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LIE ACTION OF CERTAIN SKEWS IN *-RINGS

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Introduction. A *-ring is an associative ring R with an anti-automorphism $*$ of period 2 (involution). Call $x \in R$ *skew* (*symmetric*) if $x = -x^*$ $(x = x^*)$ and let $K(S)$ be the additive subgroup of all skews (symmetrics). If $[a, b]$ denotes the Lie product of $a, b \in R$ (that is, $ab - ba$) and if $[A, B]$ denotes the Lie product of the additive subgroups A and B of R (that is, the additive subgroup generated by $[a, b]$, a and b ranging over A and B) then clearly $[K, K]$ is an additive subgroup contained in K. We shall be concerned here with the subrings A of R stable under the Lie action of $[K, K]$, that is, such that $[A, [K, K]] \subseteq A$. For R a simple 2-torsion free ring with centre $Z = 0$ or such that $\dim_Z(R) > 4$, the subrings A with the basic assumption or verifying the stronger assumption $[A, K] \subseteq A$ were studied. I. N. Herstein has shown that if $A \subseteq S$ is stable under the Lie action of $[K, K]$ and if R is, further, 3-torsion free necessarily $A \subseteq Z$. More recently, Herstein proved that if $A \subseteq S$ is stable under the Lie action of K and if R is not the 3×3 matrices over a field with characteristic 3 then $A \subseteq S$ implies $A \subseteq Z$. He then derived a general theorem for the subrings A stable under the Lie action of K of a 2-torsion free simple ring R with centre $Z = 0$ or with $\dim_Z R > 16$ namely, either $A \subseteq Z$ or $A = R$.

For R any 2-torsion free semiprime ring without necessarily an involution the subrings A of R stable under the Lie action of all of R were shown by Herstein to be either contained in Z or containing some non-zero ideal of R. This is, in a way, the best expectation for the considered subrings A. Accordingly, we shall exhibit here a clear-cut class of 2-torsion free semiprime $*$-rings R, in which, the subrings A stable under the Lie action of $[K, K]$ will behave fairly close to the non-involutive case and of course to the simple involutive case namely, in the event where A contains no ideals $\neq 0$ of R, then the closure of A under the involution (i.e. $*$-subring generated by A) has all its symmetrics central.

1. Assumptions on the ring R. In the introduction we cited some positive facts about the subrings A of a $*$-ring R stable under the Lie action of R or K or $[K, K]$, where R was, at least, 2-torsion free and semiprime. We now discuss briefly the extra assumptions we shall place on the ring R. Our first assumption will be

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Assumption (1): R is generated as a ring by skews. In symbols: $R = \bar{K}$.

Clearly every simple $*$-ring of dimension > 4 must satisfy the assumption (above). There is one more reason for requiring such a condition. Clearly all 4-dimensional $*$-division rings R with $*$ an involution of the first kind are such that $[K, K] = 0$ and every subring is trivially stable under the Lie action of $[K, K]$. In any event, if R is any semiprime ring then the subring $\bar{K}$ generated by K is a semiprime ring verifying assumption (1). We proceed to

Assumption (2): R is 3-torsion free.

Herstein has given an example of a commutative subring $A \subseteq S$, $A \nsubseteq Z$, stable under the Lie action of K itself; where R is the 3×3 matrices over a field with characteristic 3 [**4**]. This makes it clear that if we are to derive a dichotomy as cited in the introduction, we must rule out the case of characteristic 3 or, a much more difficult condition to check, we must rule out the prime images of R that are orders in 3×3 matrix rings over fields. We have decided in favour of the first restriction for two reasons. For one thing, the argumentation works much smoother. Also the 6-torsion free assumption crops up in a number of works on $*$-rings to cite only Herstein's result in the introduction. A good compromise between the two restrictions would be to assume R with *no* prime images that are orders in 3×3 matrix rings over fields with characteristic 3. As a matter of fact all of the results in this paper would still be true under this weakening of assumption (2).

Assumption (3): $R = 2R$.

Assumption (3) is for the sake of simplicity only. Some interesting consequences of Assumptions (1) and (3) are the following.

Consequences (of Assumptions (1) and (3)):

(1) Every element x of R gives rise to a symmetric $x^+ = \frac{1}{2}(x + x*)$ and a skew $x^- = \frac{1}{2}(x - x*)$ with $x = x^+ + x^-$.

