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HOLOMORPHIC MAPPINGS BETWEEN COMPACT RIEMANN SURFACES

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Dedicated to Yoichi Imayoshi on the occasion of his sixtieth birthday

Abstract There are only finitely many non-constant holomorphic mappings between two fixed compact Riemann surfaces of genus greater than 1. This result goes under the name of the de Franchis theorem. Having seen that the set of such holomorphic mappings is finite, we naturally want to obtain a bound on its cardinality. It has been known for some time that there exist various bounds depending only on the genera of the surfaces. Here we obtain 'better' bounds of the above type, using arguments based on the rigidity of holomorphic mappings and the hyperbolic geometry of surfaces.

Keywords: compact Riemann surface; holomorphic mapping; Schwartz lemma; rigidity theorems for holomorphic mappings; collar theorem; length-controlled pants decompositions

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1. Introduction

Throughout the paper, M and N are compact Riemann surfaces of genera $g_M \geqslant 2$ and $g_N \geqslant 2$, respectively, and $\mathrm{Hol}(M,N)$ denotes the set of all non-constant holomorphic mappings of M onto N. If $\mathrm{Hol}(M,N)$ is not empty, then a holomorphic mapping $f \in \mathrm{Hol}(M,N)$ is necessarily a finite-sheeted possibly ramified covering map.

Let us denote by d_f the degree of f. The Riemann–Hurwitz relation yields

$$2(g_M - 1) = 2d_f(g_N - 1) + B_f, \tag{1.1}$$

where B_f is the total branching number of f. Hence, it follows immediately that $g_M \geqslant g_N$. With this observation, $g_M \geqslant g_N \geqslant 2$ will be a blanket assumption for the rest of the paper.

Now we concisely explain some known results on upper estimates for $\#\,\mathrm{Hol}(M,N)$ below to help orient the reader. Henceforth, $\#$ denotes the cardinality of a set.

The simplest case occurs when $g_M = g_N \geqslant 2$. It is then easy to see that any holomorphic mapping $f \in \mathrm{Hol}(M,N)$ must be bijective. Hence, $\mathrm{Hol}(M,N)$ is in a natural one-to-one correspondence with $\mathrm{Aut}(M)$, the group of conformal self-maps (automorphisms) of M. A classical result of Hurwitz shows that

$$\#\,\mathrm{Hol}(M,N) = \#\,\mathrm{Aut}(M) \leqslant 84(g_M - 1) \tag{1.2}$$

and that the action of $\mathrm{Aut}(M)$ on the first homology group on M with integer coefficients, $H_1(M,\mathbb{Z})$, is faithful, i.e. a conformal self-map of M is completely determined by the automorphism

$$H_1(M,\mathbb{Z}) \to H_1(M,\mathbb{Z}) \cong H_1(N,\mathbb{Z}) \tag{1.3}$$

it induces on $H_1(M,\mathbb{Z})$.

When $g_M > g_N \geqslant 2$, the situation is complicated by the possibility that a map may have branch points. Martens [**22**] showed that Hurwitz's result can be extended; that is, $f \in \mathrm{Hol}(M,N)$ is determined by the induced homology map

$$H_1(M,\mathbb{Z}) \to H_1(N,\mathbb{Z}) \tag{1.4}$$

even if $g_M > g_N \geqslant 2$. Martens also observed that the matrices representing all such homology maps form a discrete and bounded set in the *Hurwitz space* and that, in this way, the cardinality of $\mathrm{Hol}(M,N)$ may be bounded by a constant depending only on the genera of the surfaces (see § 2).

These results were strengthened in [**29**]. Let $H_1(N,\mathbb{Z}_n)$ be the first homology group on N with coefficients in the integers $\bmod n$. Then, for $n > 8(g_M - 1)$, one knows that $f \in \mathrm{Hol}(M,N)$ can be recovered from the homomorphism

$$H_1(M,\mathbb{Z}) \to H_1(N,\mathbb{Z}_n) \tag{1.5}$$

obtained by following the homology map (1.4) by the canonical projection $H_1(N,\mathbb{Z}) \to H_1(N,\mathbb{Z}_n)$. From this it follows easily that

$$\#\,\mathrm{Hol}(M,N) \leqslant (8g_M - 7)^{4g_M g_N}. \tag{1.6}$$

Thus, we obtain an *explicit* bound on the cardinality of $\mathrm{Hol}(M,N)$ depending only on the genera of the surfaces.

But this result is (obviously) far from sharp. We now ask: can we find a better bound? Our interest in the subject originated with the question above, which constitutes one of the guiding themes for this work. See the references and notes collected at the end for more details on historical accounts.

By the uniformization theorem of Poincaré, Klein and Koebe, every Riemann surface X of genus greater than 1 is a hyperbolic complex manifold with a natural metric; the (Poincaré) hyperbolic geometry arises from the non-Euclidean Riemannian metric on the upper half-plane U given by

$$\mathrm{d}s = \frac{|\mathrm{d}z|}{2\,\mathrm{Im}\,z}, \quad z \in U. \tag{1.7}$$

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Recall that Riemannian concepts applied to this canonical hyperbolic metric on X become complex analytic invariants for the Riemann surface X. In particular, the lengths of closed geodesics on X are invariant, so if certain information was available concerning the length spectrum of the closed geodesics, this would perhaps shed new light on the set of holomorphic maps between compact Riemann surfaces. We shall apply the collar theorem in hyperbolic geometry and Bers's theorem on length-controlled pants decompositions to obtain some quantitative geometric results, and we obtain more information about the cardinality of $\mathrm{Hol}(M, N)$ with the aid of rigidity theorems for holomorphic mappings. (Also note that holomorphic maps are distance decreasing between hyperbolic manifolds. We will see that this contracting property for holomorphic maps plays an explicit and crucial role in the proof of Theorem 4.1.)

The paper is organized as follows. The main theorem and application are in § 4 and all the necessary definitions and results are contained in §§ 2 and 3. In § 5 we also point out the origin of our study and a possible direction for further research.

