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ON MODULATED TOPOLOGICAL VECTOR SPACES AND APPLICATIONS

WOJCIECH M. KOZLOWSKI

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In Memoriam Henryk Hudzik

Abstract

We introduce a notion of modulated topological vector spaces, that generalises, among others, Banach and modular function spaces. As applications, we prove some results which extend Kirk's and Browder's fixed point theorems. The theory of modulated topological vector spaces provides a very minimalist framework, where powerful fixed point theorems are valid under a bare minimum of assumptions.

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1. Introduction

Fixed point theory for contractive and nonexpansive mappings defined in Banach spaces has been extensively developed since the mid 1960s. Fixed point theory has been extended to general metric spaces and independently to modular function spaces (see [10, Ch. 2] or standard texts on metric fixed point theory such as [5, 7]). The theory of modular function spaces was introduced in the late 1980s in [14–16] as a generalisation of several classes of function and sequence spaces including l^p, L^p, Orlicz, Musielak–Orlicz, Lorentz and Marcinkiewicz spaces, and then used extensively in the fixed point theory starting from the seminal 1990 paper [11]. For recent surveys of this theory we refer to [10, 17, 18] and also [8, 9]. It has been frequently noted that modular equivalents of the norm notions of contractions and nonexpansive mappings naturally occur in applications and quite often allow results not available within the limitations of normed spaces. Despite essential differences between norms and modulars (the latter are allowed to take infinite values and may not have the triangle property), many surprising analogies have been discovered. To give just a few examples, the property (R) plays a similar role to reflexivity, ρ-a.e. convergence relates well to the weak topology, modular uniform convexity (though more complex) plays a similar role to norm uniform convexity, and major fixed point theorems like the Banach Contraction Principle and the fixed point theorems of Browder and Kirk have been expressed and proved in the language of modulars (see [10]).

The aim of this note is to introduce a large class of vector spaces that include both Banach spaces and modular function spaces (and many others) and to indicate possible applications to fixed point theory.

2. Foundations

Let X be a real vector space. Following the classical work by Nakano [20] (see also [19]), we start with the following definition.

Definition 2.1. A functional $\rho : X \to [0, \infty]$ is called a convex modular if

(1) $\rho(x) = 0$ if and only if $x = 0$;

(2) $\rho(-x) = \rho(x)$;

(3) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ for any $x, y \in X$, and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

If there exists an $s > 0$ such that for any $x, y \in X$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ the condition

(3′) $\rho(\alpha x + \beta y) \leq \alpha^s\rho(x) + \beta^s\rho(y)$

is satisfied (instead of (3)), then ρ is called an s-convex modular. Of course, a 1-convex modular is simply a convex modular.

The vector space $X_\rho = \{x \in X : \rho(\lambda x) \to 0, \text{as } \lambda \to 0\}$ is called a modular space.

Remark 2.2. In the literature devoted to topics related to modular spaces, notions of pseudomodulars and semimodulars are frequently utilised. With some care, most of the results discussed in this paper can be adapted to the language of semimodulars and a few to pseudomodulars. For both clarity and brevity we always assume here that ρ is a modular.

In fixed point theory for mappings acting in modular function spaces, the notion of ρ-convergence proved extremely useful (see [8–10]). Since ρ-convergence is a pure modular concept, we can use this idea and related notions also in the context of any modular spaces and, in particular, for modulated topological vector spaces (see Definition 2.4). Note that ρ-convergence is, in general, different from the modular convergence as defined in [19, 20]. The notions introduced in Definition 2.3 below for general modular spaces follow the same pattern as their equivalents in modular function spaces (see [10, Definition 3.4]).

Definition 2.3. Let ρ be a convex modular defined on a vector space X.

(a) We say that $\{x_n\}$, a sequence of elements of X_ρ, is ρ-convergent to x and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$.

(b) A sequence $\{x_n\}$ where $x_n \in X_\rho$ is called ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

(c) X_ρ is called ρ-complete if every ρ-Cauchy is ρ-convergent to an $x \in X_\rho$.