(2) Every symmetric $s = s*$ is a sum of squares of skews (Baxter and Herstein).

(3) $[S, S] \subseteq [K, K]$ (Herstein).

Invariably, in about every study on stable subrings under certain Lie action the following fact is exploited.

PROPOSITION 1 (Herstein). *Let A be a subring of (any ring) R and suppose that X is an additive subgroup of R such that $[A, X] \subseteq A$. If $T = T(A)$ denotes the additive subgroup of all elements $r \in R$ such that $[r, R] \subseteq A$ then*

$$\bar{X}\,[T, [T, T]]R \subseteq A;$$

where $\bar{X}$ denotes the subring generated by X.

Proof. Given any triple of elements a, b, x in R we have the Jacobi identify $[a, b]x = [a, bx] - b[a, x]$. Choose $a \in T$ and $b \in [T, R]$. Then $[a, bx] \in$

$[T, R] \subseteq A$. Also $b[a, x] \in [T, R]^2 \subseteq A^2 \subseteq A$. Thus $[a, b]x \in A$. We have shown that $[T, [T, R]]R \subseteq A$. Taking the Lie product of the latter inclusion with X we get

$$X[T, [T, R]]R \subseteq [T, [T, R]]RX + [X, (T, [T, R]]R] \subseteq A + [X, A] \subseteq A + A = A.$$

By induction on n we obtain

$$X^n[T, [T, R]]R \subseteq A,$$

resulting in

$$\bar{X}[T, [T, R]]R \subseteq A$$

as desired.

Since the Lie expressions $[T, R]$, $[T, [T, R]]$ will quite often occur in this paper we shall make the following definition.

Definition 1. Given the additive subgroup A of R let:
(1) $T(A) = \{r \in R \mid [r, R] \subseteq A\}$,
(2) $A' = [A, R]$,
(3) $A'' = [A, A'] = [A, [A, R]]$.

An interesting corollary to Proposition 1 is the following.

COROLLARY 1. *If A is a subring of R with $[A, K] \subseteq A$ then $RT(A)''R \subseteq A$.*

Proof. By Assumption (1), $\bar{K} = R$, and by Proposition 1, $\bar{K}T(A)''R \subseteq A$.

Looking at the subring $V = \overline{[K, K]}$, obviously with the condition $[V, K] \subseteq V$, one would like to ensure that at least this subring must contain a non-zero ideal of the ring R. Suppose not. By Corollary 1, $RT''(V)R = 0$. Since R is semiprime $T(V)'' = 0$ follows.

A well-known result of Herstein asserts that if $T(V)'' = 0$ then $T(V)' = 0$ ($T(V)$ can be replaced by an arbitrary subset and the 3-torsion free assumption is not required).

In other words $[r, R] \subseteq V$ implies $r \in Z$, the centre of R. Since V is stable under (the Lie action of) K it follows that $[k^2, R] \subseteq V$ for all $k \in V^- = \{\frac{1}{2}(x - x^*)\}_{x \in V}$. (Here is Herstein's argument: $[k^2, y] = [k^2, y^+] + [k^2, y^-] = (k(ky^+ + y^+k) - (ky^+ + y^+k)k) + (k[k, y^-] + [k, y^-]k) \in [k, K] + (k[k, K] + [k, K]k) \subseteq V + (kV + Vk) = V$.) Therefore $k^2 \in Z$ for all $k \in V^-$.

What can be asserted about a semiprime ring R with assumptions (1) and (3) such that $L = [K, K]$ consists entirely of square-central elements? We claim that $S \subseteq Z$ necessarily. We shall sketch a proof. We take a subdirect representation of R formed by *-prime rings (i.e. $*$-rings, in which, any two non-zero $*$-ideals have non-zero product). This reduces to the $*$-prime case, which in turn reduces to the prime case. Having in hand the polynomial identity $[k_1, k_2]^2 \in Z$ for all $k_i \in K$ we are ensured that $Z \neq 0$, and consequently $Z^+ \neq 0$. Thus R localised at Z^+ becomes a simple finite-dimensional $*$-ring with the same identity and $L(Z^+)^{-1}$ is now a Lie ideal of $[K, K]$ con-consisting entirely of square central elements. This reduces to the simple finite-dimensional case with dimension $\leqq 16$ (Amitsur). We then tensor R with the algebraic closure C of the field Z^+. This reduces to matrix rings over fields with rank $\leqq 4$. Except when the involution is transpose and the rank $= 2$, $[K, K] = K$ necessarily. But since R is, by Assumption (1), generated by K, $[K, K] = 0$ forces $R = Z$, contradicting the rank. Thus $0 \neq [K, K]$ forces $K = [K, K]$. Thus every symmetric $s = s* \in R$ is a sum of squares of elements in $[K, K]$. Thus $S \subseteq Z$ necessarily. Summarizing:

PROPOSITION 2. *If R is any 2-torsion free semiprime ring generated by skews and if $S \not\subseteq Z$, the subring V generated by $[K, K]$ contains a non-zero ideal. In fact the ideal generated by squares of the elements in $[K, K]$ must be a non-zero ideal contained in V.*

The following proposition is of independent interest [**1**, Theorem 1]:

PROPOSITION 3. *Under the assumptions of Proposition 2, $K + [K, K]^2$ contains a non-zero ideal I. Consequently every symmetric $s = s* \in I$ is a sum of squares of commutators in $[K, K]$.*

Proof. We show first that $K + [K, K]^2$ contains $[K, K]^3$. Given the triple $k_1, k_2, k_3 \in [K, K]$ we have

$$\begin{aligned} 2k_1k_2k_3 &= (k_3k_1k_2 + k_3k_2k_1) + (k_3k_1k_2 - k_3k_2k_1) \\ &= k_3[k_1, k_2] - ([k_2, k_3]k_1 + k_1[k_2, k_3]) \\ &\qquad + (k_1k_2k_3 + k_3k_2k_1) \in [K, K]^2 + K. \end{aligned}$$

As observed earlier, if $k_0 \in [K, K]$ then $[k_0^2, R] \subseteq [K, K] + [K, K]^2$, that is, $k_0^2 \in T(K + [K, K]^2) = T$. Now

$$K[T, [K, K]]R \subseteq K + [K, K]^2.$$

For, let $k_0 \in [K, K]$ and $r \in T$. For x an arbitrary element of R:

$$\begin{aligned} [r, k_0]x &= [r, k_0x] - k_0[r, x] \in [r, R] + [K, K][r, R] \\ &\subseteq [K, K] + [K, K]^2 + [K, K]([K, K] + [K, K]^2) \\ &\subseteq [K, K] + [K, K]^2 + [K, K]^3 \subseteq K + [K, K]^2. \end{aligned}$$

Taking the Lie product with K we get

$$\begin{aligned} K[T, [K, K]]R &\subseteq [T, [K, K]]RK + [K, K + [K, K]^2] \\ &\subseteq K + [K, K]^2 + [K, K] + [K, K]^2 = K + [K, K]^2. \end{aligned}$$

An induction on n gives that $R[T, [K, K]]R \subseteq K + [K, K]^2$. If the latter ideal is zero then $[T, [K, K]] = 0$. In particular $[k_0^2, [K, K]] = 0$, for all $k_0 \in [K, K]$. If R is a $*$-prime ring with the latter property and if $S \not\subseteq Z$ then

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k_0^2 centralizes $V = \overline{[K, K]}$ which contains a non-zero ideal (Proposition 2). It follows that $k_0^2 \in Z$, for all k_0, and we are back to Proposition 2.

We shall conclude this preparatory section by a study of the equation $a[K, K] = 0$. We assume, further, that $a^2 = 0$ and $a \in S \cup K$.

PROPOSITION 4. (i) *The equation* $a[K, K]a = 0$, *with* $a^2 = 0$ *and* $a = -a^*$, *implies* $a = 0$.

(ii) *The equation* $a[K, K]a = 0$, *with* $a^2 = 0$ *and* $a = a*$, *implies* $aKa = 0$ *and consequently, if* $a \neq 0$, $ak_0^2a \neq 0$, *for some* $k_0 \in [K, K]$.