2. Rigidity of holomorphic mappings

We start with a compact Riemann surface X of genus $g \geqslant 2$, and a canonical homology basis $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ on X; this homology basis has the intersection numbers

$$\left.\begin{aligned} &a_i \cdot b_j = \delta_{ij} \quad \text{(Kronecker delta)},\\ &a_i \cdot a_j = 0 = b_i \cdot b_j, \end{aligned}\right\} \tag{2.1}$$

for $1 \leqslant i, j \leqslant g$. By abuse of language we do not distinguish between homology classes and their representatives.

We briefly review the basic facts to fix notation. Since the vector space $A(X)$ of holomorphic abelian differentials on X has dimension g, we choose a basis $\{\omega_1, \dots, \omega_g\}$ for $A(X)$. With these choices of basis we call the following $g \times 2g$ complex matrix the *period matrix* of X:

$$\Pi = \begin{pmatrix} \int_{a_1} \omega_1 & \cdots & \int_{a_g} \omega_1 & \int_{b_1} \omega_1 & \cdots & \int_{b_g} \omega_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \int_{a_1} \omega_g & \cdots & \int_{a_g} \omega_g & \int_{b_1} \omega_g & \cdots & \int_{b_g} \omega_g \end{pmatrix}. \tag{2.2}$$

The columns of Π form $2g$ vectors in $\mathbb{C}^g$ which are linearly independent over the reals. Let the ith column be denoted by Π_i. These $2g$ column vectors therefore generate a lattice in $\mathbb{C}^g$, namely, the following discrete subgroup of $\mathbb{C}^g$:

$$L(\Pi) = \{n_1 \Pi_1 + \cdots + n_{2g} \Pi_{2g};\ n_1, \dots, n_{2g} \in \mathbb{Z}\}. \tag{2.3}$$

We shall call

$$J(X) = \mathbb{C}^g / L(\Pi)$$

the *Jacobian variety* of X. It is a compact, commutative g-dimensional complex Lie group. We also define a map

$$\kappa : X \to J(X) \tag{2.4}$$

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by choosing a point $P_0 \in X$ and setting

$$\kappa(P) = \left(\int_{P_0}^{P} \omega_1, \dots, \int_{P_0}^{P} \omega_g \right)^{\mathrm{T}}.$$

(The transpose of the matrix x will be denoted x^{T}; vectors are usually written as columns and thus x^{T} is a row vector for $x \in \mathbb{C}^g$.) The map κ is a well-defined injective holomorphic mapping of X into $J(X)$ and is known as the *Abel–Jacobi embedding* of X. It will be denoted by κ_{P_0}, when its dependence on the base point is to be emphasized. (See, for example, [**9**] for more details on the concepts discussed here.)

Assumption 2.1. *From now on a basis $\{\omega_1, \dots, \omega_g\}$ for holomorphic abelian differentials will be assumed to be dual to the given canonical homology basis; that is,*

$$\int_{a_i} \omega_j = \delta_{ij}, \quad i, j = 1, 2, \dots, g.$$

Let M and N be any two compact Riemann surfaces with $g_M \geqslant g_N \geqslant 2$. Let $a_1^M, \dots, a_{g_M}^M, b_1^M, \dots, b_{g_M}^M$ (respectively, $a_1^N, \dots, a_{g_N}^N, b_1^N, \dots, b_{g_N}^N$) be a canonical homology basis on M (respectively, N) and let $\omega_1^M, \dots, \omega_{g_M}^M$ (respectively, $\omega_1^N, \dots, \omega_{g_N}^N$) be a basis of the holomorphic abelian differentials dual to the canonical homology basis. A holomorphic mapping $f \in \mathrm{Hol}(M, N)$ induces a homology map

$$H_1(M, \mathbb{Z}) \to H_1(N, \mathbb{Z}).$$

Thus, using obvious vector notation we immediately see that there exists a $2g_N \times 2g_M$ integer matrix H_f such that

$$(f(a_1^M), \dots, f(a_{g_M}^M), f(b_1^M), \dots, f(b_{g_M}^M)) = (a_1^N, \dots, a_{g_N}^N, b_1^N, \dots, b_{g_N}^N) H_f.$$

On the other hand, there is a $g_N \times g_M$ complex matrix A_f relating the pull-back of ω_i^N via f, $f^*\omega_i^N$, to the basis $\omega_1^M, \dots, \omega_{g_M}^M$; that is,

$$(f^*\omega_1^N, \dots, f^*\omega_{g_N}^N) = (\omega_1^M, \dots, \omega_{g_M}^M) A_f^{\mathrm{T}}.$$

Now it is clear that

$$\int_{f(a_i^M)} \omega_j^N = \int_{a_i^M} f^*\omega_j^N, \quad \int_{f(b_i^M)} \omega_j^N = \int_{b_i^M} f^*\omega_j^N, \quad 1 \leqslant i \leqslant g_M,\ 1 \leqslant j \leqslant g_N.$$

Therefore, if Π_M and Π_N denote the respective period matrices with respect to the canonical homology bases and the dual bases for their holomorphic abelian differentials, we then have the *Hurwitz relation*

$$A_f \Pi_M = \Pi_N H_f. \tag{2.5}$$

The matrix H_f now induces a holomorphic map F of $J(M)$ into $J(N)$. It is not difficult to show that the diagram

$$\begin{array}{ccccccc} \mathbb{R}^{2g_M} & \xrightarrow{\Pi_M} & \mathbb{C}^{g_M} & \longrightarrow & J(M) & \xhookleftarrow{\kappa_{P_0}} & M \\ {\scriptstyle H_f}\big\downarrow & & {\scriptstyle A_f}\big\downarrow & & \big\downarrow{\scriptstyle F} & & \big\downarrow{\scriptstyle f} \\ \mathbb{R}^{2g_N} & \xrightarrow[\Pi_N]{} & \mathbb{C}^{g_N} & \longrightarrow & J(N) & \xhookleftarrow[\kappa_{f(P_0)}]{} & N \end{array} \tag{2.6}$$

is commutative. Here (and hereafter) unmarked horizontal arrows denote natural projections (which do not depend on the base points).

We have therefore established most of the following.

