$; and since M is a minimal normal subgroup, J a normal subgroup of H, we have $M = M \cap J \subseteq J$ so that

$$M \cap L \subseteq J \cap L = L_H.$$

It follows that

$$1 \subset M \cap L = M \cap L_H.$$

Since M is a minimal normal subgroup of H and L_H is a normal subgroup of H, we conclude that $M = M \cap L_H \subseteq L$. Recalling that $\mathsf{n}_H S \subseteq L$ we find that

$$H = M\mathsf{n}_H S = ML = L \subset H,$$

a contradiction. This proves that M is primary. Application of II(c) shows that $H/\mathsf{c}_H M$ is finite. We note that we have proved:

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(1) If M is a minimal normal subgroup of the epimorphic image H of G, then M is primary and $H/\mathfrak{c}_H M$ is finite.

A combination of (1), II(d), and the Proposition shows next that

(2) G is artinian and almost abelian.

Consequently, there exists an abelian normal subgroup A of G with finite G/A. The principal factors of the finite group G/A are, because of (1), primary so that the finite group G/A is soluble. Since A is abelian and G/A is soluble, G is soluble. G is, by (2), artinian, completing the proof of the equivalence of I and II.

Proof of the Corollary. If, first, G is a finite soluble group, then every subgroup of G is a finite soluble group. As a consequence of Lemma 3 every subgroup of G is an $\mathfrak{s}$-group; and now it is clear that conditions (ii) and (iii) are consequences of (i).

If, secondly, (ii) is satisfied by G, then G meets requirement II of the Main Theorem. We have already verified the equivalence of conditions I and II of the Main Theorem: G is soluble, conditions (i) and (ii) are equivalent.

If, thirdly, (iii) is satisfied by G, then we form the intersection $\mathfrak{J}G$ of all the subgroups X of G with finite index $[G: X]$. This is a well-determined characteristic subgroup of G. It is a consequence of (iii.b) and Baer (**2**, p. 3, Proposition 2) that $G/\mathfrak{J}G$ is finite too. Since G is an $\mathfrak{s}$-group by (iii.c), so is its epimorphic image $G/\mathfrak{J}G$, by Lemma 2. Thus $G/\mathfrak{J}G$ is a finite $\mathfrak{s}$-group. We have already verified the equivalence of (i) and (ii), proving the solubility of $G/\mathfrak{J}G$. Since G is, by (iii.a), finitely generated and $G/\mathfrak{J}G$ is finite, $\mathfrak{J}G$ is likewise finitely generated; see, e.g. (**9**, p. 153, Satz 4). Assume now by way of contradiction that G is not finite and soluble. This implies that $\mathfrak{J}G \neq 1$, since $G/\mathfrak{J}G$ is finite and soluble. Since $\mathfrak{J}G$ is finitely generated, there exists a maximal normal subgroup W of $\mathfrak{J}G$ (Maximum Principle of Set Theory). It is a consequence of (iii.c) that $\mathfrak{J}G$ is an $\mathfrak{s}$-group. By Lemma 2 its simple epimorphic image $\mathfrak{J}G/W$ is likewise an $\mathfrak{s}$-group. The finitely generated group $Q = \mathfrak{J}G/W \neq 1$ possesses a maximal subgroup S (Maximum Principle of Set Theory). There exists a normal subgroup T of the $\mathfrak{s}$-group Q with $Q = ST$ and $S \cap T = (S \cap T)^T$. Since T is a normal subgroup of the simple group Q, we have $T = 1$ or $T = Q$. In the first case we would find $S \subset Q = ST = S$, a contradiction. Hence $T = Q$ so that $S = S \cap Q = (S \cap Q)^Q = S^Q$ is a normal subgroup of the simple group Q. From $S \subset Q$ we deduce that $S = 1$ so that 1 is a maximal subgroup of Q. Hence Q is cyclic of order a prime. From the finiteness of $G/\mathfrak{J}G$ and of $Q = \mathfrak{J}G/W$ we deduce the finiteness of $[G: W]$, implying the contradiction $W \subset \mathfrak{J}G \subseteq W$. Hence G is finite and soluble, proving the equivalence of (i) and (iii).

Derivation of I *from* III. Assume that G meets requirement III. Then we deduce from III(b), (c) and the Proposition that G is artinian and almost abelian. If F is a finite epimorphic image of G, then F is an $\mathfrak{s}$-group because of III(a) and hence soluble by the Corollary, which has already been proved. Thus G is an extension of an abelian normal subgroup by a finite soluble group, and hence artinian and soluble, proving the equivalence of I and III.

Derivation of I *from* IV. If G meets requirement IV, then it is an immediate consequence of IV(a) that

(3) G is a torsion group.

Next denote by C the product of all the hyperabelian normal subgroups of G. Then C is a characteristic subgroup of G; and every epimorphic image $H \neq 1$ of C is the product of its hyperabelian normal subgroups. Clearly there exists at least one hyperabelian normal subgroup, not 1, of H; and this implies the existence of an abelian subnormal subgroup, not 1, of H. Because of IV(a) condition (3) of (**4**, pp. 359–360, Satz 8.15, **A**) is satisfied by C. Hence

(4) C is soluble and artinian.

If A is an abelian subgroup of G/C, then there exists a subgroup B of G with $C \subseteq B$ and $B/C = A$. The group B is soluble because of (4) and its abelian subgroups are artinian because of IV(a). Hence condition (3) of (**4**, pp. 359–360, Satz 8.15, **A**) is satisfied by B, proving that B is artinian. Hence $A = B/C$ is likewise artinian; and we have shown that

(5) abelian subgroups of G/C are artinian.