Proof. (i) We may take R to be a $*$-prime ring. If R is not prime there must be a non-zero ideal I of R with $I \cap I^* = 0$. Choosing $x, y \in I$ we get $[x, y] - [x, y]^* = [x - x*, y - y*]$, so, $a[x, y]a - a[x, y]*a = a[x - x*, y - y*]a = 0$, whence, $a[x, y]a = 0 = a[x, y]*a = a[y*, x*]a$. From this $a[J, J]a = 0$; where $J = I + I*$ is a $*$-ideal of R. Then for $u, v \in J$, $a(u(av) - (av)u)a = auava = 0$. Thus $(aJ)^3 = 0$ giving $aJ = 0$, whence $a = 0$. If, on the other hand, R is prime with involution of the second kind take any skew $0 = \sigma \in Z$. Then $a[K, \sigma K]a = a[\sigma K, \sigma K]a = 0$. Without loss of generality σ can be assumed invertible giving thus that $a[R, R]a = 0$, and by the above $a = 0$ necessarily.

Next suppose that R is a regular ring. For any $x, y \in R$,

$$[x, y] - [x, y]* = 2[x^+, y^+] + 2[x^-, y^-] \in [K, K].$$

Therefore, $a[x, y]a = a[x, y]*a$. Write $a = aba$ and set $x = ba = x^2$. For any $y \in R$:

$$\begin{aligned} a[x, y]a &= a(xy - yx)a \\ &= (ax)ya - ay(xa) \\ &= aya - ayba^2 = aya; \\ a[x, y]*a &= (a[x, y]a)* = ay*a = aya; \\ aKa &= 0, \quad \text{for all } k \in K. \end{aligned}$$

Since a was a skew element the latter equation gives $a = 0$.

We go back to a prime ring R with involution of the first kind. Exactly as in the case $aKa = 0$ this gives for $a \neq 0$ that R has a central closure Q, which is a primitive ring containing in its socle a [**5**], Since the property of a carries over to Q, and hence to $R_0 = \text{socle }(Q)$, and since the latter ring is certainly a regular ring we are done.

(ii) For (ii) observe that for any $k \in K$, aka is a square-zero skew inheriting the property of a and apply (i). If $a \neq 0$, but $ak^2a = 0$ for all $k \in [K, K]$, it follows that $a(k_1k_2 + k_2k_1)a = 0$, all $k_i \in [K, K]$. Because $a[k_1, k_2]a = 0$ this gives $ak_1k_2a = 0$, and consequently, $a(K + [K, K])a = 0$, contrary to Proposition 3.

2. The dichotomy in *R*. We announced in the introduction that under the assumptions (1)-(3) we were to obtain that every subring A with $[A, [K, K]] \subseteq$

A either contains a non-zero ideal of R or the closure of A under $*$ is such that all symmetrics are central. We wish to sketch a plan of attack we shall follow here. In the closed case ($A = A*$) we associate to A a *canonical* ideal $J(A)$ (i.e. if f maps R onto $\underline{R}$ then f maps $J(A)$ onto $J(\underline{A})$) such that if this is the zero ideal necessarily $A^+ \subseteq Z$. In the general case we associate to A the canonical ideal $J((\overline{A + A*})^+)$. (Clearly if $[A, [K, K]] \subseteq A$ then $(\overline{A + A*})^+$ is a *subring* generated by symmetrics inheriting the basic property of A, so, the dichotomy above will apply (at full force). The Lie relation between the latter subring and A, together with some results of Herstein to be indicated in time will yield the desired dichotomy.

Definition 2. Let $A = A^*$ be such that $[A, [K, K]] \subseteq A$. If G denotes the ideal of R generated by all k^2, ranging over $[K, K]$, then set

$$J(A) = G[[A^+, A^+], [A^+, A^+]]''G.$$

PROPOSITION 5. *$J(A)$ is a canonical ideal of R contained in A provided $A = A^*$ is preserved under the Lie product with $[K, K]$.*

Proof. That $J(A)$ is a canonical ideal of R is evident from Definition 2. For the inclusion $J(A) \subseteq A$ proceed as follows:

$$\begin{aligned}
&[A^+, [R, R]] = [A^+, [K, K]] + [A^+, [S, S]] + [A^+, [S, K]] \\
&\qquad \subseteq [A^+, [K, K]] + [A^+, S] \subseteq A^+ + [A^+, S]; \\
&[[A^+, A^+], [R, R]] \subseteq [A^+, [A^+, [R, R]]] \subseteq [A^+, (A^+ + [A^+, S])] \\
&\qquad \subseteq [A^+, A^+] + [A^+, [A^+, S]] \subseteq [A^+, A^+] + [A^+, [K, K]] \\
&\qquad \subseteq [A^+, A^+] + A^+ \subseteq A; \\
&[[[A^+, A^+], [A^+, A^+]], R] \subseteq [[A^+, A^+], [[A^+, A^+], R]] \\
&\qquad\qquad \subseteq [[A^+, A^+], [R, R]] \subseteq A.
\end{aligned}$$

By Proposition 1, $\overline{[K, K]}(T(A))''\overline{[K, K]} \subseteq A$, which gives immediately the desired inclusion.

