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OPTIMAL CONTROL THEORY WITH GENERAL CONSTAINTS

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Abstract

We consider an optimal control problem with, possibly time-dependent, constraints on state and control variables, jointly. Using only elementary methods, we derive a sufficient condition for optimality. Although phrased in terms reminiscent of the necessary condition of Pontryagin, the sufficient condition is logically independent, as can be shown by a simple example.

1. Introduction

The standard theory of optimal control (Pontryagin *et al.* [8]) starts from the assumption that the postulated control problem *has* an optimal solution, and that the functions $f_i(x, v, t)$, $i = 0, 1, \ldots, n$, $x \in E^n$, $v \in E^s$, $0 \leqslant t \leqslant T$, which enter that control problem are differentiable with respect to all components of the vector x. The basic problem of the standard theory, which we shall call *problem A*, reads

$$\text{Minimize } \int_0^T f_0(x, v, t)\, dt \tag{1}$$

subject to

$$dx_i/dt - f_i(x, v, t) = 0, \quad \text{for } i = 1, 2, \ldots, n, \tag{2}$$

$$v(t) \in \Omega, \quad \text{for } t \text{ in } [0, T], \tag{3}$$

and

$$x_i(0) = a_i \text{ and } x_i(T) = b_i, \quad \text{for } i = 1, 2, \ldots, n. \tag{4}$$

For this problem, the Pontryagin theory provides a necessary, but not sufficient, condition that a state function $x^*(t)$ and associated control function $v^*(t)$ be optimal.

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From a purely practical point of view, there are several points of concern:

1. The theory provides no existence proof; there are well-known counter examples, in which the problem is feasible (that is, (2)–(4) have solutions) but in which no *optimal* solution exists. The case of "chattering control" is annoyingly frequent.
2. Even when a solution to the Pontryagin conditions exists, there is no guarantee that the solution is optimal (since the conditions are necessary but not sufficient). In practice, the "intermediate thrust" solutions of the Pontryagin equations for a certain optimal control problem with rockets took several *years* for a decision regarding optimality (the eventual decision was negative), see Bell and Jacobson [2].
3. The restrictive condition (3) involves only the control vector, not the state vector. In many problems of practical importance, there are 'state space constraints' as well. The straightforward theory does not allow for these, and extensions of that theory to include space state constraints have proved rather intractable (for example, see Chapter VI of [8] or Chapter 8 of Hestenes [4]).
4. A set of space state constraints, say of the form $x(t) \in X \subset E^n$ superimposed on (3) is not really of sufficient generality for practical purposes. State space and control space constraints may interact, that is, the effective control set Ω may depend upon the current state of the system. Thus, what we really require is a constraint of the form

$$\{x(t), v(t)\} \in Q(t) \subset E^n \times E^s. \tag{3a}$$

We shall denote the problem defined by (1), (2), (3a) and (4) by *problem B*. For some relevant literature, see Asher and Sebesta [1], Hadley and Kemp [3], Hestenes [4, Chapter 11], Long and Vousden [5], Mangasarian [6], Neustadt [7, Chapters 5 and 6], Schwartzkopf [9], Sebesta and Clark [10], and Teo and Craven [11]. However, it is fair to say that the results available for problem B are either hard to apply, or require very stringent conditions for their validity, or both.

5. Pontryagin theory admits controls which are measurable functions of time, and then proceeds to use the heavy apparatus of real analysis. The function which is 1 at every rational point, 0 at every irrational point, is measurable. But only a pure mathematician would ever declare it to be admissible as a control–try telling an engineer to build the controller! Pontryagin's own justification [8, page 76] for introducing measurable control functions as admissible is based on his desire to prove an existence theorem valid for a special class of control problems. Since no existence theorem can be proved for all, or even most, control problems of practical interest (there are lots of counterexamples!), this justification is rather weak. The generality used by pure mathematicians has resulted in admitting the practically nonsensical.

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Optimal control theory, which should be firmly a part of applied mathematics, has become the almost exclusive preserve of pure mathematicians.