Assume now by way of contradiction that G is not soluble and artinian. Then $C \subset G$ by (4). We apply IV(b) on the epimorphic image $H = G/C \neq 1$ of G. Consequently, there exists a normal subgroup $N \neq 1$ of H such that every finitely generated subgroup F of N meets the following two requirements:

(6′) the minimum condition is satisfied by the normal subgroups of F;

(6″) to every maximal subgroup S of F there exists a normal subgroup T of F with $F = ST$ and $S \cap T = (S \cap T)^T$.

But then every finitely generated subgroup of N satisfies condition (iii) of the Corollary, already proved. Hence we have shown that

(6) every finitely generated subgroup of N is finite and soluble.

Tschernikow (**10**, p. 128, Teorema 3) has proved the beautiful theorem that a group is artinian and soluble if its abelian subgroups are artinian and its finitely generated subgroups are finite and soluble. Because of (5) every abelian subgroup of N is artinian. Because of (6) it follows that

(7) N is artinian and soluble.

Now there exists a normal subgroup K of G with $C \subset K$ and $K/C = N$.

Since N and C are soluble (by (7) and (4)), K is soluble. Because of the definition of C we have

$$K \subseteq C \subset K,$$

a contradiction proving that G is soluble and artinian, completing the proof of the equivalence of I and IV.

Derivation of IV *from* V. If a group G meets requirement V(b), and if σ is an epimorphism of G upon H, then there exists to every finitely generated subgroup S of H a finitely generated subgroup T of G with $T^\sigma = S$. By V(b′) the minimum condition is satisfied by the normal subgroups of T and hence also by the normal subgroups of S. By V(b″) the finitely generated subgroup T of G is an $\mathfrak{s}$-group so that S is, by Lemma 2, an $\mathfrak{s}$-group. We have shown that a strong form of IV(b) is a consequence of V(b). Hence V implies IV. But IV has been shown to be equivalent to I, proving the equivalence of I and V.

Derivation of I *from* VI. Assume that G meets requirement VI. Denote by G^* the product of all soluble normal subgroups of G. This is a well-determined characteristic subgroup of G. If $H \neq 1$ is an epimorphic image of G^*, then H is the product of soluble normal subgroups. Consequently there exists a soluble normal subgroup, not 1, of H; and this implies the existence of an abelian normal subgroup, not 1, of H. Hence G^* is hyperabelian. Combine this with VI(a) and (**4**, pp. 359–360, Hauptsatz 8.15, **A**) and it follows that

(8) G^* is artinian and soluble.

If N/G^* is a soluble normal subgroup of G/G^*, then N is, by (8), a soluble normal subgroup of G so that $N \subseteq G^*$ by the definition of G^*. Hence

(9) 1 is the only soluble normal subgroup of G/G^*.

If F is a finite epimorphic image of G, then a combination of VI(c) and the Corollary, already proved, shows the solubility of F. Hence

(10) finite epimorphic images of G are soluble.

Assume now by way of contradiction that G is not artinian and soluble. Then we deduce from (8) that $G \neq G^*$. If G/G^* were finite, then G/G^* would be soluble by (10) so that $G/G^* = 1$ by (9), a contradiction showing the infinity of G/G^*. By VI(b) there exists, therefore, a normal subgroup $N \neq 1$ of $G/G^* = H$ such that $H/\mathfrak{c}_H N$ is finite. Since $\mathfrak{z}N = N \cap \mathfrak{c}_H N$ is an abelian normal subgroup of H, we deduce that $\mathfrak{z}N = 1$ from (9). Consequently,

$$N = N/(N \cap \mathfrak{c}_H N) \cong N\mathfrak{c}_H N/\mathfrak{c}_H N \subseteq H/\mathfrak{c}_H N.$$

The last of these groups is a finite epimorphic image of G and hence, by (10), soluble. Consequently N is a soluble normal subgroup of H; and we deduce the contradiction $N = 1$ from (9). Hence G is soluble and artinian, showing the equivalence of I and VI.

Derivation of I *from* VII. If the group G could meet requirement VII without being soluble, then there would exist among the non-soluble subgroups of G a minimal one M. This group M has the following properties:

(11) M is artinian, but not soluble.

(12) Every proper subgroup of M is soluble.

If M were not finitely generated, then every finitely generated subgroup of M would be a proper subgroup of M and, as such, artinian and soluble. It follows from (1.1, c) that finitely generated subgroups of M are finite and soluble; and this would imply the solubility of M itself—see, for instance, (**3**, p. 18, Lemma 3.3), a contradiction. Thus we have shown:

(13) M is finitely generated.

Since M is not soluble, but finitely generated, there exists a maximal normal subgroup N of M (Maximum Principle of Set Theory). Then $E = M/N$ is a finitely generated simple group. Since N is soluble by (12), and since M is not soluble, E is not soluble. Every maximal subgroup of E has the form S/N with S a maximal subgroup of M. Since M/N is simple, N is the product of all normal subgroups of M which are part of S. It follows that any two maximal subgroups of M which contain N actually contain the same normal subgroups of M. They are therefore conjugate in M by VII (b); and this implies that

(14) any two maximal subgroups of E are conjugate in E.

Because of (14) we may apply Lemma 4 to the finitely generated, simple, non-cyclic group E; and this leads to an immediate contradiction because of VII (c) and (d). Hence I is a consequence of VII.

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