PROPOSITION 6. *$J(A) = 0$ implies that $A^+ \subseteq Z$.*

Proof. Since $J(A)$ is a canonical ideal it suffices to establish the result for R, a $*$-prime ring. From Definition 2 follows that $K_1 = [[A^+, A^+], [A^+, A^+]] \subseteq Z$. We separate the rest of the proof according as R is prime with involution of the first kind (I) or not (II).

Case (I): Necessarily $K_1 = [[A^+, A^+], [A^+, A^+]] = 0$. Since $[A^+, A^+]$ is itself a Lie ideal of $[K, K]$, it follows by an adaptation of Erickson's [**2**] or directly that $[A^+, A^+] = 0$ necessarily. The rest of the argumentation is routine but, in view of the incidence on the final theorem, we give it explicitly. We prove actually a slightly more general result: any commutative subring $A = A*$ with $[A, [K, K]] \subseteq A$ must be in fact contained in Z.

Given $a \in A^+$, let $d_a : x \rightarrow [a, x]$. From $[a, d_a(S)] \subseteq [S, S]$ follows $d_a^2(S) \subseteq A$, and $d_a^3(S) = 0$, whence $d_a^4(R) = d_a^4(S) + d_a^4(K) = 0 + d_a^3(d_a(K)) = 0$.

From this either $a^3 = 0$ or a is a non-zero divisor. Actually if $a^3 = 0$ then $a^2Sa^2 = a(aSa^2) = a(a^2Sa) = 0$, so, $a = 0$, and all symmetric nilpotents in A are then square-zero elements. We shall show that A contains no such elements (but 0). For suppose the contrary. The element under consideration a is such that $[a, [a, [K, K]]] = 0$, giving $a[K, K]a = 0$. By Proposition 4 ii) $aKa = 0$ follows and $ak_0^2a \neq 0$, some $k_0 \in [K, K]$. Now $c = [a, k_0] \in A$ and $c^2 = -ak_0^2a \neq 0$. Thus c is a non-zero divisor on R having the square-zero factor a, which is a contradiction. This shows that A^+ consists entirely of non-zero divisors. But, $0 = d_a^4(k_0x) \equiv 4d_a(k)d_a^3(x)$, $a \neq 0$ in A, $k \in [K, K]$, forces $d_a^3 \equiv 0$. (For if $d_a(k) \neq 0$, as $a \in A$, it must be a non-zero divisor. If, on the other hand, $d_a([K, K]) = 0$, by Proposition 2, $a \in Z$ necessarily, whence $d_a^3(x) \equiv 0$.) Then $0 \equiv d_a^3(k_0x) \equiv 3d_a(k) \cdot d_a^2(x)$ gives $d_a^2(x) \equiv 0$, so, $a \in Z$. Summarizing: $A^+ \subseteq Z$. Let $\sigma \in A^-$. If $\sigma^2 = 0$, then $0 = [\sigma, [\sigma, [K, K]]]$ gives $\sigma[K, K]\sigma = 0$, and by Proposition 4, i), $\sigma = 0$ necessarily. From this $0 \neq \sigma$ forces $0 \neq \sigma^2 \in Z$, and σ is a non-zero divisor. Then for any $k \in [K, K]$:

$$0 = [\sigma^2, k] = \sigma[\sigma, k] + [\sigma, k]\sigma = 2\sigma[\sigma, k];$$
$$[\sigma, k] = 0.$$

This shows that $[\sigma, [K, K]] = 0$. Either $S \subseteq Z$ or the latter relation forces $\sigma = 0 \in Z$ (Proposition 2). In the first case $[\sigma, [\sigma, x^-)) \equiv 0$ and $[\sigma, [\sigma, x^+]] \in [\sigma, S] = 0$ gives $[\sigma, [\sigma, x^+]] \equiv 0$. Thus $[\sigma, [\sigma, x]] \equiv 0$ resulting in $\sigma \in Z$. This shows that $A^-(= 0) \subseteq Z$, whence $A = A^+ + A^- \subseteq Z$.

