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A NOTE ON A SAMPLE-PATH RATE CONSERVATION LAW AND ITS RELATIONSHIP WITH $H = \lambda G$

KARL SIGMAN*, *Columbia University*

Abstract

We present a simple sample-path version of the rate conservation law (of Miyazawa) and then show that the $H = \lambda G$ law (of Heyman and Stidham) is essentially the same law, that is, either one can be derived from the other. As a final remark we illustrate the use of both laws jointly to quickly obtain a queueing result.

QUEUE; TIME AVERAGE

1. Introduction

Miyazawa [7], [8] gave the *rate conservation law* (RCL) for a time stationary stochastic process $\{X(t) : t \in \mathcal{R}\}$:

$$EX'(t) = \lambda E^0\{X(0-) - X(0+)\}, \tag{1.1}$$

where λ denotes the intensity of an underlying stationary point process, $\psi = \{t_n\}$, which includes the discontinuities of $X(t)$ and E^0 denotes expectation under the Palm distribution of X with respect to ψ.

RCL has been used to derive a variety of useful and interesting relations between time and customer averages in queues (see for example [7], [8], [14], [11], [4]). Recently, it has been generalized to cover certain non-stationary stochastic processes using local martingales (Mazumdar et al. [6]), as well as to multiple jump situations (Miyazawa [9]), where it is also shown to imply Little's formula. It also has been shown to admit the Palm inversion formula for stationary point processes (Brémaud [1]). The purpose of the present note is to give a simple *sample-path* version of (1.1) and then show that the well-known sample-path law, $H = \lambda G$, of Heyman and Stidham [5] (originally proved by Brumelle [3] in a stochastic setting) is really the same law: either law can be viewed as a special case of the other. Finally we remark how the two laws can be used jointly to derive queueing results.

2. The rate conservation law and $H = \lambda G$

Let $x : \mathcal{R}_+ \to \mathcal{R}$ be a function and let $\psi = \{t_n : n \geqq 0\}$ denote a simple point process on $\mathcal{R}_+$ that contains the discontinuities (if any) of x. We assume:

(i) x is right continuous with left-hand limits.

(ii) x is piecewise continuously differentiable with a right derivative existing at all points: on any bounded interval there exists (at most) a finite number of points where x is not differentiable, but at any such point x is right differentiable and between any two successive such points, x has a continuous derivative.

The above properties (which can be relaxed) are virtually always satisfied by functionals of interest of any queueing system modeled in continuous time where ψ typically (but not

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* Postal address: Department of Industrial Engineering and Operations Research, Columbia University, Mudd Building, New York, NY 10027, USA.

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necessarily) represents the arrival and or departure epochs of a queueing system. Throughout, we make no stochastic assumptions: (x, ψ) is assumed a fixed sample path (from perhaps some underlying probability space). Let $N(t)$ denote the corresponding counting process. Let $-J_n \overset{\text{def}}{=} x(t_n+) - x(i_n-)$ denote the nth jump size. We define (when the limits exist)

$$\lambda \overset{\text{def}}{=} \lim_{t\to\infty} \frac{N(t)}{t},$$

$$E(J) \overset{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} J_n, \tag{2.1}$$

$$E(x') \overset{\text{def}}{=} \lim_{t\to\infty} \frac{1}{t} \int_0^t x'(x)\, ds.$$

Here, $x'(t)$ denotes the right derivative at time t. Even though no stochastic set-up is assumed, we use the expectation notation, E, for convenience.

Theorem 2.1. If λ and $E(J)$ exist and are finite and $x(t)/t \to 0$ as $t \to \infty$ then $E(x')$ exists and

$$E(x') = \lambda E(J). \tag{2.2}$$

Proof. Using Assumptions (i) and (ii) we can rewrite any interval $[0, t]$ as the finite union of disjoint subintervals within each of which x has a continuous derivative. From elementary integration theory, we thus have

$$\int_0^t x'(s)\, ds = x(t) - x(0) + \sum_{n=1}^{N(t)} J_n.$$

Dividing by t and taking the limit as $t \to \infty$ gives the result.

Deriving $H = \lambda G$ from RCL. We consider $H = \lambda G$ in the set-up found in Wolff [13], Theorem 5, p. 290. We have non-negative functions f_n (non-negativity is not essential) defined on the interval $I_n \overset{\text{def}}{=} [t_n, t_n + l_n)(f_n(t) = 0$ for $t \notin I_n$ and the technical condition $l_n/n \to 0$ is assumed), $G_n \overset{\text{def}}{=} \int_0^\infty f_n(s)\, ds$, $H(t) \overset{\text{def}}{=} \sum_{n=1}^\infty f_n(t)$, $H \overset{\text{def}}{=} \lim_{t\to\infty} 1/t \int_0^t H(s)\, ds$, $G \overset{\text{def}}{=} \lim_{n\to\infty} 1/n \sum_{j=1}^n G_j$, and λ is defined as in (2.1) using $\psi \overset{\text{def}}{=} \{t_n\}$. The theorem states that $H = \lambda G$ if both λ and G exist and are finite. To use our Theorem 2.1 RCL to prove this, let $G_n(t) \overset{\text{def}}{=} \int_t^\infty f_n(s)\, ds$; $t \in I_n$ (0 otherwise), and $x(t) \overset{\text{def}}{=} \sum_{n=1}^\infty G_n(t)$. The discontinuities of x are contained in ψ and $G_n'(t) = -f_n(t)$ so that $x'(t) = -H(t)$. Moreover, $-J_n = G_n$; thus, we need only verify the technical condition that $x(t)/t \to 0$. To this end, let $t_n^d = t_n + l_n$ and let ψ^d, $N^d(t)$ denote the corresponding point process (with the points put in ascending order). Then $x(t) \leqq y(t) \overset{\text{def}}{=} \sum_{n=1}^{N(t)} G_n - \sum_{n=1}^{N^d(t)} \tilde{G}_n$ where the $\tilde{G}_n$ denote the reordering of the G_n with respect to the order of the points in ψ^d. The technical assumption $l_n/n \to 0$ implies that $N^d(t)/t \to \lambda$, thus (since both λ and G are assumed finite), we obtain $x(t)/t \leqq y(t)/t \to \lambda G - \lambda G = 0$.