The purpose of this paper is to show that a much simpler theory can be formulated when one looks for *sufficient*, rather than necessary, conditions for optimal control. Sufficient conditions exist, for example, Managasarian [6] for problem B, but they require some kind of convexity assumption, which is often very difficult or impossible to establish, and may simply not be true for the application in question. For example, the set $Q(t)$ in (3a) may not be a convex set, or even a connected set. The condition established here does not depend upon convexity at all. Being a sufficient condition, there is no guarantee that it can be satisfied whenever an optimal control exists. As against this, however, we can guarantee that a solution to the condition, when it can be found, is indeed optimal; we can handle the full generality of (3a); and, perhaps best of all, we use only elementary mathematics throughout.

2. The new condition

DEFINITION 1. *Let* $x(t)$, $R \to R^n$, *be continuous, piecewise differentiable, and satisfy* (4); *let* $v(t)$, $R \to R^s$, *be piecewise continuous on* $[0, T]$; *and let* $\{x(t), v(t)\}$ *satisfy* $(3a)$ *for all* $t \in [0, T]$. *Then the time path* $\{x(t), v(t)\}$ *will be called admissible.*

DEFINITION 2. *The time path* $\{x(t), v(t)\}$ *is called feasible if it is admissible and satisfies* (2) *almost everywhere in* $[0, T]$.

DEFINITION 3. *Let* $p(t)$, $R \to R^n$, *be continuous and piecewise differentiable; the Pontryagin Hamiltonian is defined by*

$$H(x, v, t, p) = \sum_{i=1}^{n} p_i f_i(x, v, t) - f_0(x, v, t). \tag{5}$$

THEOREM 1. *Let* $f_i(x, v, t)$, $i = 0, 1, \ldots, n$ *be piecewise continuous functions* $R^n \times R^s \times R \to R$. *Suppose there exists a continuous, piecewise differentiable function* $p(t)$, $R \to R^n$, *and a feasible path* $\{x^*(t), v^*(t)\}$ *such that, for almost every* t *in* $[0, T]$, *and for every feasible path* $\{x(t), v(t)\}$, *it is true that*

$$\sum_{i=1}^{n} x_i^*(t)(dp_i/dt) + H(x^*, v^*, t, p) \geqslant \sum_{i=1}^{n} x_i(t)(dp_i/dt) + H(x, v, t, p). \tag{6}$$

Then $\{x^*(t), v^*(t)\}$ *is an optimal path. (In this context, "almost every* t*" means all but a finite number of* t*-values in* $0 \leqslant t \leqslant T$.)

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PROOF. For any feasible path $\{x(t), v(t)\}$ and for almost every t,

$$f_0(x, v, t) = f_0(x, v, t) + \sum_{i=1}^{n} p_i(t)[dx_i/dt - f_i(x, v, t)]. \tag{7}$$

Since $x_i(t)$ and $p_i(t)$ are continuous and piecewise differentiable, and $x_i(t)$ satisfies (4), we obtain

$$\int_0^T p_i(t)(dx_i/dt)\,dt = b_i p_i(T) - a_i p_i(0) - \int_0^T x_i(t)(dp_i/dt)\,dt. \tag{8}$$

We combine (5), (7), and (8) for the two paths $\{x^*(t), v^*(t)\}$ and $\{x(t), v(t)\}$ to obtain

$$\int_0^T [f_0(x, v, t) - f_0(x^*, v^*, t)]\,dt \tag{9}$$

$$= \int_0^T \left\{ \sum_{i=1}^{n} [x_i^*(t) - x_i(t)](dp_i/dt) + H(x^*, v^*, t, p) - H(x, v, t, p) \right\} dt.$$

By condition (6), the integrand on the right hand side is non-negative for almost all t, and hence the value of the integral is non-negative.

NOTES

1. The integration by parts, (8), is the only point at which we need any condition of differentiability.
2. Although condition (6) may look formidable at first sight, it is actually of the same order of difficulty as the conventional Pontryagin maximum condition: one needs to find a maximum subject to given constraints. The Kuhn-Tucker theory can be applied directly if the functions involved are differentiable, which is the usual case. We shall show that (6) implies certain much simpler conditions; these simpler conditions can be used to limit the possible choices for $x^*(t)$ and $v^*(t)$, before checking that (6) is satisfied.
3. For examples, see the Appendix.