Case (II): 1) If R is not prime, we shall prove that every subring A with $[[K, K], A] \subseteq A$ is either central or must contain some non-zero ideal. This will clearly take care of the case under consideration. Let P be a non-zero ideal of R with $P \cap P* = 0$. If $A \cap P \not\subseteq Z$ (or $A \cap P* \not\subseteq Z$) then $A \cap P$ is a subring of P thought of as a ring, such that $[A \cap P, [P, P]] \subseteq A \cap I$. By [**7**, Theorem 3], $A \cap P$ contains a non-zero ideal of P, and hence of R. If, on the other hand, $A \cap P \subseteq Z$ and $A \cap P* \subseteq Z$, this forces $A \cap P \subseteq Z$ in $R/P*$ and $A \cap P* \subseteq Z$ in R/P. Since the latter rings are prime rings we may assume that $A \cap P = A \cap P* = 0$. It follows that $[A, [P, P]] = [A, [P*, P*]] = 0$, whence $[A, [P + P*, P + P*]] = 0$ forcing $[A, P + P*] = 0$ [**7**, Lemma 2], whence $A \subseteq Z$.

2) If R is prime with involution of the second kind we shall then prove that every subring A with $[A, [K, K]] \subseteq A$ is either contained in Z or such that AZ^{-1} contains an ideal in the localisation $R(Z^+)^{-1}$ of R. This will show that for the subring A under consideration necessarily $A \subseteq Z$. Take any central skew $\sigma \neq 0$. Then

$$[ZA, [\sigma S, K]] \subseteq ZA.$$

It follows that $(ZA)Z^{-1} = A_0Z^{-1}$ is a subring of $R(Z^+)^{-1}$ satisfying $[AZ^{-1}, [R, R]] \subseteq AZ^{-1}$. By [**7**, Theorem 3], A_0Z^{-1} has the desired property.

Definition 3. Given the subring A (not necessarily a $*$-subring) let $U = (A + A*)^+$, and let $W = \overline{[U, U]}$. Define $J_1(A) = J(W)$. Then $J_1(A)$ is a canonical ideal of R and $[A, [J_1[A], R]] \subseteq A$.

Justification. Clearly W depends canonically on A, and as observed earlier $J(W)$ depends canonically on W. Thus $J_1(A)$ depends canonically on A. For the inclusion in the statement, let us first show that $[A, [U, [R, R]]] \subseteq A$. Now

$$[U, [S, K]] \subseteq [S, S] \subseteq [K, K];$$
$$[U, [K, K]] \subseteq [A, [K, K]] + [A*, [K, K]] \subseteq A + [A*, [K, K]];$$
$$[A, [U, [R, R]]] \subseteq [A, [K, K]] + [A, A] + [A, [A*, [K, K]]].$$

To get the desired inclusion we must then show that $[A, [A*, [K, K]]] \subseteq A$. Let $a \in A$, $b* \in A*$, and $c = [k_1, k_2] \in [K, K]$. Clearly

$$d = [b*, c] - [b*, c]* = 2[(b*)^-, c] \in [K, K].$$

Then

$$\begin{aligned}[a, [b*, c]] &= [a, d] + [a, [b*, c]*] = [a, d] + [a, [c*, b]] \\ &= [a, d] + [a, [b, [k_1, k_2]]] \in [A, [K, K]] + [A, [A, [K, K]]] \subseteq A.\end{aligned}$$

Now

$$\begin{aligned}[A, [\overline{[U, U]}, R]] = [A, [[U, U], R]] &\subseteq [A, [U, [U, R]]] \\ &\subseteq [A, [U, [R, R]]] \subseteq A.\end{aligned}$$

Since $W = \overline{[U, U]}$ is a $*$-subring with $[W, [K, K]] \subseteq W$, it was observed earlier that $J_1(A) = J(W) \subseteq W$. Thus $[A, [J_1(A), R]] \subseteq [A, [W, R]] \subseteq A$, as wished.