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(d) A set $B \subset X_\rho$ is called ρ-closed if for any sequence of $x_n \in B$, the convergence $x_n \xrightarrow{\rho} x$ implies that x belongs to B.

(e) A set $B \subset X_\rho$ is called ρ-bounded if its ρ-diameter $\delta_\rho(B)$ is finite, where $\delta_\rho(B) = \sup\{\rho(x - y) : x \in B, y \in B\}$.

(f) A set $K \subset X_\rho$ is called ρ-compact if for any $\{x_n\}$ in K, there exists a subsequence $\{x_{n_k}\}$ and an $x \in K$ such that $\rho(x_{n_k} - x) \to 0$.

(g) Let $x \in X_\rho$ and $C \subset X_\rho$. The ρ-distance $d_\rho(x, C)$ between x and C is defined as $d_\rho(x, C) = \inf\{\rho(x - y) : y \in C\}$.

(h) A ρ-ball $B_\rho(x, r)$ is defined by $B_\rho(x, r) = \{y \in X_\rho : \rho(x - y) \leq r\}$.

It can be easily proved that the ρ-limit is well (uniquely) defined (see [10, Proposition 3.2]), and that if a sequence $\{x_n\}$ of elements of X_ρ is ρ-convergent to $x \in X$ then $x \in X_\rho$ (see [10, Proposition 3.2]). However, there are some essential differences between ρ-convergence and convergence in the sense of norm spaces. First of all, ρ-convergence does not necessarily imply the ρ-Cauchy condition. Also, $x_n \xrightarrow{\rho} x$ does not imply in general that $\lambda x_n \xrightarrow{\rho} \lambda x$, where $\lambda > 1$.

Let us introduce the main concept of this work.

Definition 2.4. Let ρ be a convex modular defined on X and let τ be a linear, Hausdorff topology on X_ρ. The triplet (X_ρ, ρ, τ) is called a modulated topological vector space if the following two conditions are satisfied:

(i) ρ is τ-lower semicontinuous on X;

(ii) if $x_n \xrightarrow{\rho} x$ then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \xrightarrow{\tau} x$, where $x, x_n \in X$.

Proposition 2.5. *Let (X_ρ, ρ, τ) be a ρ-complete modulated topological vector space. The following assertions follow immediately from the above definitions.*

(i) *Every τ-closed set is also ρ-closed.*

(ii) *Every ρ-compact set is also sequentially τ-compact.*

(iii) *Every ρ-ball $B_\rho(x, r)$ is τ-closed (and hence also ρ-closed).*

Remark 2.6. The theory of modulated topological vector spaces provides a very minimalist framework compared with the theory of Banach or modular function spaces. As shown in the next section, in this framework powerful theorems are valid under a bare minimum of assumptions.

Remark 2.7. In this note we assume for the sake of simplicity that ρ is a convex modular. However parallel results can be obtained for s-convex modulars.

In the Introduction we claimed that the surprising similarity between concepts and methods of Banach spaces and modular function spaces was the primary inspiration for the introduction of modulated topological vector spaces. By a quick inspection one can easily verify that in the case where ρ is a norm in a Banach space X and τ is the corresponding weak topology in X, all conditions of Definition 2.4 are satisfied (recall that the norm is weak lower semicontinuous in Banach spaces) and obviously the completeness comes by definition. Hence every Banach space is a ρ-complete modulated topological vector space. Similarly $(X^*, \|\cdot\|_{X^*}, \tau)$, where τ is the weak*-topology is a complete modulated topological vector space.