THEOREM 2. (*Final state free.*) *Let condition* (4) *be replaced by the weaker one*

$$x_i(0) = a_i \text{ for } i = 1, 2, \ldots, n; \quad \text{no condition on } x_i(T). \tag{4a}$$

In the statement of Theorem 1, *replace* (4) *by* (4a) *throughout, and demand the additional condition*

$$p_i(T) = 0 \text{ for } i = 1, 2, \ldots, n. \tag{10}$$

Then the starred solution is optimal.

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PROOF. This is just as for Theorem 1, except that the integration by parts in (8) gives a term $x_i(T)p_i(T)$ which no longer reduces to $b_ip_i(T)$. However condition (10) causes this term to vanish, and hence (9) still follows.

At this stage, we develop a number of useful corollaries to the basic theorem. These corollaries give practical methods of simplifying condition (6), in the sense that they are conditions which follow from (6) and must therefore be satisfied for every candidate optimal solution. The conditions can be used to limit possible candidates, before proceeding to a full test of (6); but this full test must still be carried out. We start with two definitions.

DEFINITION 4.

$$X(v, t) = \{x \mid \text{There exists a feasible path with } x(t) = x \text{ and } v(t) = v\}. \quad (11)$$

DEFINITION 5.

$$\Omega(x, t) = \{v \mid \text{There exists a feasible path with } x(t) = x \text{ and } v(t) = v\}. \quad (12)$$

For problem A, $X(v, t)$ is independent of v, though not necessarily independent of the time t (for example, points x far from the initial point a may be inaccessible by any feasible path if t is small); $\Omega(x, t) = \Omega$ is independent of both x and t.

COROLLARY 1. *Condition* (6) *implies*

$$H(x^*, v^*, t, p) \geqslant H(x^*, v, t, p) \text{ for all } v \in \Omega(x^*, t). \quad (13)$$

PROOF. Put $x = x^*$ on the right-hand side of (6).

Before stating Corollary 2, we remind the reader of the definition of a *subgradient*.

DEFINITION 6. *Let $h(u)$, $R^n \to R$, be defined on the set $U \subset R^n$. An n-vector q is said to belong to the subgradient set $\partial h(u)$ at the point u if and only if*

$$h(u') \geqslant h(u) + \sum_{i=1}^{n} q_i(u_i' - u_i) \text{ for all } u' \in U. \quad (14)$$

Although this definition does not involve derivatives, it is clearly closely related to the idea of a gradient. Indeed, if U is a convex set and $h(u)$ is a convex and differentiable function defined on this set, then $\partial h(u)$ contains only a single vector q which is equal to $\nabla h(u)$. However, the relationship is by no means precise. For example, if $h(u)$ is not locally convex at u, then the subgradient is the empty set, but the gradient may exist. Conversely, if $h(u)$ is convex but has a cusp at u, then the gradient does not exist but the subgradient $\partial h(u)$ exists and contains a multitude of n-vectors q, namely all those q for

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which a hyperplane through the point $\{u, h(u)\} \in R^{n+1}$ with normal vector $\{q, -1\} \in R^{n+1}$ lies everywhere below the function value.

By letting u' approach the point u, assumed to be an interior point of U, we conclude that $h(u)$ differentiable implies that $\partial h(u)$ contains *at most* the single vector $\nabla h(u)$, under these conditions.

Corollary 2. *Condition* (6) *implies*

$$dp/dt \in \partial_x[-H(x^*, v^*, t, p)] \text{ for all } x \in X(v^*, t). \tag{15}$$

Proof. Put $v = v^*$ on both sides of (6), whereupon the inequality can be written in the form

$$[-H(x, v^*, t, p)] \geq [-H(x^*, v^*, t, p)] + \sum_{i=1}^{n} (dp_i/dt)(x_i - x_i^*)$$

$$\text{for all } x \in X(v^*, t).$$

Corollary 3. *Let x^* be an interior point of $X(v^*, t)$ and let H be differentiable with respect to x at $x = x^*$. Then* (6) *implies*

$$dp/dt = -\nabla_x H(x^*, v^*, t, p). \tag{16}$$

Proof. Under the stated conditions, the subgradient set in (15) contains, at most, this one element.

At this stage we pause for discussion.

