Can. J. Math., Vol. XXVIII, No. 6, 1976, pp. 1121–1131

THE NORM OF THE L^p-FOURIER TRANSFORM, II

BERNARD RUSSO

1. Introduction. Let G be a locally compact separable unimodular group. The general theory [**18**] assigns to G a measure space (Λ, μ) whose points λ index a family of unitary factor representations of G in such a way that if U_λ corresponds to λ and $U_\lambda(f) = \int_G f(x)U_\lambda(x)dx$ then

$$(1.1) \quad \int_G |f(x)|^2 dx = \int_\Lambda \operatorname{tr}\,(U_\lambda(f)^* U_\lambda(f))d\mu(\lambda)$$

for all $f \in L^1(G) \cap L^2(G)$. Here tr denotes the Murray-vonNeumann trace on the factor generated by the operators $U_\lambda(x)$, $x \in G$.

In the case when G is a group of type I the measure μ, called Plancherel measure, is unique, the U_λ are irreducible representations, and tr denotes the usual trace. The expression (1.1), which is called the *Plancherel Formula* for G, is proved in this case in [**4**, p. 328].

This paper, which is a continuation of [**17**], is concerned with the problem of sharpness in the Hausdorff Young inequality for the class of connected simply connected real nilpotent Lie groups. The inequality in question, stated for separable locally compact unimodular groups of type I is the assertion

$$(1.2) \quad \left\{\int_\Lambda ||U_\lambda(f)||_{p'}{}^{p'} d\mu(\lambda)\right\}^{1/p'} \leqq \left\{\int_G |f(x)|^p dx\right\}^{1/p}$$

for $f \in L^p(G)$ ([**9**; **8**]). Here of course $1 < p \leqq 2$, $1/p + 1/p' = 1$, and $||U_\lambda(f)||_{p'}{}^{p'} = \operatorname{tr}\,((U_\lambda(f)^* U_\lambda(f))^{p'/2})$. By rewriting (1.2) as $||\hat{f}||_{p'} \leqq ||f||_p$ and defining

$$||\mathscr{F}_p(G)|| = \sup_{||f||_p \leqq 1} ||\hat{f}||_{p'},$$

one can express the Hausdorff Young theorem for G by $||\mathscr{F}_p(G)|| \leq 1$.

The work in [17] made it plausible that

$$(1.3) \quad ||\mathscr{F}_p(G)|| < 1$$

for any connected, non-compact, locally compact unimodular group G. In fact, using the remarkable theorem of Babenko [**2**] : $||\mathscr{F}_p(\mathbf{R})|| < 1$ for $1 < p < 2$, it was shown in [**17**] that (1.3) holds whenever G contains the real line $\mathbf{R}$ as a direct factor or G contains $\mathbf{R}^n$, $n \geqq 1$, as a semi-direct factor with compact quotient.

According to a letter from J. Fournier (cf. [**10**]), (1.3) holds if and only if G

Received July 12, 1974 and in revised form, August 5, 1976.

https://doi.org/10.4153/CJM-1976-110-6 Published online by Cambridge University Press

has no compact open subgroups. This confirms a conjecture of the author [**17**]. However, Fournier's estimate, while universal, is very crude, on the order of .999999 . . . On the other hand, the estimates made in [**17**] are sharp enough to compute the number $||\mathscr{F}_p(G)||$ for the two classes of groups considered in [**17**].

Returning to the discussion of nilpotent groups, it follows from [**17**a, Proposition 13] that

$$(1.4) \qquad ||\mathscr{F}_p(G)|| \leq ||\mathscr{F}_p(\mathbf{R}^l)||, \quad 1 < p < 2$$

if G is a connected simply connected real nilpotent Lie group, where l is the dimension of the center of G and of course $l \geqq 1$. The present paper constitutes a step in the direction of the computation of $||\mathscr{F}_p(G)||$ in that (1.4) is improved, with one exception ($\Gamma_{5,4}$), for all of the connected, simply connected, real nilpotent Lie groups whose Plancherel measures are explicitly known (to the author). This includes all the (non-commutative) examples of dimension ≤ 5, denoted by Γ_3, Γ_4, $\Gamma_{5,1}$, $\Gamma_{5,2}$, $\Gamma_{5,3}$, $\Gamma_{5,4}$, $\Gamma_{5,5}$, $\Gamma_{5,6}$ in [**3**, III], all Heisenberg groups N_k, $k \geqq 1$, which are of dimensions $2k + 1$ ([**11**; **16**]) and the groups G_n, $n \geqq 3$ of real n by n triangular matrices with ones on the diagonal, which are of dimensions $\frac{1}{2}n(n-1)$ ([**3**, IV; **6**; **14**]).