Proposition 2.2 (Martens's rigidity theorem). *Let* $f,\, g \in \mathrm{Hol}(M, N)$. *If*

$$f(a_i^M) = g(a_i^M), \quad f(b_i^M) = g(b_i^M), \quad i = 1, \dots, g_M,$$

in $H_1(N, \mathbb{Z})$, *then* $f = g$.

Proof. It is evidently sufficient to assume that $H_f = H_g$. Assume that $P_0 \in M$ is the fixed base point of the Abel–Jacobi embedding $\kappa_{P_0} : M \to J(M)$. If F and G are holomorphic maps from $J(M)$ to $J(N)$ induced by f and g, respectively, then the two diagrams

$$\begin{array}{ccccc} \mathbb{R}^{2g_M} & \longrightarrow & J(M) & \xhookleftarrow{\kappa_{P_0}} & M \\ {\scriptstyle H_f}\big\downarrow & & \big\downarrow{\scriptstyle F} & & \big\downarrow{\scriptstyle f} \\ \mathbb{R}^{2g_N} & \longrightarrow & J(N) & \xhookleftarrow[\kappa_{f(P_0)}]{} & N \end{array}$$

and

$$\begin{array}{ccccc} \mathbb{R}^{2g_M} & \longrightarrow & J(M) & \xhookleftarrow{\kappa_{P_0}} & M \\ {\scriptstyle H_g}\big\downarrow & & \big\downarrow{\scriptstyle G} & & \big\downarrow{\scriptstyle g} \\ \mathbb{R}^{2g_N} & \longrightarrow & J(N) & \xhookleftarrow[\kappa_{g(P_0)}]{} & N \end{array}$$

commute. By a simple diagram chase, we conclude that $F = G$.

We also have $\kappa_{f(P_0)}(N) = \kappa_{g(P_0)}(N)$, which implies that $f(P_0) = g(P_0)$, since distinct base points must give distinct imbedded images of the surface in the Jacobian variety by Abel's theorem. From here it is easy to obtain the desired result. □

We need a slight (but important) improvement. Let $e_1, \dots, e_{2g_M}$ be the standard basis of $\mathbb{R}^{2g_M}$; e_i has 1 in the ith place and 0 elsewhere. Since $\Pi_M e_1, \dots, \Pi_M e_{g_M}$ form a standard basis for $\mathbb{C}^{g_M}$ (clearly), we have almost obtained the following.

Lemma 2.3. *Let f, $g \in \mathrm{Hol}(M,N)$. If*

$$f(a_i^M) = g(a_i^M), \quad i = 1, \dots, g_M,$$

in $H_1(N,\mathbb{Z})$, then $f = g$.

Proof. Using the same notation as before we assume that

$$H_f e_i = H_g e_i, \quad i = 1, \dots, g_M.$$

It is sufficient to show that $A_f = A_g$ in view of the fact exhibited in the proof of Proposition 2.2. But this is practically obvious by (2.5). Indeed, we have already seen that $\Pi_M e_1, \dots, \Pi_M e_{g_M}$ form a basis for $\mathbb{C}^{g_M}$, so the equation

$$A_f(\Pi_M e_1, \dots, \Pi_M e_{g_M}) = A_g(\Pi_M e_1, \dots, \Pi_M e_{g_M}) \tag{2.7}$$

implies $A_f = A_g$. □

We now explain, omitting some details, the Martens [**22**] approach to establishing a special case of a result of de Franchis [**8**]. With the foregoing notation, let $\mathcal{S}(\Pi_M, \Pi_N)$ denote the vector space (over $\mathbb{Q}$) of $2g_N \times 2g_M$ matrices H with rational entries which satisfy

$$A\Pi_M = \Pi_N H \quad \text{for some } g_N \times g_M \text{ matrix } A. \tag{2.8}$$

The space $\mathcal{S}(\Pi_M, \Pi_N)$ is known as the *Hurwitz space* of Π_M and Π_N. We can define the *Castelnuovo–Severi inner product* of $H_1, H_2 \in \mathcal{S}(\Pi_M, \Pi_N)$ by

$$\langle H_1, H_2 \rangle = \mathrm{tr}(J_M H_1^{\mathrm{T}} J_N^{-1} H_2), \tag{2.9}$$

where J_M and J_N are the respective intersection matrices with respect to the fixed canonical homology bases (see the lemma and its corollary in [**24**, p. 534]). We also define the norm of $H \in \mathcal{S}(\Pi_M, \Pi_N)$ by

$$\|H\| = \langle H, H \rangle^{1/2}. \tag{2.10}$$

It has been shown that

$$H_f J_M H_f^{\mathrm{T}} = d_f J_N,$$

where d_f is the degree of $f \in \mathrm{Hol}(M,N)$. In particular, using the Riemann–Hurwitz formula, we have

$$\|H_f\|^2 = 2g_N d_f \leqslant 4(g_M - 1). \tag{2.11}$$

Now it follows that the possible matrices under consideration form a set of lattice points in a bounded set of finite-dimensional space and hence the number of holomorphic maps of M onto N is finite by Proposition 2.2. We shall return to this topic in § 4.

3. Closed geodesics on compact Riemann surfaces

This section explores the hyperbolic geometric nature of compact Riemann surfaces. We shall be especially concerned with some quantitative explicit information about closed geodesics on the surface. The reader is referred to [**5**] for this material.

We introduce now some notation that we will follow for the remainder of the paper. Again, X is a compact Riemann surface of genus $g \geqslant 2$. We use $\mathrm{dist}(\cdot,\cdot)$ to denote the hyperbolic distance function and $l(c)$ to denote the length of a curve c. If E is any Lebesgue-measurable subset of X, then $\mathrm{area}(E)$ will denote the area of E.