It is also easy to see that every modular function space L_ρ, where ρ is a regular convex function modular (see [10, Definition 3.1]), is a ρ-complete modulated topological vector space. In this case, (i) of Definition 2.4 follows from the Fatou property of ρ [10, Proposition 3.4], while (ii) is proved in [10, Theorem 3.1]. The ρ-completeness of L_ρ is actually proved in [10, Theorem 3.2]. A surprising fact is that all these definitions and proofs significantly depend on the order and orthogonal subadditivity properties of function modulars, concepts completely absent from the theory of modulated topological vector spaces. However, the absence of order-related assumptions in the definition of modulated topological vector spaces allows this theory to be used for the study of a large variety of modular spaces, for example Fenchel–Orlicz spaces introduced by Turett [22] or generalised Orlicz–Sobolev spaces introduced by Hudzik [6], and many others.

The following example illustrates the flexibility of modulated topological vector spaces.

Example 2.8. Let $X_\rho = L^1[0, 1]$, the space of all functions summable with respect to the Lebesgue measure m, where $\rho = \|\cdot\|_{L^1}$. We can consider X_ρ as a Banach space and hence modulated topological vector space (with τ being the weak topology) or as a modular function space and hence modulated topological vector space (in which case τ will be topology of convergence m-almost everywhere). However it is also easy to check (using Fatou's Lemma) that the triplet (X_ρ, ρ, τ_m), where τ_m is the topology of convergence in measure m, also satisfies the conditions of Definition 2.4 and hence is a modulated topological vector space as well. Interestingly, it is known that $L^1[0, 1]$ fails the w-fpp (weak fixed point property) (see [1, 10]) but it has the τ_m-fpp.

3. Normal structure in modulated topological vector spaces

For the last 30 years, the evolution of fixed point theory has demonstrated the great usefulness of modular space techniques. Therefore, the fundamental fixed point existence theorems serve as an excellent example of the application of the theory of modulated topological vector spaces introduced in the previous section. Because of the important role played by normal structure during the first 20 years (since the 1965 paper by Kirk [12]) in fixed point theory, we have chosen this property as an illustration of the power of the theory introduced in the current note. Observe that an analogous property in modular function spaces was defined and investigated in [11] as early as 1990 (see also [10]).

Let us start with some basic definitions and results valid generally in modular function spaces. Most of the references below refer to the context of modular function spaces but can be easily applied in the general case of modular spaces. As in the previous section, X is a vector space and ρ is a convex modular defined on X.

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Definition 3.1 (see [10, Definitions 5.2 and 5.6] for function modular equivalents). Let C be a ρ-bounded subset of X_ρ.

(1) A mapping $T : C \to C$ is called ρ-nonexpansive if $\rho(T(x) - T(y)) \leq \rho(x - y)$ for any $x, y \in C$.
(2) The quantity $r_\rho(x, C) = \sup\{\rho(x - y) : y \in C\}$ will be called the *ρ-Chebyshev radius of C with respect to x*.
(3) The *ρ-Chebyshev radius of C* is defined by $R_\rho(C) = \inf\{r_\rho(x, C) : x \in C\}$.
(4) The *ρ-Chebyshev centre of C* is defined by $C_\rho(C) = \{x \in C : r_\rho(x, C) = R_\rho(C)\}$.

Note that $R_\rho(C) \leq r_\rho(x, C) \leq \delta_\rho(C)$, for any $x \in C$ and any ρ-bounded nonempty subset C of X_ρ. Observe that $C_\rho(C)$ may be empty.

Definition 3.2 (see [10, Definition 5.7] for function modular equivalents). Let C be a ρ-bounded subset of X_ρ.

(1) We say that A is a *ρ-admissible subset of C* if $A = \bigcap_{i\in I} B_\rho(x_i, r_i) \cap C$, where $x_i \in C$, $r_i \geq 0$ and I is an arbitrary index set. The family of all ρ-admissible subsets of C will be denoted by $\mathcal{A}_\rho(C)$. If D is a subset of C, we write

$$co_C(D) = \bigcap_{x\in D} B_\rho(x, r_\rho(x, D)) \cap C.$$

Note that $co_C(D) \in \mathcal{A}_\rho(C)$ and is the smallest ρ-admissible subset of C which contains D.
(2) We say that $\mathcal{A}_\rho(C)$ is countably compact if any decreasing sequence $\{A_n\}_{n\geq 1}$ of nonempty elements of $\mathcal{A}_\rho(C)$, has a nonempty intersection.
(3) $\mathcal{A}_\rho(C)$ is said to be normal (or equivalently, that C has ρ-normal structure) if for each ρ-admissible subset A of C, not reduced to a single point, we have $R_\rho(A) < \delta_\rho(A)$.