The improved inequality is

$$(1.5) \qquad ||\mathscr{F}_r(G)|| \leq ||\mathscr{F}_r(\mathbf{R}^{p+q})||, \quad 1 \leq r \leq 2,$$

where p is the defect of commutativity of G, and $q = n - 2p$ where n is the dimension of G. This terminology is taken from [**3**, II].

Note that if G is commutative then $p = 0$, $n = q$, $G = \mathbf{R}^n$ and equality holds in (1.4) and (1.5). On the other hand, for G non-commutative, $p + q \geqq l$ (see Table I) and evidence seems to indicate that equality does not hold even in (1.5) (cf. [**17**a, Proposition 9]).

The techniques of [**17**] for semi-direct products rely heavily on the compactness of the quotient and hence do not apply to a nilpotent group, which can always be written as a semi-direct product of a normal nilpotent group of smaller dimension whose quotient group is the real line **R**. What works here for nilpotent groups and what is needed for other classes of groups is the explicit knowledge of the Plancherel measure for the group.

Except for Section 4, (Heisenberg groups) this paper depends heavily on the treatment of nilpotent groups given by Dixmier in [**3**, I–VI]. It seems possible that simplifications or improvements might be made using later treatments of the theory ([**6**; **14**; **12**]).

Future papers will deal with this problem for solvable groups (cf. [**1**; **15**; **5**]) semi-simple groups (cf. [**13**; **19**]) and non-unimodular groups (cf. [**7**]).

2. The three dimensional Heisenberg group. This group belongs to each of the classes of examples mentioned in Section 1, will be denoted by Γ_3 as in [**3**, III], and illustrates the method to be described in Section 3.

The underlying set of Γ_3 is $\mathbf{R}^3$ and the multiplication is given by $(\sigma_1, \sigma_2, \sigma_3)(\rho_1, \rho_2, \rho_3) = (\rho_1 + \sigma_2, \rho_2 + \sigma_2, \rho_3 + \sigma_3 - \rho_2\sigma_3)$ [**3**, III, p. 330]. According to [**3**, III, Proposition 3], for each $\lambda \neq 0$ in $\mathbf{R}$ there is an irreducible unitary representation U_λ of Γ_3 on $L^2(\mathbf{R})$ given by

$$(2.1) \qquad (U_\lambda(\gamma)f)(\theta) = \exp i\lambda(\rho_3 - \rho_2\theta)f(\theta + \rho_1)$$

for $\gamma = (\rho_1, \rho_2, \rho_3) \in \Gamma_3, f \in L^2(\mathbf{R}), \theta \in \mathbf{R}$; and

$$(2.2) \qquad \int_{\Gamma_3} |F(\gamma)|^2 d\gamma = c_3 \int_{\lambda \neq 0} ||U_\lambda(F)||_2{}^2 |\lambda| d\lambda \quad \text{for all } F \in L^1(\Gamma_3) \cap L^2(\Gamma_3).$$

Here $d\gamma$ denotes Lebesgue measure on $\mathbf{R}^3$, $U_\lambda(F) = \int_{\Gamma_3} F(\gamma) U_\lambda(\gamma) d\gamma$, and c_3 is a constant.