A fundamental property of compact Riemann surfaces, known as the collar theorem, is that the small closed geodesics have large tabular neighbourhoods which are topological cylinders. It is now convenient to review this fact. Let $\beta_1,\dots,\beta_k$ be the set of all simple closed geodesics of length less than or equal to $2\operatorname{arcsinh} 1$ on X. We then have the following:

(i) $k \leqslant 3g-3$;

(ii) $\beta_1,\dots,\beta_k$ are pairwise disjoint;

(iii) the collars
$$\mathcal{C}(\beta_j) = \{p \in X;\ \mathrm{dist}(p,\beta_j) \leqslant w(\beta_j)\}$$
of widths $w(\beta_j) = \operatorname{arcsinh}\{1/\sinh \frac{1}{2} l(\beta_j)\}$ are pairwise disjoint for $j = 1,\dots,k$ and each is a topological cylinder; and

(iv) for all $p \in X \setminus (\mathcal{C}(\beta_1) \cup \cdots \cup \mathcal{C}(\beta_k))$,
$$r_p(X) > \operatorname{arcsinh} 1,$$
where $r_p(X)$ denote the *injectivity radius* of X at p.

It is also known that there exist $3g-3-k \geqslant 0$ simple closed geodesics which, together with $\beta_1,\dots,\beta_k$, decompose X into $2g-2$ pairs of pants, i.e. spheres with three disjoint open discs removed. We can derive these results on the 'small' simple geodesics on X as an easy consequence of [**5**, Theorems 4.1.1 and 4.1.6]. The details are left to the reader.

We proceed to obtain quantitative estimates for the number of closed geodesics on X of length less than or equal to L for given $L > 0$.

Lemma 3.1. *Let X be a compact Riemann surface of genus $g \geqslant 2$ and let $L > 0$. There are at most*
$$\frac{(g-1)\sinh^2 \frac{1}{2} L'}{\sinh^2 \frac{1}{2}\operatorname{arcsinh} 1 \sinh^2 \frac{1}{4} \operatorname{arcsinh} 1}$$
oriented closed geodesics of length less than or equal to L on X which are not iterates of closed geodesics of length less than or equal to $2\operatorname{arcsinh} 1$, *where* $L' = L + 3\operatorname{arcsinh} 1$.

Remark 3.2. The parametrization of closed curves will be on the real line with period 1 or on the quotient $\mathcal{S}^1 = \mathbb{R}/[t \mapsto t+1]$. (Here $\mathbb{R}$ is, of course, the real axis.) Let γ and δ be closed curves. We say that γ is an iterate of δ if (for suitable parametrization) $\gamma(t) = \delta(mt)$, $t \in \mathcal{S}^1$, for some $m \in \mathbb{Z} \setminus \{0\}$.

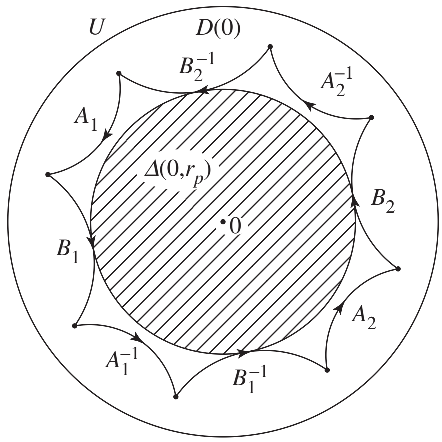

Figure 1. Illustration for genus 2. The sides are labelled in consecutive anticlockwise order as A_1, B_1, A_1^{-1}, B_1^{-1}, A_2, B_2, A_2^{-1} and B_2^{-1}. Note that $\Delta(0, r_p) = \{z \in U;\ d(z, 0) < r_p\}$ is the largest disc around 0 contained in the Dirichlet polygon $D(0)$ for G with centre 0, where d denotes hyperbolic distance on U. (It is now convenient to assume that U is the unit disc.)

Proof of Lemma 3.1. The discussion found in [**5**, § 6.6] (more precisely, see the proof of Lemma 6.6.4 therein) gives us a substantial portion of this lemma. Nevertheless, we sketch a proof, for the convenience of the reader.

Let $\mathcal{L}(p, L)$ denote the set of oriented geodesic loops at $p \in X$ of length less than or equal to L. For any $p \in X \setminus (\mathcal{C}(\beta_1) \cup \cdots \cup \mathcal{C}(\beta_k))$, we claim that

$$\#\mathcal{L}(p, L) \leqslant \frac{\sinh^2 \frac{1}{2}(L + \operatorname{arcsinh} 1)}{\sinh^2 \frac{1}{2} \operatorname{arcsinh} 1}. \tag{3.1}$$

Represent X as U/G, where G is a fixed-point free Fuchsian group. To prove (3.1), we use a universal covering $\pi : U \to U/G$. Without loss of generality we may assume that $\pi(0) = p$. Then we lift the geodesic loops at p onto geodesic arcs in U with a common initial point 0 (see Figure 1). Clearly, in the light of the definition of injectivity radius, the end points of these arcs have pairwise distances greater than or equal to $2r_p$. (Often we use r_p if there is no confusion about the surface.) Comparison of the areas now shows that there are at most

$$\frac{4\pi \sinh^2 \frac{1}{2}(L + \operatorname{arcsinh} 1)}{4\pi \sinh^2 \frac{1}{2} \operatorname{arcsinh} 1} = \frac{\sinh^2 \frac{1}{2}(L + \operatorname{arcsinh} 1)}{\sinh^2 \frac{1}{2} \operatorname{arcsinh} 1}$$

such geodesic loops. This concludes the proof of (3.1).

From now on $\Delta(p)$ will denote the disc of radius $\frac{1}{2} \operatorname{arcsinh} 1$ around $p \in X$. Let $\{\Delta(p_\lambda)\}_{\lambda \in \Lambda}$ be a maximal set of discs such that

(i) $p_\lambda \in X \setminus (\mathcal{C}(\beta_1) \cup \cdots \cup \mathcal{C}(\beta_k))$ for each $\lambda \in \Lambda$, and

(ii) if $\lambda_1 \neq \lambda_2$, then $\Delta_{p_{\lambda_1}}$ and $\Delta_{p_{\lambda_2}}$ are mutually disjoint.

It follows that

$$\#\Lambda \leqslant \frac{g-1}{\sinh^2 \frac{1}{4} \operatorname{arcsinh} 1}, \tag{3.2}$$

since

$$\#\Lambda 4\pi \sinh^2 \tfrac{1}{4} \operatorname{arcsinh} 1 \leqslant \operatorname{area}(X) = 4\pi(g-1)$$

(to obtain the above identity use the Gauss–Bonnet theorem).