Next we discuss a constructive result discovered for Banach spaces by Kirk [13] and evolved for modular function spaces in [11]. The main ingredient in this constructive proof is a technical lemma due to Gillespie and Williams [4]. The next lemma is the modular version of this technical result.

Lemma 3.3. *Let C be a ρ-bounded subset of X_ρ. Let $T : C \to C$ be a ρ-nonexpansive mapping. Assume that $\mathcal{A}_\rho(C)$ is normal. Let $A \in \mathcal{A}_\rho(C)$ be nonempty and T-invariant, that is, $T(A) \subset A$. Then there exists a nonempty $A_0 \in \mathcal{A}_\rho(C)$ such that A_0 is T-invariant, $A_0 \subset A$ and*

$$\delta_\rho(A_0) \leq \frac{\delta_\rho(A) + R_\rho(A)}{2}.$$

Proof. The proof follows the same path as the original one, and remains essentially the same as in the case of modular function spaces (see [11, Lemma 5.4]). □

We are now ready to exhibit a modular analogue of Kirk's fixed point theorem [13].

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Theorem 3.4 (see [10, Theorem 5.10] for a function modular equivalent). *Let ρ be a convex modular on X. Let C be a ρ-bounded and ρ-closed nonempty subset of X_ρ. Assume that $\mathcal{A}_\rho(C)$ is normal and countably compact. If $T : C \to C$ is ρ-nonexpansive, then T has a fixed point.*

Proof. Let $\mathcal{F} = \{D \in \mathcal{A}_\rho(C) : D \neq \emptyset \text{ and } T(D) \subset D\}$. Note that $\mathcal{F} \neq \emptyset$ since $C \in \mathcal{F}$. Define $\tilde{\delta} : \mathcal{F} \to [0, +\infty)$ by

$$\tilde{\delta}(D) = \inf\{\delta_\rho(B) : B \in \mathcal{F} \text{ and } B \subset D\}.$$

Set $D_1 = C$. By definition of $\tilde{\delta}(D_1)$, there exists $D_2 \in \mathcal{F}$ such that $D_2 \subset D_1$ and $\delta_\rho(D_2) < \tilde{\delta}(D_1) + 1$. Using the same argument we can inductively construct a sequence $\{D_n\}$ such that $D_{n+1} \in \mathcal{F}$, $\delta_\rho(D_{n+1}) < \tilde{\delta}(D_n) + 1/n$ and $D_{n+1} \subset D_n$. Since $\mathcal{A}_\rho(C)$ is countably compact, $D_\infty = \bigcap_{n\geq 1} D_n$ is not empty. Clearly, $D_\infty \in \mathcal{F}$. It remains to be proved that D_∞ is reduced to one point. Using Lemma 3.3, there exists $D^* \in \mathcal{F}$ and $D^* \subset D_\infty$ such that

$$\delta_\rho(D^*) \leq \frac{R_\rho(D_\infty) + \delta_\rho(D_\infty)}{2}. \tag{3.1}$$

Since $D^* \subset D_n$,

$$\delta_\rho(D^*) \leq \delta_\rho(D_\infty) \leq \delta_\rho(D_{n+1}) \leq \tilde{\delta}(D_n) + \frac{1}{n} \leq \delta_\rho(D^*) + \frac{1}{n},$$

for any $n \geq 1$. If we let $n \to \infty$, we get $\delta_\rho(D^*) = \delta_\rho(D_\infty)$. Then the inequality (3.1) implies $\delta_\rho(D_\infty) \leq R_\rho(D_\infty)$. Since $\mathcal{A}_\rho(C)$ is normal, this is only possible if D_∞ is reduced to one point. Since D_∞ is T-invariant, this point is a fixed point of T. □

As a corollary we get the following generalisation of Kirk's theorem which covers both norm and modular cases.