Clearly (16) is the co-state equation of the Pontryagin theory and, *if* $p(t)$ satisfies that equation, *then* (13) is Pontryagin's maximum principle. However, note the condition that x^* must be an *interior* point of $X(v^*, t)$; in the appendix, we shall exhibit an example in which $x^*(t)$ is on the boundary of the feasible set, and in which both the Pontryagin theory and the theory of this paper are applicable, but only with *different* functions $p(t)$. Furthermore, the function $p(t)$ appropriate for the sufficient condition is not defined uniquely, there being infinitely many choices for it.

Returning to the sufficient condition (6), conditions (13) and (15) must be satisfied, but these auxiliary conditions do *not* imply that (6) is satisfied. Yet (13) and (15) are most useful for practical purposes, to limit possible candidates before testing (6), which is the true sufficient condition. This final test must not be omitted, however.

When (x^*, v^*) is a boundary point of $Q(t)$, condition (15) taken by itself may permit many choices for the vector dp/dt. However, this condition should not be taken by itself. It must be coupled, at the very least, with condition (13); in the first example of the appendix, this results in a well-defined differential

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equation for $p(t)$, different from the co-state equation (16). But even if choices for dp/dt are permitted (the second example), all we need is *one* choice that works, in the sense that (6) is satisfied. The result is then optimal and cannot be improved no matter how many other solutions there might be. We can therefore stop searching at this stage.

COROLLARY 4. *Assume that, in problem B, condition* (4) *is replaced by*

$$x_i(0) = a_i,\ i = 1, 2, \dots, n,\ \textit{but}\ x(T) \in S \subset E^n. \tag{4b}$$

Then we get a sufficient condition for optimality by demanding, in addition to (6), *the condition*

$$\sum_{i=1}^{n} p_i(T)[x_i(T) - x_i^*(T)] \geqslant 0\ \textit{all}\ x \in S \cap X(v^*(T), T). \tag{17}$$

PROOF. This is the same as for Theorem 1, except that (9) now contains the integrated term shown on the left side of (17). Condition (17) together with (6) then ensures that (9) is non-negative.

NOTE. This is the appropriate generalization of the transversality condition of the usual Pontryagin theory. For problem A, with (4) replaced by (4b), it reduces to a generalized transversality condition; it implies the transversality condition itself if the set S is sufficiently smooth so that it has a tangent space at x^*. However, even then the transversality condition, which is local, is less restrictive than (17).

A practical person, having to choose between a necessary condition which is not sufficient, and a sufficient condition which is not necessary, normally prefers the sufficient condition. In practice, one tries to satisfy conditions (13) and (15) first. When the problem is at all complicated, it becomes impossible to explore all solutions of (13) and (15). A particular, numerical solution is the most that one can achieve. However, if the particular solution is also a solution to (6), then it is certain to be optimal.

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Appendix. Two illustrative examples

EXAMPLE 1.

$$\text{Minimize } \int_0^2 x^2\, dt \tag{A.1}$$

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subject to

$$dx/dt = v, \tag{A.2}$$

$$\{x, v\} \in Q(t): \{x, v | \tfrac{1}{4} \leqslant x \leqslant t + 1 \text{ and } |v| \leqslant t + 1 - x\}, \tag{A.3}$$

and

$$x(0) = 1; \quad \text{(there is no condition on } x(2)). \tag{A.4}$$

Condition (6) assumes the form

$$\text{Maximize } W(x, v) = (dp/dt)x + pv - x^2 \text{ over } \{x, v\} \in Q(t). \tag{A.5}$$

It is easiest to use (13) to cut down on possible candidates (that is, to maximize over v first). This yields:

$$\text{If } p > 0, v = t + 1 - x. \tag{A.6a}$$

$$\text{If } p < 0, v = x - t - 1. \tag{A.6b}$$

$$\text{If } p = 0, \text{ there is no further condition on } v \text{ beyond (A.3).} \tag{A.6c}$$

We investigate the case of singular control, (A.6c), first. Let $p(t) = 0$ over some neighbourhood of t; then $dp/dt = 0$ as well, so that $W(x, v) = 0 + 0 - x^2$. This has its maximum within $Q(t)$ when $x = 1/4$. But $x = 1/4$ implies $dx/dt = 0 = v$, by (A.2). Hence a singular arc exists of the form

$$x(t) = 1/4 \qquad \text{and} \qquad v(t) = p(t) = 0 \text{ (singular control arc).} \tag{A.7}$$

Such an arc can join smoothly at $t = T = 2$, by condition (10). But it cannot be the full solution, since it violates (A.4) at $t = 0$.