A closed geodesic γ of length less than or equal to L which is not an iterate of β_j for $1 \leqslant j \leqslant k$ cannot be contained in the collar $\mathcal{C}(\beta_j)$, so γ passes through a point p of $X \backslash (\mathcal{C}(\beta_1) \cup \cdots \cup \mathcal{C}(\beta_k))$. By the maximality of $\{\Delta(p_\lambda)\}_{\lambda \in \Lambda}$, we see that $d(p, p_\lambda) < \operatorname{arcsinh} 1$ for some p_λ. Thus, γ is freely homotopic to some element of $\mathcal{L}(p_\lambda, L + 2 \arcsin 1)$.

We have already seen that any oriented closed geodesic of length less than or equal to L which is not an iterate of β_j for $1 \leqslant j \leqslant k$ must be freely homotopic to an oriented geodesic loop at some p_λ of length less than or equal to $L + 2 \operatorname{arcsinh} 1$. Hence, we can finally complete the calculation: the number of oriented closed geodesics of length less than or equal to L on X which are not iterates of closed geodesics of length less than or equal to $2 \operatorname{arcsinh} 1$ is at most

$$\frac{\sinh^2 \frac{1}{2} L'}{\sinh^2 \frac{1}{2} \operatorname{arcsinh} 1} \frac{g-1}{\sinh^2 \frac{1}{4} \operatorname{arcsinh} 1} = \frac{(g-1) \sinh^2 \frac{1}{2} L'}{\sinh^2 \frac{1}{2} \operatorname{arcsinh} 1 \sinh^2 \frac{1}{4} \operatorname{arcsinh} 1}$$

from (3.1) and (3.2). □

As mentioned above, every compact Riemann surface X can be decomposed into $2g-2$ pairs of pants. We can use admissible graphs as the combinatorial skeleton for the pasting of pairs of pants, and for this purpose the very rudiments of graph theory, as explained below, are sufficient. For more about graph theory, see [**3**].

Let (g, n) be a pair of non-negative integers with $v = 2g - 2 + n > 0$. Then $d = 3g - 3 + n \geqslant 0$. Consider a connected labelled graph $\mathcal{G}$ with $v(\mathcal{G}) = v$ vertices and $d(\mathcal{G}) = d$ edges. Each edge connects two vertices; an edge may connect a vertex with itself. We assume that at most three edges meet at any one vertex. A graph $\mathcal{G}$ of the above type will be called an *admissible* graph of type (g, n).

We associate a sphere with three holes to each vertex of $\mathcal{G}$. If two vertices are joined by an edge, then we glue the corresponding spheres along boundary curves. We thus obtain a (topological) surface S of type (g, n) and a set of d curves Σ that partitions S into a union of v pairs of pants. The pair of boundary curves forms a *partition* curve. The topological data (S, Σ) is uniquely determined by the graph $\mathcal{G}$; we shall call it the *surface* (*with maximal partition*) *corresponding to the graph*. Conversely, every surface S of type (g, n) with a maximal partition Σ determines an admissible graph $\mathcal{G}$ of type (g, n). A glance at Figure 2 will convey what is intended.

What are the estimates for lengths of geodesics in a maximal partition? Bers proved that every compact Riemann surface of genus $g \geqslant 2$ has a maximal partition $\{\gamma_1, \dots, \gamma_{3g-3}\}$ with geodesics of length

$$l(\gamma_j) \leqslant L_g, \quad j = 1, \dots, 3g-3, \tag{3.3}$$

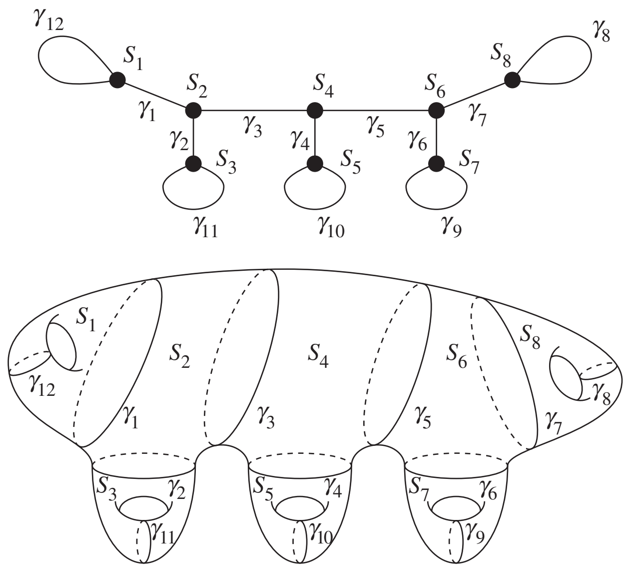

Figure 2. A trivalent graph and corresponding maximal partition for a surface of genus 5.

where L_g is a constant depending only on g. The best possible constant with this property is called Bers's constant and is denoted by L_g (by abuse of notation we also denote the best possible constant by L_g).

Lemma 3.3. *Let X be a compact Riemann surface of genus $g \geqslant 2$. Then there exist $2g$ closed loops $a_1, \dots, a_g$, $b_1, \dots, b_g$ on X providing a canonical homology basis for X such that*

$$l(a_i) \leqslant L_g, \quad i = 1, \dots, g. \tag{3.4}$$

Furthermore, each a_i can be selected to be a simple closed geodesic.

Proof. Let $\Sigma = \{\gamma_1, \dots, \gamma_{3g-3}\}$ be a maximal partition on X with geodesics satisfying (3.3). Let $\mathcal{G}$ be the admissible graph of type $(g, 0)$ associated with Σ, and let $\mathcal{G}'$ be a connected subgraph of $\mathcal{G}$ obtained by deleting g edges of $\mathcal{G}$.* Note that $\mathcal{G}'$ has $2g - 3$ edges and $2g - 2$ vertices. It is evident from our discussions that $\mathcal{G}'$ is admissible of type $(0, 2g)$.