A routine calculation using (2.1) shows that $U_\lambda(F)$, $F \in C_c^\infty(\Gamma_3)$ is an integral operator on $L^2(\mathbf{R})$ with kernel k_λ given by

$$(2.3) \qquad \begin{aligned} k_\lambda(\rho_1, \theta) &= \int\int F(\rho_1 - \theta, \rho_2, \rho_3) \exp (i\lambda\rho_3 - i\lambda\rho_2\theta) d\rho_2 d\rho_3 \\ &= 2\pi \cdot F(\rho_1 - \theta, \cdot\,, \cdot)^\wedge(\lambda\theta, -\lambda) \end{aligned}$$

i.e.

$$(U_\lambda(F)f)(\theta) = \int k_\lambda(\rho_1, \theta) f(\rho_1) d\rho_1, \quad \text{a.e. } \theta.$$

In order to determine the constant c_3 we first rewrite (2.2) for suitable F in the form

$$(2.4) \qquad F(e) = c_3 \int_{\lambda \neq 0} \operatorname{tr} (U_\lambda(F)) |\lambda| d\lambda, \quad e = \text{identity element.}$$

Next, since $\operatorname{tr}(U_\lambda(F)) = \int_{\mathbf{R}} k_\lambda(\theta, \theta) d\theta$, (2.3) yields

$$(2.5) \qquad F(e) = c_3 \int_{\lambda \neq 0} \int_{\mathbf{R}} 2\pi F(0, \cdot\,, \cdot)^\wedge(\lambda\theta, -\lambda) d\theta |\lambda| d\lambda$$

and substituting $F(\gamma) = \exp(-\frac{1}{2}||\gamma||^2) = \exp(-\frac{1}{2}(\rho_1^2 + \rho_2^2 + \rho_3^2))$ into (2.5) yields

$$\begin{aligned} 1 &= c_3 \int_{\lambda \neq 0} \int_{\mathbf{R}} 2\pi F(0, \cdot\,, \cdot)^\wedge(\lambda, \theta) d\lambda d\theta \\ &= c_3 \cdot 2\pi \int_{\lambda \neq 0} \int_{\mathbf{R}} \exp(-\tfrac{1}{2}(\lambda^2 + \theta^2)) d\lambda d\theta = c_3 \cdot 2\pi \cdot (2\pi)^{1/2} (2\pi)^{1/2}. \end{aligned}$$

Hence $c_3 = (2\pi)^{-2}$.

https://doi.org/10.4153/CJM-1976-110-6 Published online by Cambridge University Press

By (1.2) and [**17**, Theorem 3] we obtain for $1 < p \leqq 2$,

$$(2.6)\quad \begin{aligned}(2\pi)^2\|\hat{F}\|_{p'}^{p'} &= \int_{\lambda\neq 0} \|U_\lambda(F)\|_{p'}^{p'}|\lambda|d\lambda \\ &\leqq \int_{\lambda\neq 0} (\|k_\lambda\|_{p,p'}^{p'}\|k_\lambda^*\|_{p,p'}^{p'})^{1/2}|\lambda|d\lambda \\ &\leqq \left\{\int_{\lambda\neq 0}\|k_\lambda\|_{p,p'}^{p'}|\lambda|d\lambda\right\}^{1/2}\left\{\int_{\lambda\neq 0}\|k_\lambda^*\|_{p,p'}^{p'}|\lambda|d\lambda\right\}^{1/2}.\end{aligned}$$

By definition,

$$\begin{aligned}\|k_\lambda\|_{p,p'}^{p'} &= \int\left\{\int|k_\lambda(\rho_1,\theta)|^p d\rho_1\right\}^{p'/p}d\theta \\ &= \int\left\{\int|F(\rho_1-\theta,\cdot,\cdot)^\wedge(\lambda\theta,-\lambda)|^p(2\pi)^p d\rho_1\right\}^{p'/p}d\theta.\end{aligned}$$

Hence

$$(2.7)\quad \begin{aligned}&\int_{\lambda\neq 0}\|k_\lambda\|_{p,p'}^{p'}|\lambda|d\lambda \\ &= (2\pi)^{p'}\int_{\lambda\neq 0}\int\left\{\int|F(\rho_1,\cdot,\cdot)^\wedge(\lambda\theta,-\lambda)|^p d\rho_1\right\}^{p'/p}d\theta|\lambda|d\lambda \\ &\leqq (2\pi)^{p'}\left\{\int\left\{\int_{\lambda\neq 0}\int|F(\rho_1,\cdot,\cdot)^\wedge(\lambda\theta,-\lambda)|^{p'}|\lambda|d\lambda d\theta\right\}^{p/p'}d\rho_1\right\}^{p'/p}\end{aligned}$$