Obviously, by (A.2) and (A.7), we want v to be negative so as to cut down $x(t)$ from its initial value, 1, to its eventual value, 1/4. We therefore investigate (A.6b). The co-state equation (16) is *not* applicable, because we are on a boundary of $Q(t)$. On this boundary, $W(x, v)$ takes the form

$$W = (dp/dt)x + pv - x^2 = (dp/dt)x + p(x - t - 1) - x^2$$

which must be maximized subject to $1/4 \leqslant x \leqslant t + 1$, with x an interior point except at $t = 0$ and at the changeover to solution (A.7). Straightforward differentiation yields

$$dp/dt + p - 2x = 0. \tag{A.8}$$

This is valid, according to (A.6b), as long as $p(t) < 0$. Under this same asumption, (A.6b) and (A.2) yield

$$dx/dt = v = x - t - 1. \tag{A.9}$$

When this differential equation is solved subject to (A.4), we get

$$x(t) = t + 2 - e^t \text{ and hence } v(t) = 1 - e^t. \tag{A.10}$$

This solution is valid as long as $x(t)$ remains in $Q(t)$, that is, as long as $x(t) \geqslant 1/4$. This condition fails when $t > s = 1.01819$, where $s + 2 - e^s = 1/4$.

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We substitute $x(t)$ from (A.10) into (A.8), and impose the condition that the co-state function $p(t)$ must be continuous at $t = s$, where we change over to (A.7); that is, we require $p(s) = 0$. The desired solution is

$$p(t) = 2t + 2 - e^t - (s + 1/4)e^{s-t} \text{ for } 0 \leqslant t \leqslant s = 1.01819. \quad \text{(A.11)}$$

The full optimal solution is given by (A.10) and (A.11) for $0 \leqslant t \leqslant s$ and by (A.7) for $s < t \leqslant 2$. This solution satisfies conditions (6) and (10) by construction, and is therefore optimal.

We note the following:

1. Unlike the usual theory, with only necessary conditions, there is no need to investigate any other possibilities. Other solutions have not been excluded (they can be excluded if one wishes), but no other solution can improve on the one we have found.
2. There is a singular arc within our solution, but there is no need to invoke higher order conditions to establish optimality. The criterion (6), which has been satisfied along this singular arc, is global, not local.
3. The constraint boundary (A.6b), which has been used, is not handled by the straightforward Pontryagin theory, and the ordinary co-state equation (16) does not apply along this segment of our optimal solution (compare (16) with (A.8)).
4. Our entire solution lies on, one or another, boundary of $Q(t)$. Thus condition (15) taken by itself does not define a unique value for dp/dt. But taken in conjunction with the maximum principle (13), the value of dp/dt is given uniquely. For example, a boundary solution with $x(t) = 1/4$ over a neighborhod of t must have $v(t) = 0$, by (A.2), and hence only $p(t) = 0$ will work, by (A.6); hence $dp/dt = 0$, uniquely.

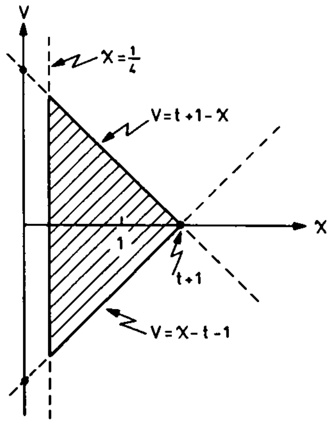

FIGURE 1. The admissible set $Q(t)$ in (x, v)-space, at a particular time t (= 0.6). The triangle retains its shape, but grows larger, as the time increases.