Let $a_1, \dots, a_g$ be the partition curves corresponding to the edges on $\mathcal{G} \setminus \mathcal{G}'$ (the deleted edges). Then this system is extendable to a 'standard dissection' $a_1, b_1, \dots, a_g, b_g$. Indeed, if X is depicted as a sphere with g handles, a_i may be thought of as a loop 'across' the ith handle (see Figure 3), so b_i can be chosen to be a loop 'along' the same handle. □

* The emphasis here is on the word *connected*. Existence of such $\mathcal{G}'$ is easily established by induction on d.

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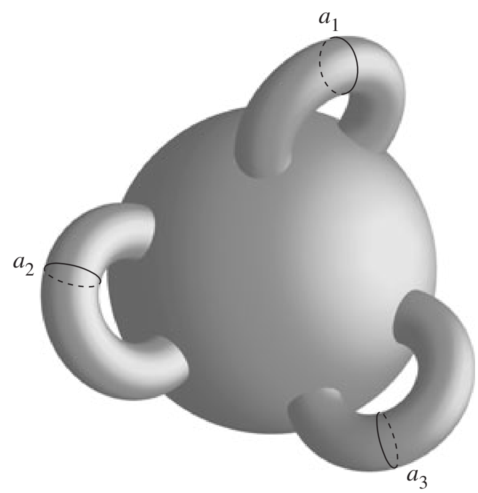

Figure 3. This figure illustrates the case where $g = 3$ and there are three 'a' curves. Here each a_i is a loop 'across' the ith handle.

Buser has gone further and showed that every compact Riemann surface of genus $g \geqslant 2$ has a maximal partition $\{\gamma_1, \dots, \gamma_{3g-3}\}$ with geodesics satisfying

$$l(\gamma_j) \leqslant 4j \log \frac{8\pi(g-1)}{j}, \quad j = 1, \dots, 3g-3. \tag{3.5}$$

This yields the following upper bound on Bers's constant:

$$L_g \leqslant 26(g-1). \tag{3.6}$$

How much can it be improved? Since many of the finiteness theorems resulting from Bers's theorem on length-controlled pants decompositions, described above, involve bounds which are rapidly growing functions of L_g, it would be desirable to have a (fairly) better bound for L_g. Yet a torus with 'thin hairs' was constructed to show that the bound cannot be improved too much. In fact, the hairy torus gives the following *lower* bound for L_g:

$$L_g \geqslant \sqrt{6g} - 2 \tag{3.7}$$

(see [**5**], in particular §§ 5.1 and 5.3).

4. Upper estimates for the cardinality of Hol(M, N)

In this section we finally establish the following result.

Theorem 4.1. *Let M and N be compact Riemann surfaces with $g_M \geqslant g_N \geqslant 2$. Let $\{a_1^M, \dots, a_g^M, b_1^M, \dots, b_g^M\}$ be a canonical homology basis on M and let $k(N)$ be the number of simple closed geodesics of length less than or equal to $2 \operatorname{arcsinh} 1$ on N. Then*

$$\# \operatorname{Hol}(M, N) \leqslant \prod_{i=1}^{g_M} \left[\frac{(g_M - 1)\sinh^2 \frac{1}{2} l_i}{\sinh^2 \frac{1}{2} \operatorname{arcsinh} 1 \sinh^2 \frac{1}{4} \operatorname{arcsinh} 1} + (8g_M - 7)k(N) \right],$$

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where

$$l_i = l(a_i^M) + 3 \operatorname{arcsinh} 1, \quad i = 1, \dots, g_M. \tag{4.1}$$

Proof. In the light of Lemma 2.3, we now wish to study the homology classes of the images of 'a' curves under holomorphic mappings of M onto N. Let $f \in \mathrm{Hol}(M, N)$. Then f, being holomorphic, must be distance decreasing in their respective Poincaré metrics by the Schwartz lemma (holomorphic maps are distance decreasing between hyperbolic manifolds). We conclude that

$$l(f(a_i^M)) \leqslant l(a_i^M), \quad i = 1, \dots, g_M. \tag{4.2}$$

Standard arguments based on the topological theory of covering spaces imply that there is an oriented closed geodesic of length less than or equal to $l(a_i^M)$ on N which is freely homotopic (thus homologous) to $f(a_i^M)$ (see, for example, the proof of [**6**, Lemma 2.3]).

We study the homology classes of the images of a_i^M under holomorphic mappings of M onto N. To this end, the index i will be fixed and we will denote by $\beta_1^N, \dots, \beta_{k(N)}^N$ the set of all simple closed geodesics of length less than or equal to $2 \operatorname{arcsinh} 1$ on N.

There are now two possibilities to consider. Either the image of a_i^M is homologous to an iterate of some β_j^N or it is not. Exclude for a moment the former case. Then we conclude that (using Lemma 3.1) the number of homology classes of the images of a_i^M under holomorphic mappings of M onto N is at most

$$\frac{(g_M - 1) \sinh^2 \frac{1}{2} l_i}{\sinh^2 \frac{1}{2} \operatorname{arcsinh} 1 \sinh^2 \frac{1}{4} \operatorname{arcsinh} 1}.$$

Next we need to consider the former case. Suppose that the image of a_i^M is homologous to $m\beta_j^N$, $m \in \mathbb{Z}$. It can be shown that m is bounded in absolute value by a constant depending only on the genera of the surfaces (see [**34**]). However, a much simpler algebraic proof can be given and it is sufficient for our purposes. We shall therefore be content with the following assertion.

Assertion 4.2. *We say that two closed curves c_1 and c_2 on the surface are n-homologous, in symbols $c_1 \overset{n}{\sim} c_2$, if there is an integer n such that $c_1 - c_2$ is homologous to the n-fold iterate of some closed curve c. Let f, $g \in \mathrm{Hol}(M, N)$. If*

$$f(a_i) \overset{n}{\sim} g(a_i), \quad i = 1, \dots, g_M,$$

for some integer $n > 8(g_M - 1)$, then $f = g$.