PROPOSITION 7. *$J_1(A) = 0$ forces the closure of A under $*$ to be such that every symmetric is central.*

Proof. Case (I). We suppose that R is a prime ring with $*$ of the first kind. We are given that

$$J_1(A) = J(W) = 0.$$

Since W is a $*$-subring with $[W, [K, K]] \subseteq W$ we get, using Proposition 5, that $W^+ \subseteq Z$. If $W^- = 0$, then $[U, U] \subseteq W^-$ gives $[U, U] = 0$. By Proposition 6 applied to $\bar{U}$, $U \subseteq Z$ follows. Let $B = Z^+A + Z^+ + A$. We claim that B is a $*$-subring containing obviously A all of whose symmetrics are central. For let $x \in B$. Write

$$x = \sum z_i a_i + z + a;$$
$$x* = \sum z_i a_i* + z + a*;$$
$$x + x* = \sum z_i(a_i + a_i*) + 2z + (a + a*).$$

Now $a_i + a_i* \in (A + A*)^+ = U \subseteq Z$. Also $a + a* \in U \subseteq Z$. Thus $x + x* \in Z^+ \subseteq B$. Since x was in B, $x* \in B$ follows. From this B is a $*$-subring with the desired property for its symmetric part B^+. Thus the closure of A under $*$ has the desired property.

If, on the other hand, $W^- \neq 0$, we shall reduce to the case $K = [K, K]$ and will finish up this part of the proof. Without loss of generality we may assume that Z^+ (if $\neq 0$) is a field. Now W^- is a Lie ideal of $[K, K]$ consisting entirely of square-central elements. By a result of Erickson [**2**] this can happen only if R is an order in a simple ring Q of dimension $\leqq 16$. By the assumption on Z^+, $R = Q$ follows. Tensoring R with the algebraic closure C of the field Z^+ we we reach the situation

$$\bar{R} = R \otimes_{Z^+} C \approx C_n, \quad n \leqq 4.$$

If $n = 2$, either the involution is symplectic in which case there is nothing to prove or the involution is transpose in which case $[\bar{K}, \bar{K}] = 0$, so we may take $n > 2$. But for $n = 3$ or 4, we repeat that $\bar{K} = [\bar{K}, \bar{K}]$ necessarily. Now $A \otimes C$ is a subring verifying $[A \otimes C, K] \subseteq A \otimes C$. If $A \otimes C = \bar{R}$, then for every pair of symmetrics $s, d \in R$:

$$\begin{aligned} s &= \sum a_i \otimes c_i = \sum a_i{}^* \otimes c_i \\ &= \sum \tfrac{1}{2}(a_i + a_i*) \otimes c_i; \\ d &= \sum b_i \otimes d_i = \sum \tfrac{1}{2}(b_i + b_i*) \otimes d_i; \\ [s, d] &= \sum_{i,j} [a_i + a_i*, b_j + b_j* \otimes c_i d_j \in [U, U] \otimes C \end{aligned}$$

Thus $[s, d]^2 \in Z(\bar{R}) = C$, all $s, d \in \bar{R}^+$ contradicting the restriction $n \geqq 3$. We have to agree that $A \otimes C$ cannot be $\bar{R}$. Thus $A \otimes B$ contains no ideals $\neq 0$ since $\bar{R}$ is simple.

We go back to any prime ring R and consider any subring B such that $[B, K] \subseteq B$. We prove that either B contains an ideal or $B^+ \subseteq Z$. This will finish up case (I). By an argument similar to the one used in the justification of Definition 3, we get that $[B, [(B + B*)^+, R]] \subseteq B$. Thus $[B, [(\overline{B + B*})^+, R]] \subseteq B$. If $(\overline{B + B*})^+$ contains a non-zero ideal I, then $[I, R] \neq 0$ is a Lie ideal of R with $[B, [I, R]] \subseteq B$. By [**7**, Theorem 3], B must contain a non-zero ideal. If, on the other hand, $(\overline{B + B*})^+$ contains no ideals, this must be central (Proposition 6). As in the above the closure of B under $*$ must have all the symmetrics central.

Case (II): 1) We suppose that R is not prime. Since $J_1(A) = J(W) = 0$ it follows that $W \subseteq Z$. Thus $[U, U] \subseteq W \subseteq Z$, and hence $[U, U] = 0$ giving $U \subseteq Z$, and we are done.

2) We suppose that R is prime with an involution of the second kind. It can be easily checked that the expansion WZ^{-1} of W in $R(Z^+)^{-1}$ contains no ideals and consequently $W \subseteq Z$. As in the above this gives $U \subseteq Z$, and the proof is now complete.

We have all the pieces to prove the following result.