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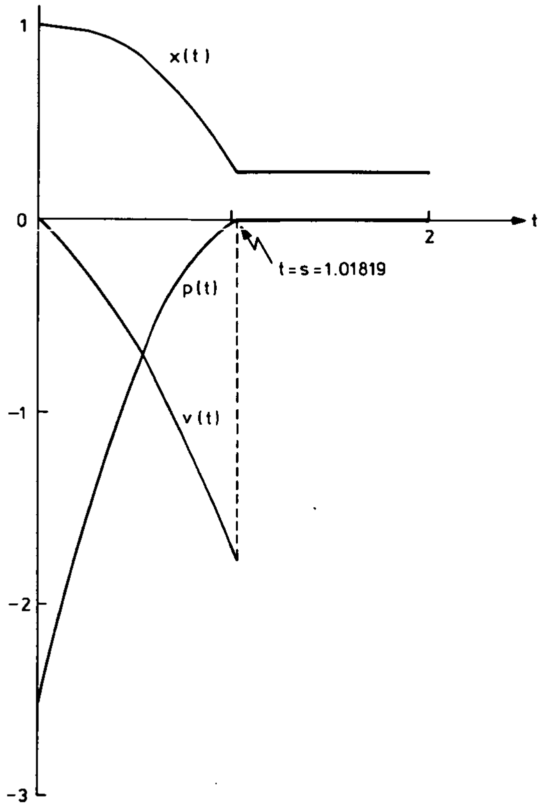

FIGURE 2. The optimal solution, showing $x(t)$, $v(t)$ and $p(t)$ plotted as functions of the time t; $x(t)$ and $p(t)$ are continuous, piecewise differentiable functions; $v(t)$ is a piecewise continuous function, with a jump at $t = s = 1.01819$.

5. A sketch of the admissible region $Q(t)$, at $t = 0.6$, is given in Figure 1. The time paths of $x(t)$, $v(t)$, and $p(t)$ for the optimal solution are shown in Figure 2.
6. Only elementary mathematics has been used throughout.

EXAMPLE 2.

$$\text{Minimize } J = \int_0^T (-x^2)\,dt \tag{A.12}$$

subject to

$$dx/dt = v, \tag{A.13}$$

$$0 \leqslant v \leqslant 1, \tag{A.14}$$

and

$$x(0) = 1. \tag{A.15}$$

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The inequality, immediately deduced form (A.13)–(A.14),

$$1 \leqslant x(t) \leqslant 1 + t, \tag{A.16}$$

defines the set we have called $X(v^*, t)$ earlier, and clearly the optimal result is attained for

$$x^*(t) = 1 + t \text{ and } v^*(t) = 1 \text{ for all } 0 \leqslant t \leqslant T. \tag{A.17}$$

Pontryagin theory is applicable to this problem, and yields

$$dp/dt = -\partial H/\partial x = -2x^* \text{ and } p(t) = 2T + T^2 - 2t - t^2, \tag{A.18}$$

However, this particular function $p(t)$ is *unsuitable* for the sufficient condition (6). The quantity which should be maximized is

$$W(x, v) = (dp/dt)x + H(p, x, v, t) = (dp/dt)x + pv + x^2. \tag{A.19}$$

Substitution of (A.17) and (A.18) into (A.19) shows that

$$W(x^*, v^*) \leqslant W(x, v^*) \text{ for all } x \in X(v^*, t). \tag{A.20}$$

Inequality (A.20) is in the wrong direction for condition (6).

However, this does not mean that the sufficient condition is inapplicable. Maximizing W with respect to x alone, for the moment, we are to maximize the quadratic function

$$W_1(x) = (dp/dt)x + x^2 \tag{A.21}$$

subject to (A.16). The function $W_1(x)$ is convex; hence the maximum must occur on the boundary of (A.16). The maximum occurs on the desired boundary, $x = 1 + t$, if and only if $W_1(1 + t) \geqslant W_1(1)$, which reduces to

$$dp/dt + 2 + t \geqslant 0. \tag{A.22}$$

This condition is violated by the Pontryagin solution (A.18), but there is no difficulty satisfying it with some other choice of $p(t)$. For example, $dp/dt = -1$ clearly satisfies (A.22). It leads to $p(t) = T - t$, which is positive for $t < T$, and is therefore consistent with $v^* = 1$ in a maximization of (A.19) with respect to both x and v, jointly.

Here, then, is a case in which the sufficient condition (6) is applicable, and the necessary condition of Pontryagin is also applicable, but the Pontryagin $p(t)$ is not suitable for (6), and any $p(t)$ suitable for (6) must be different from Pontryagin's $p(t)$. Furthermore, there are infinitely many functions $p(t)$ suitable for (6).

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