(by Minkowski's integral inequality)

$$= (2\pi)^{p'}\left\{\int\left\{\int_{\lambda\neq 0}\int|F(\rho_1,\cdot,\cdot)^\wedge(\lambda,\theta)|^{p'}d\lambda d\theta\right\}^{p/p'}d\rho_1\right\}^{p'/p}$$

(by change of variables as $|\lambda| = |\text{Jacobian }(\lambda,\theta)\to(\lambda\theta,\lambda)|$)

$$\begin{aligned}&= (2\pi)^{p'+1}\left\{\int\|F(\rho_1,\cdot,\cdot)^\wedge\|_{p'}^{p}d\rho_1\right\}^{p'/p} \\ &\leqq (2\pi)^{p'+1}\|\mathscr{F}_p(\mathbf{R}^2)\|^{p'}\left\{\int\|F(\rho_1,\cdot,\cdot)\|_p^p d\rho_1\right\}^{p'/p}\end{aligned}$$

(by Hausdorff Young Theorem for $\mathbf{R}^2$)

$$\begin{aligned}&= (2\pi)^{p'+1}\|\mathscr{F}_p(\mathbf{R}^2)\|^{p'}\left\{\int(2\pi)^{-1}\int|F(\rho_1,\rho_2,\rho_3)|^p d\rho_2\,d\rho_3\,d\rho_1\right\}^{p'/p} \\ &= (2\pi)^2\|\mathscr{F}_p(\mathbf{R}^2)\|^{p'}\|F\|_p^{p'}.\end{aligned}$$

By exactly the same argument

$$(2.8)\quad \int_{\lambda\neq 0}\|k_\lambda^*\|_{p,p'}^{p'}|\lambda|d\lambda \leqq (2\pi)^2\|\mathscr{F}_p(\mathbf{R}^2)\|^{p'}\|F\|_p^{p'}.$$

Combining (2.6), (2.7) and (2.8) yields $\|\hat{F}\|_{p'} \leq \|\mathscr{F}_p(\mathbf{R}^2)\|\,\|F\|_p$, and since $C_c^\infty(\Gamma_3)$ is dense in $L^p(\Gamma_3)$ this proves:

https://doi.org/10.4153/CJM-1976-110-6 Published online by Cambridge University Press

PROPOSITION 1. *Let* Γ_3 *denote the unique (up to topological isomorphism) non-commutative connected simply connected real nilpotent Lie group of dimension* 3. *Then* $||\mathscr{F}_p(\Gamma_3)|| \leqq ||\mathscr{F}_p(\mathbf{R}^2)||$ *for all* p, $1 \leqq p \leqq 2$. *Hence* $||\mathscr{F}_p(\Gamma_3)|| < 1$ *for all* p, $1 < p < 2$.

3. The method of integral operators. Let Γ be a connected simply connected nilpotent real Lie group of dimension n. Write $n = 2p + q$ where p is the defect of commutativity of Γ and let $\mathscr{H} = L^2(\mathbf{R}^p)$. According to [**3**, II, Théorème 4] there is a (Zariski open) set $\Omega \subset \mathbf{R}^q$ and for each $\lambda = (\lambda_1, \ldots, \lambda_q) \in \Omega$ there is an irreducible unitary representation U_λ of Γ on $\mathscr{H}$ such that

$$(3.1) \qquad \int_\Gamma |f(\gamma)|^2 d\gamma = \int_\Omega ||U_\lambda(f)||_2^2 |F(\lambda_1, \ldots, \lambda_q)| d\lambda_1 \ldots d\lambda_q$$

for all $f \in L^1(\Gamma) \cap L^2(\Gamma)$, where F is a real valued rational function with no singularities on Ω. The underlying point set of Γ is $\mathbf{R}^n$ and it is important for us that the Haar measure $d\gamma$ be chosen specifically. For simplicity, we take $d\gamma$ to be n-dimensional Lebesgue measure (not normalized in any way). $C_c^\infty(\Gamma)$ will denote the infinitely differentiable functions on Γ with compact support.