Remarks 4.3. (1) The $\overset{n}{\sim}$ equivalence classes of closed curves (homology classes) will be called the n-homology classes. It is not too difficult to see that the set of n-homology classes is isomorphic to the finite group of homology classes mod n. Hence, $f \in \mathrm{Hol}(M, N)$ can be recovered, of course, from the homomorphism

$$H_1(M, \mathbb{Z}) \to H_1(N, \mathbb{Z}_n)$$

obtained by following the homology map $H_1(M, \mathbb{Z}) \to H_1(N, \mathbb{Z})$, which is induced on the usual first homology groups, by the canonical projection $H_1(N, \mathbb{Z}) \to H_1(N, \mathbb{Z}_n)$.

(2) This assertion is thus an improvement over Lemma 2.3 rather than just the Martens 'rigidity' theorem (Proposition 2.2); see also the results of Tanabe [**29**] referred to in § 1.

Proof of Assertion 4.2. We continue to use the notation introduced near the end of § 2. Let $D = H_f - H_g$. If we write

$$D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \tag{4.3}$$

in $g_N \times g_M$ blocks, then

$$J_M D^{\mathrm{T}} J_N^{-1} D = \begin{pmatrix} -D_2^{\mathrm{T}} D_3 + D_4^{\mathrm{T}} D_1 & * \\ * & D_1^{\mathrm{T}} D_4 - D_3^{\mathrm{T}} D_2 \end{pmatrix}. \tag{4.4}$$

Suppose now that

$$D_1 \equiv D_3 \equiv 0 \pmod n \quad \text{for } n > 8(g_M - 1). \tag{4.5}$$

We immediately see that $2n \mid \|D\|^2$ ($2n$ divides $\|D\|^2$). However, it is easy to verify that

$$\|D\|^2 \leqslant 2(\|H_f\|^2 + \|H_g\|^2) \leqslant 16(g_M - 1) \tag{4.6}$$

(see § 2, especially inequality (2.11)). Thus, $D = 0$, and hence $f = g$ by Proposition 2.2. □

To complete the proof of Theorem 4.1 it suffices to study the n-homology classes of the images of a_i^M under holomorphic mappings of M onto N for $n > 8(g_M - 1)$. Since the simple homology equivalence relation is stricter than n-homology equivalence, the number of n-homology classes of the images of a_i^M which are not homologous to the iterates of β_j^Ns is less than or equal to

$$\frac{(g_M - 1)\sinh^2 \frac{1}{2} l_i}{\sinh^2 \frac{1}{2} \operatorname{arcsinh} 1 \sinh^2 \frac{1}{4} \operatorname{arcsinh} 1}$$

by our previous considerations. Thus, the number of n-homology classes of the images of a_i^M is less than or equal to

$$\frac{(g_M - 1)\sinh^2 \frac{1}{2} l_i}{\sinh^2 \frac{1}{2} \operatorname{arcsinh} 1 \sinh^2 \frac{1}{4} \operatorname{arcsinh} 1} + nk(N),$$

because the number of n-homology classes of the iterates of β_j^Ns is at most $nk(N)$. The assumption that $n > 8(g_M - 1)$ allows us to apply the assertion. We conclude that

$$\# \operatorname{Hol}(M, N) \leqslant \prod_{i=1}^{g_M} \left[\frac{(g_M - 1)\sinh^2 \frac{1}{2} l_i}{\sinh^2 \frac{1}{2} \operatorname{arcsinh} 1 \sinh^2 \frac{1}{4} \operatorname{arcsinh} 1} + nk(N) \right]$$

for $n > 8(g_M - 1)$. Choosing the integer n to be as small as possible, the theorem follows. □

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Let us be a little more explicit. As an application of Lemma 3.3, we take

$$\{a_1^M, \dots, a_{g_M}^M, b_1^M, \dots, b_{g_M}^M\}$$

to be a canonical homology basis on M such that $l(a_i^M) \leqslant L_{g_M}$ for $i = 1, \dots, g_M$. Using (3.6) we see that

$$l(a_i^M) \leqslant 26(g_M - 1), \quad i = 1, \dots, g_M. \tag{4.7}$$

We can now state the following as a consequence of our main theorem.

Theorem 4.4. *Let M and N be compact Riemann surfaces with $g_M \geqslant g_N \geqslant 2$, then*

$$\#\,\mathrm{Hol}(M,N) \leqslant \left[\frac{(g_M - 1)\sinh^2 \frac{1}{2} L'_{g_M}}{\sinh^2 \frac{1}{2} \operatorname{arcsinh} 1 \sinh^2 \frac{1}{4} \operatorname{arcsinh} 1} + 24 g_M^2\right]^{g_M},$$

where

$$L'_{g_M} = 26(g_M - 1) + 3 \operatorname{arcsinh} 1. \tag{4.8}$$

Proof. Use the fact that $k(N) \leqslant 3g_N - 3$ (see § 3). □

We thus have an upper bound on the cardinality of $\mathrm{Hol}(M,N)$ depending only on g_M, denoted $B(g_M)$. It may be worthwhile to obtain some information on the behaviour of $B(g_M)$ for very large values of g_M. The reader can easily verify that the exponential bound $\log B(g_M)$ grows like a constant multiple of g_M^2 as $g_M \to \infty$ and hence that

$$\log B(g_M) = O(g_M^2), \quad g_M \to \infty. \tag{4.9}$$

In contrast, [**29**, Theorem 2] gives an exponential bound depending only on g_M which is of order $g_M^2 \log g_M$.

Remark 4.5. We have made use of the inequality (3.6) to derive Theorem 4.4 from Theorem 4.1. As mentioned at the end of § 3, this estimate for Bers's constant cannot be improved too much. However, if we have a better bound on the sum of the lengths of the 'a' curves, the result of Theorem 4.4 will be strengthened. Imayoshi asked whether there is a canonical homology basis $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ on every compact Riemann surface X of genus $g \geqslant 2$ such that

$$l(a_1) + \cdots + l(a_g) \leqslant K g \log g, \tag{4.10}$$

where K is a constant (which does *not* depend on g, of course). If this conjecture is true, our condition shows that there will have to be an upper bound $B'(g_M)$ on the cardinality of $\mathrm{Hol}(M,N)$ such that

$$\log B'(g_M) = O(g_M \log g_M), \quad g_M \to \infty. \tag{4.11}$$

This is the strongest result that can be obtained by essentially the same method.