THEOREM. *Let $R = 2R$ be any 6-torsion free $*$-ring generated by skews. If A is a subring of R with $[A, [K, K]] \subseteq A$, then A must be a semiprime subring with center contained in the centre of R. In fact either A contains an ideal or the closure of A under $*$ has all its symmetrics central elements of R.*

Proof. We establish first the dichotomy about A. Let $W = [\overline{U, U}]$ and let $J_1(A) = J(W)$ be the ideal associated to W. It was observed earlier that $[A, [J(W), R]] \subseteq A$. Now $[J(W), R]$ is a Lie ideal of R. By [**7**, Theorem 3] either $[A, [J(W), R]] = 0$ or A must contain a non-zero ideal of R. Suppose that $[A, [J(W), R]] = 0$. Since this equation is preserved by homomorphisms we may assume that R is a $*$-prime ring (for we shall prove that the closure of A under $*$ has the desired property on symmetrics, and centrality can be reduced to that case). If $J(W) \neq 0$ and $R \neq Z$, then $[J(W), R]$ is a non-zero $*$-Lie ideal of R having for centralizer Z necessarily. This shows that $J(W) = J_1(A) = 0$ necessarily. We then quote Proposition 7.

Next we establish the desired properties about the prime radical and centre of A. Clearly the centre B of A is a commutative subring such that $[B, [K, K]] \subseteq B$. From this $D = Z^+A + A + A^+$ is a $*$-subring of all whose symmetrics are central. It follows that $D \subseteq Z$ necessarily, whence $D \subseteq Z$ (for D is commutative).

Take $a \in A$ with $aAa = 0$ and $a^2 = 0$. For every $k_1, k_2 \in [K, K]$

$$0 = a[[a, k_1], k_2]a = ak_1ak_2a + ak_2ak_1a.$$

Setting $k_1 = k_2$ we get $akaka = 0$. Thus

$$ak_1ak_2ak_1a = -(ak_2ak_1a)k_1a = 0, \quad \text{for all } k_i \in K.$$

It follows that $ak_1a[K, K]ak_2a = 0$. We may suppose R a $*$-prime ring. If R is not prime it was observed earlire that $a = 0$ necessarily. The same conclusion holds if R is prime with involution of the second kind (Proposition 4). Finally if R is prime with involution with the first kind, clearly A cannot contain an ideal $I \neq 0$. By the dichotomy above $a + a* \in Z$. Thus $aa* = a*a$ and consequently $a + a*$ is a central nilpotent, so, $a = -a*$ necessarily. Since ak_1a is also a skew, this gives $ak_1a = 0$ (Proposition 4), which in turn forces $a = 0$, and the theorem is proved.

We make some further remarks about the assumptions in the theorem. The generation by skews (Assumption (1)) was to ensure that the centralizer of $[K, K]$ in R must be the centre Z. More generally, if we do not insist on this generation then the conclusion in the theorem becomes modulo $[K, K]$. More precisely, in the event that the considered subring A contains no ideals $\neq 0$, then the closure of A under $*$ centralizes $[K, K]$. The 6-torsion freeness (Assumption (2)) was discussed earlier (§ 1, Assumption (2)). Removing it totally will impair the conclusion of Theorem 1. The best we can say in this event is that if A contains no ideals then the closure of A under $*$ is such that the symmetrics commute. (This requires some work.) Notice however that A will then satisfy a polynomial identity of fairly low degree.

To close, here is an example, as in the theorem.

Example. Let D be a 2-torsion free 4-dimensional $*$-division ring with centre $Z(D)$ having a 1-dimensional additive subgroup $K(D)$. Let R be the 2×2 matrices over D with the $*$-transpose involution. If A is the subset of all matrices of the form

$$\begin{bmatrix} z+k & z_1+k_1 \\ -z_1+k_1 & z-k \end{bmatrix}; \quad z, z_1 \in Z(D), k, k_1 \in K(D)$$

the centralizer B of R is then all matrices of the form

$$\begin{bmatrix} z+k & t \\ -t & z+k \end{bmatrix};$$

where $z, k \in Z(D), K(D)$; t any element anti-commuting with the skews in D. It can be verified that A is a $*$-subalgebra (and hence B), and $A^- \oplus B^- = K$. Consequently A (and hence B) is stable under the Lie action of K. But $A^- \not\subseteq Z$. This example shows also that the centre of A and the centralizer of A are different.

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