For each $f \in C_c^\infty(\Gamma)$, $U_\lambda(f)$ is an integral operator on $L^2(\mathbf{R}^p)$ ([**14**, p. 108] or [**3**, *V*, Corollary 1]). Denote its kernel by $k_\lambda = k_\lambda(f) : \mathbf{R}^p \times \mathbf{R}^p \to \mathbf{C}$. Also write $f : \Gamma \to \mathbf{C}$ as $f(\rho, \sigma, \mu)$, $\rho, \sigma \in \mathbf{R}^p$, $\mu \in \mathbf{R}^q$.

PROPOSITION 2. *Let* Γ *be a connected simply connected nilpotent real Lie group. With the notation of this section suppose that for all* $f \in C_c^\infty(\Gamma)$

$$(3.2) \qquad k_\lambda(\rho, \theta) = (2\pi)^{(p+q)/2} f(\rho - \theta, \cdot, \cdot)\hat{}\,(T(\lambda, \theta)), \quad \rho, \theta \in \mathbf{R}^p, \lambda \in \mathbf{R}^q$$

where $T : \mathbf{R}^{q+p} \to \mathbf{R}^{q+p}$ *is a transformation with Jacobian* J_T *satisfying*

$$(3.3) \qquad |J_T(\lambda, \theta)| = (2\pi)^{p+q} |F(\lambda)|, \quad \lambda \in \mathbf{R}^q, \theta \in \mathbf{R}^p.$$

Then $||\mathscr{F}_r(\Gamma)|| \leqq ||\mathscr{F}_r(\mathbf{R}^{p+q})||$ *for all* r, $1 \leqq r \leqq 2$. *Hence* $||\mathscr{F}_r(\Gamma)|| < 1$ *for all* r, $1 < r < 2$.

The proof is exactly the same as for Proposition 1. The only point to remember is that application of Babenko's theorem to $\mathbf{R}^{p+q}$ requires the proper normalization, i.e.

$$||f(\rho, \cdot, \cdot)\hat{}\,||_{r'} \leqq ||\mathscr{F}_r(\mathbf{R}^{p+q})||\, ||f(\rho, \cdot, \cdot)||_r \quad \text{means}$$

$$\left[\int_{\mathbf{R}^{p+q}} |f(\rho, \cdot, \cdot)\hat{}\,|^{r'} (2\pi)^{-(p+q)/2} d\theta d\lambda\right]^{1/r'}$$

$$\leqq ||\mathscr{F}_r(\mathbf{R}^{p+q})|| \left[\int_{\mathbf{R}^{p+q}} |f(\rho, \sigma, \tau)|^r (2\pi)^{-(p+q)/2} d\sigma d\tau\right]^{1/r}.$$

PROPOSITION 3. *Let* Γ *be a connected simply connected real nilpotent Lie group of dimension* $\leqq 5$. *Then* $\|\mathscr{F}_r(\Gamma)\| < 1$ *for all* r, $1 < r < 2$. *Precisely,* $\|\mathscr{F}_r(\Gamma)\| \leqq \|\mathscr{F}_r(\mathbf{R}^{p+q})\|$ *if* $\Gamma \neq \Gamma_{5,4}$ *and* $\|\mathscr{F}_r(\Gamma_{5,4})\| \leqq \|\mathscr{F}_r(\mathbf{R}^2)\|$.

Proof. Up to a constant the Plancherel formula for each of these groups has been given in [**3**, III]. Table I summarizes the relevant information where we assume non-normalized n-dimensional Lebesgue measure on each group.

Table I

Γ	n	p	q	$\lvert F(\lambda)\rvert d\lambda$	dimension of center
Γ_3	3	1	1	$C_3\lvert\lambda\rvert d\lambda$	1
Γ_4	4	1	2	$C_4 d\lambda d\mu$	1
$\Gamma_{5,1}$	5	2	1	$C_{5,1}\lambda^2 d\lambda$	1
$\Gamma_{5,2}$	5	1	3	$C_{5,2} d\lambda d\mu d\nu$	2
$\Gamma_{5,3}$	5	2	1	$C_{5,3}\lambda^2 d\lambda$	1
$\Gamma_{5,4}$	5	1	3	$C_{5,4} d\lambda d\mu d\nu$	2
$\Gamma_{5,5}$	5	1	3	$C_{5,5}\lambda^{-2} d\lambda d\mu d\nu$	3
$\Gamma_{5,6}$	5	2	1	$C_{5,6}\lambda^2 d\lambda$	3