5. Notes on references

We end this paper by organizing some of the references that have not been specifically cited in the text, since a brief survey of such publications can serve as a reader's guide to those results which are not as well known as they should be. However, we must emphasize that we have not made an exhaustive search of the literature; no claims of completeness are implied or intended. Omissions are the result of ignorance, not judgment, by the authors.

In 1913 de Franchis [**8**] showed that the number of holomorphic mappings between two fixed compact Riemann surfaces is finite and that the number of possible target surfaces N for a fixed source surface M is also finite. As expected, much interest has centred on finding relevant bounds. This is a rather difficult problem, even in order to obtain a very crude bound.

Indeed, more explicitly, it was regarded as an open question whether the bounds depend only on the genera of the surfaces. A proof of this for the case of two fixed surfaces, which uses an idea of Weil [**33**], is alluded to in [**22**] (see § 2), but the reader should note that Martens's paper [**24**] credits Howard and Sommese [**10**] with the initial analysis of the general case. Both of these results yield an exponential bound on the number of holomorphic mappings between two fixed surfaces M and N, which depends only on g_M and is of order $g_M^2 \log g_M$.

Before we continue with our review, we remark that the second statement of de Franchis's theorem referred to in the preceding paragraph gives a remarkable finiteness theorem concerning the number of possible target surfaces for a fixed source surface. This statement is often attributed to Severi. In retrospect it seems that Severi [**27**] also obtained the relevant result. See below for fuller details.

Continuing a brief survey of facts about estimates on the cardinality of $\mathrm{Hol}(M, N)$, we mention Kani's paper [**17**] although it approaches the subject from a more algebraic-geometric point of view. We can use his result to sharpen the bound on the number of holomorphic mappings between M and N, and this provides an exponential bound which is of order g_M^2.

Now recall from § 4 that our exponential bound $\log B(g_M)$ is of order g_M^2, while Tanabe [**29**] gives an exponential bound which is of order $g_M^2 \log g_M$.

Remark 5.1. We mention here the work of Imayoshi [**14**], which came to our attention during completion of the preparation of the present paper. It presents a more complex-analytic approach to the subject; he also gave a successful application of Martens's arguments (see § 2) to 'improve' the count of holomorphic mappings.

These estimates are not the whole story. We conclude by mentioning a few more topics and references.

(i) First of all, significant progress is still being made, and it is impossible to predict where and when it will end. Recently, Tanabe [**30**] has proved that

$$\#\,\mathrm{Hol}(M, N) \leqslant 2\left\{\frac{4(g_M - 1)}{g_N - 1} + 1\right\}^{2g_M} (g_M - 1)(2g_N - 1).$$

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From the above inequality an exponential bound depending only on g_M which is of order $g_M \log g_M$ can easily be derived. As far as we know, the theorem in [**30**] is new and one of the main results of his investigation (recall the discussion at the end of § 4).

(ii) Very little is known about the sharpness of the estimates. However, we can discuss some of what is known about the simplest case of groups of conformal self-maps. Let $N(g)$ be the order of the largest group of conformal self-maps that a Riemann surface of genus $g \geqslant 2$ can admit. Hurwitz [**11**], as given in the course of the introduction, established (among other things) that $N(g) \leqslant 84(g-1)$. Macbeath [**20**] proved that $N(g) = 84(g-1)$ for an infinite number of integers g, so this upper bound is sharp for an infinite number of gs. Furthermore, Accola [**1**] showed that $N(g) \geqslant 8(g+1)$, and this lower bound is sharp for an infinite number of gs. In order to extend the considerations of these papers to the general case, however, completely different methods seem (at our present state of knowledge) absolutely necessary. For a survey of facts about the Hurwitz group see [**7**].

(iii) As we remarked earlier, de Franchis's second theorem, which also bears the name of Severi, asserts that the number of possible target Riemann surfaces for a fixed source Riemann surface is finite. But this is hardly the place to enter into a discussion. (In our context we have only encountered situations where the source and target surfaces are both fixed.) Moreover, the authors do not feel capable of crediting everyone who should be mentioned, so the interested reader is referred to the papers of Martens (see [**23**] and others), Imayoshi [**12**], and Tanabe [**31**] for more details and bibliographic references.

(iv) What if we worked in the category of (not necessarily compact) Riemann surfaces of finite analytic type? The situation for this slightly general case is considered by Ito and Yamamoto [**16**]. See also [**13**], which covers a noteworthy finiteness theorem concerning the number of holomorphic mappings between higher-dimensional manifolds.

(v) In fact, the various approaches to the subject, discussed here, are one instance of the interplay between topology, complex analysis and algebraic geometry. Some of the methods in this paper are specific to surfaces, while others apply in more general circumstances. The reader should seek out related results of a great deal of work by a number of mathematicians which have been published or which will be in forthcoming publications. The authors are aware of the papers by Bandman and Libgober [**2**], Borel and Narasimhan [**4**], Imayoshi and Shiga [**15**], Kobayashi and Ochiai [**18**], Noguchi [**25**], Noguchi and Sunada [**26**], Szabó [**28**] and Urata [**32**], and this is probably a very incomplete list of all such publications.

(vi) Finally, we mention that Imayoshi and others have suggested that one should picture the mappings $f \in \mathrm{Hol}(M, N)$ not just as holomorphic mappings between Riemann surfaces, but as holomorphic *sections* of a globally trivial N-bundle

$$M \times N \to M$$

fibreing over M, determined by the projection of $M \times N$ onto M, which deletes the second coordinate. The well-known *geometric Mordell conjecture*, which was formulated by Lang [**19**] and proved by Manin [**21**], states that an analytic family of Riemann surfaces C/B (truly varying and with fibres of genus $g \geqslant 2$) has only a finite number of sections $s : B \to C$. Can one, mimicking the preceding analysis, bound the number of holomorphic sections of such a truly varying family in purely topological terms?

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