Deriving RCL from $H = \lambda G$. Let $l_n = T_n \overset{\text{def}}{=} t_{n+1} - t_n$ and $I_n \overset{\text{def}}{=} [t_n, t_{n+1})$ and assume the hypothesis of Theorem 2.1; we shall also assume that $\lambda > 0$ (to help our derivation proceed at top speed) so that in particular, $l_n/n \to 0$. Define $f_n(s) \overset{\text{def}}{=} x'(s)$, $s \in I_n$ (0 otherwise), $G_n \overset{\text{def}}{=} \int_0^\infty f_n(s)\, ds = \int_{t_n}^{t_{n+1}} f_n(s)\, ds = x(t_{n+1}-) - x(t_n) = J_{n+1} + x(t_{n+1}) - x(t_n)$ and $H(s) \overset{\text{def}}{=} \sum_{n=1}^\infty f_n(s) = x'(s)$. From $H = \lambda G$ we thus obtain our result if we can show that $1/n \sum_{j=1}^n x(t_{j+1}) - x(t_j) = x(t_{n+1}) - x(t_1)/n \to 0$. But this follows from the assumption that $x(t)/t \to 0$ because

$$\frac{x(t_n)}{n} = \frac{x(t_n)}{t_n} \left\{ \frac{t_n}{n} \right\}$$

and $t_n/n \to \lambda^{-1} < \infty$.

Remark 1. We can also get an iff version of Theorem 2.1, as follows. Assume that λ exists and $0 < \lambda < \infty$ and that $x(t)/t \to 0$ as $t \to \infty$. Then $E(x')$ exists (and is finite) if and only if $E(J)$

exists (and is finite), in which case (2.2) holds. This version can be thought of as the iff version of $H = \lambda G$ found in Whitt [12].

Remark 2. The connection between RCL and *level-crossing* methods (see for example, [2], and [15]) for a process $Y(t)$ (satisfying conditions (i) and (ii) say, with underlying point process t_n with rate λ) can be easily realized by considering for a fixed level x, the indicator process $X(t) = I\{Y(t) \leqq x\}$ and then applying RCL. In this case $X'(t) = 0$ and we additionally shall assume that all jumps are non-negative and that $Y(t)$ is non-increasing between consecutive t_n (as is typically the case for queues where the t_n denote arrival times). We must therefore consider the additional point process $\{t_m(x): m \geqq 1\}$ denoting the consecutive times at which $Y(t)$ makes a downcrossing at level x (assumed to have rate $\lambda(x)$); these additional points are discontinuities for X. Applying RCL (and observing that jumps of X from $t_m(x)$ are of magnitude 1) we obtain $0 = -\lambda(x) + \lambda(P^0(Y(0-) \leqq x) - P^0(Y(0+) \leqq x))$, where P^0 denotes the empirical distribution of Y with respect to the points t_n (which under stochastic assumptions is the Palm distribution of Y). Rewriting gives

$$\lambda(x) = \lambda(P^0(Y(0-) \leqq x) - P^0(Y(0+) \leqq x)).$$

Remark 3. Typically, in practice, one of the two laws is better suited for the problem at hand depending on the particular model and process under consideration. Nevertheless, as the following example illustrates, sometimes one can gain by using both laws at the same time on the same model.

Example. Let $V(t)$ denote total work at time t in a FIFO single-server queue with input the (simple) marked point process $\{(t_n, S_n); n \geqq 0\}$ (assumed a fixed sample path of arrival times and service times, $t_0 \equiv 0$). Let $W_a(t)$ denote the *attained waiting time* of the customer in service at time t (how long the customer currently in service has been in the system; set to 0 if system is empty). It is known that the empirical distributions of V and W_a are identical:

$$P(V > x) \stackrel{\text{def}}{=} \lim_{t \to \infty} \frac{1}{t} \int_0^t I\{V(s) > x\}\, ds = P(W_a > x) \stackrel{\text{def}}{=} \lim_{t \to \infty} \frac{1}{t} \int_0^t I\{W_a(s) > x\}\, ds; \; x \geqq 0.$$

See Sengupta [11], Sakasegawa and Wolff [10] (also see [1]). We shall now quickly prove this result (only to illustrate the quick joint use of the two laws; a new proof of this result is certainly not needed). On the one hand we apply $H = \lambda G$ to $H(t) = I(W_a(t) > x)$ with $f_n(t) = I(t - t_n > x;\, t_n + D_n \leqq t < t_n + D_n + S_n)$, where $D_n \stackrel{\text{def}}{=} V(t_n-)$ denotes the delay in queue of the nth customer. Then

$$G_n = \int_0^\infty f_n(t)\, dt = \min\{(D_n + S_n - x)^+, S_n\},$$

and hence

$$P(W_a > x) = \lambda E \min\{(D + S - x)^+, S\},$$

where the expectation denotes the empirical average of the G_n's. On the other hand, by applying RCL to $x(t) \stackrel{\text{def}}{=} (V(t) - x)^+$, we have $x'(t) = -I\{V(t) > x\}$, and $-J_n = (D_n + S_n - x)^+ - (D_n - x)^+ = \min\{(D_n + S_n - x)^+,\; S_n\} = G_n$ from which we immediately obtain our result.

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