Table II

Γ	constant	$T(\lambda, \theta)$	$\lvert J_T\rvert$
Γ_3	$(2\pi)^{-2}$	$(\lambda\theta, -\lambda)$	$\lvert\lambda\rvert$
Γ_4	$\frac{1}{2}(2\pi)^{-3}$	$\left(\frac{\mu}{2\lambda} - \frac{1}{2}\lambda\theta^2, \lambda\theta, -\lambda\right)$	$\frac{1}{2}$
$\Gamma_{5,1}$	$(2\pi)^{-3}$	$(\lambda\theta_1, \lambda\theta_2, -\lambda)$	λ^2
$\Gamma_{5,2}$	$(2\pi)^{-4}$	$\left(\frac{\mu\nu}{\lambda^2+\mu^2} + \lambda\theta, -\frac{\lambda\nu}{\lambda^2+\mu^2} + \mu\theta, -\lambda, -\mu\right)$	1
$\Gamma_{5,3}$	$(2\pi)^{-3}$	$(\lambda\theta_2, \lambda\theta_1, -\lambda)$	λ^2
$\Gamma_{5,4}$	?	?	?
$\Gamma_{5,5}$	$\frac{1}{6}(2\pi)^{-4}$	$\left(\frac{\nu}{3\lambda^2} - \frac{\mu\theta}{2\lambda} + \frac{\theta^3\lambda}{6}, \frac{\mu}{2\lambda} - \frac{\lambda\theta^2}{2}, \lambda\theta, -\lambda\right)$	$(6\lambda^2)^{-1}$
$\Gamma_{5,6}$	$(2\pi)^{-3}$	$(-\frac{1}{2}\lambda\theta_1^2 + \lambda\theta_2, \lambda\theta_1, -\lambda)$	λ^2

Using the explicit formulas for the representations U_λ given in [**3**, III] we determine as in Section 2 the constants $c_3, c_4 \ldots c_{5,6}$ and the kernels k_λ of the integral operators $U_\lambda(f)$, $f \in C_c^\infty(\Gamma)$. These kernels are shown to satisfy the properties (3.2) and (3.3) of Proposition 2. The details were carried out in Section 2 for Γ_3.

This procedure can be carried out for each of the groups in question except for $\Gamma_{5,4}$ which resists calculation. The straight forward calculations are omitted and the results summarized in Table II. The validity of Proposition 3 for $\Gamma_{5,4}$ was shown in Section 1.

4. The Heisenberg groups. Let N_k be the $2k + 1$ dimensional Heisenberg group, i.e. N_k is the connected simply connected real nilpotent Lie group which can be characterized by the fact that its Lie algebra has a basis $\{x_1, \ldots, x_k, y_1, \ldots, y_k, z\}$ with $[x_j, y_j] = z$, $1 \leqq j \leqq k$ and all other brackets are zero.

The Plancherel formula for N_k is described as follows ([**11**; **16**]). In the notation of Section 3, $n = 2k + 1$, $p = k$, $q = 1$ and for each $\lambda \neq 0$ in $\mathbf{R}$ there is an irreducible unitary representation U_λ of N_k on $L^2(\mathbf{R}^k)$ satisfying

$$(4.1) \qquad \int_{N_k} |f(\gamma)|^2 d\gamma = c_{2k+1} \int_{\lambda \neq 0} ||U_\lambda(f)||_2^{\,2} |\lambda|^k d\lambda$$

for all $f \in L^1(N_k) \cap L^2(N_k)$. Again we take $d\gamma$ to be $(2k + 1)$ – dimensional Lebesgue measure and U_λ is given by

$$(4.2) \qquad (U_\lambda(\gamma_0)\varphi)(x) = \exp\,(i\lambda(u_0 - \langle y_0, x\rangle + \tfrac{1}{2}\langle x_0, y_0\rangle))\varphi(x - x_0)$$

where $\gamma_0 = (x_0, y_0, u_0)$, $x, x_0, y_0 \in \mathbf{R}^k$, $u_0 \in \mathbf{R}$, $\varphi \in L^2(\mathbf{R}^k)$, and $\langle , \rangle$ is the usual inner product in $\mathbf{R}^k$.