Theorem 3.5. *Let (X_ρ, ρ, τ) be a ρ-complete modulated topological vector space. Let C be a ρ-bounded and ρ-closed nonempty subset of X_ρ. Assume that C is τ-sequentially compact and $\mathcal{A}_\rho(C)$ is normal. If $T : C \to C$ is ρ-nonexpansive, then T has a fixed point.*

Proof. Recall that ρ-balls are τ-closed. Hence any element of $\mathcal{A}_\rho(C)$ is a τ-closed subset of C. Since C is τ-sequentially compact, we deduce that $\mathcal{A}_\rho(C)$ is countably compact. Therefore the conclusion of Theorem 3.5 follows from Theorem 3.4. □

As discussed by Brailey Sims in [21], Banach spaces which are uniformly convex in every direction (*UCED*) have weak normal structure (that is, every weak compact convex set has normal structure), an important result that originated in work of Garkavi [3]. Hence, via Kirk's theorem, *UCED* spaces enjoy the weak fixed point property. As it turns out, the same can be said of modulated topological vector spaces. To see this, let us first introduce a relevant notion of *UCED*, in which we will follow the relevant notion introduced for modular function spaces in [11] (see also a more recent application of *UCED* for function modulars in [2]).

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Definition 3.6. Let ρ be a convex modular. For any nonzero $u \in X_\rho$ and $r > 0$, we define the r-modulus of uniform convexity of ρ in the direction of u as

$$\delta(r, u) = \inf\Big\{1 - \frac{1}{r}\rho\Big(y + \frac{1}{2}u\Big)\Big\},$$

where the infimum is taken over all $y \in X_\rho$ such that $\rho(y) \le r$ and $\rho(y + u) \le r$.

We say that X_ρ is ρ uniformly convex in every direction (ρ-*UCED*) if $\delta(r, u) > 0$ for every nonzero $u \in X_\rho$ and all $r > 0$.

Proposition 3.7. *Let a modular space X_ρ be ρ-UCED and let $C \subset X_\rho$ be convex, ρ-bounded and not a singleton. Then C has a ρ-nondiametral point.*

Proof. Take any distinct elements x, y of C and define $\varepsilon = \delta_\rho(C)\rho((x - y)/2)$. Notice that $0 < \varepsilon < \infty$. Fix temporarily any $h \in C$ and set $u = x - y$, $w = y - h$ and $r = \delta_\rho(C)$. Then $\rho(w) = \rho(y - u) \le r$ and $\rho(w + u) = \rho(x - h) \le r$. By *UCED*,

$$\rho(w + \tfrac{1}{2}u) \le r(1 - \delta(r, u)),$$

which by a straightforward calculation gives

$$\rho\Big(\frac{x + y}{2} - h\Big) \le r(1 - \delta(r, u)).$$

Hence,

$$\sup_{h \in C} \rho\Big(\frac{x + y}{2} - h\Big) \le \delta_\rho(C)(1 - \delta(r, u)) < \delta_\rho(C),$$

because $\delta(r, u) > 0$, and consequently $(x + y)/2$ is not a ρ-diametral point in C. □

By combining Theorem 3.5 with Proposition 3.7, we obtain the following result which is an extension of the Browder fixed point theorem to the case of modulated topological vector spaces.

Theorem 3.8. *Let (X_ρ, ρ, τ) be a ρ-UCED ρ-complete modulated topological vector space. Let $C \subset X_\rho$ be convex, ρ-bounded and τ-sequentially compact. If $T : C \to C$ is ρ-nonexpansive, then T has a fixed point.*

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WOJCIECH M. KOZLOWSKI, School of Mathematics and Statistics,
University of New South Wales, Sydney, NSW 2052, Australia
e-mail: w.m.kozlowski@unsw.edu.au

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