The usual calculation using (4.2) shows that $U_\lambda(f)$ is an integral operator with kernel k_λ given by

$$(4.3) \qquad k_\lambda(x', x) = (2\pi)^{(k+1)/2} f(x' - x, \cdot, \cdot)^\wedge(\tfrac{1}{2}\lambda(x + x'), -\lambda).$$

Note that this differs in form from (3.2) in that the argument depends on x, λ and x'. Thus we cannot apply Proposition 2 directly. However, exactly as in Section 2 we determine that $c_{2k+1} = (2\pi)^{-k-1}$. Then arguing as in Section 2 starting with (2.6):

$$\begin{aligned}
\int_{\lambda \neq 0} ||k_\lambda||_{r,r'}^{\,r'} |\lambda|^k d\lambda &= (2\pi)^{r'(k+1)/2} \\
&\times \int_{\lambda \neq 0} \int \left\{ \int |f(x' - x, \cdot, \cdot)^\wedge(\tfrac{1}{2}\lambda(x + x'), -\lambda)|^r dx' \right\}^{r'/r} dx |\lambda|^k d\lambda \\
&\leqq (2\pi)^{r'(k+1)/2} \\
&\times \left\{ \int \left\{ \int_{\lambda \neq 0} \int |f(-x', \cdot, \cdot)^\wedge(\tfrac{1}{2}\lambda(2x - x'), -\lambda)|^{r'} |\lambda|^k dx d\lambda \right\}^{r/r'} dx' \right\}^{r'/r} \\
(4.4) \qquad &= (2\pi)^{r'(k+1)/2} \left\{ \int \left\{ \int_{\lambda \neq 0} \int |f(-x', \cdot, \cdot)^\wedge(x, \lambda)|^{r'} dx d\lambda \right\}^{r/r'} dx' \right\}^{r'/r} \\
&\leqq (2\pi)^{r'(k+1)/2} (2\pi)^{(k+1)/2} ||\mathscr{F}_r(\mathbf{R}^{k+1})||^{r'} \left\{ \int ||f(-x', \cdot, \cdot)||_r^{\,r} dx' \right\}^{r'/r} \\
&= (2\pi)^{k+1} ||\mathscr{F}_r(\mathbf{R}^{k+1})||^{r'} ||f||_r^{\,r'}, \quad \text{and similarly}
\end{aligned}$$

$$(4.5) \qquad \int_{\lambda \neq 0} ||k_\lambda^*||_{r,r'}^{\,r'} |\lambda|^k d\lambda \leqq (2\pi)^{k+1} ||\mathscr{F}_r(\mathbf{R}^{k+1})||^{r'} ||f||_r^{\,r'}.$$

Finally, by (4.1), [**17**, Theorem 3], (4.4) and (4.5)

$$(2\pi)^{k+1}||\hat{f}||_{r'}^{r'} = \int_{\lambda\neq 0} ||U_\lambda(f)||_{r'}^{r'}|\lambda|^k d\lambda$$
$$\leqq \int_{\lambda\neq 0} (||k_\lambda||_{r,r'}^{r'}||k_\lambda^*||_{r,r'}^{r'})^{1/2}|\lambda|^k d\lambda$$
$$\leqq \left\{\int_{\lambda\neq 0} ||k_\lambda||_{r,r'}^{r'}|\lambda|^k d\lambda\right\}^{1/2}\left\{\int_{\lambda\neq 0} ||k_\lambda^*||_{r,r'}^{r'}|\lambda|^k d\lambda\right\}^{1/2}$$
$$\leqq (2\pi)^{k+1}||\mathscr{F}_r(\mathbf{R}^{k+1})||^{r'}||f||_r^{r'}.$$

This proves:

PROPOSITION 4. *Let N_k be the $(2k + 1)$-dimensional Heisenberg nilpotent group. Then $||\mathscr{F}_r(N_k)|| \leqq ||\mathscr{F}_r(\mathbf{R}^{k+1})||$ for all r, $1 \leqq r \leqq 2$. Hence $||\mathscr{F}_r(N_k)|| < 1$ for all r, $1 < r < 2$.*

5. Nilpotent groups of triangular matrices. For $n \geqq 3$ let G_n be the (connected simply connected nilpotent Lie) group of all n real matrices $x = (\xi_{jk})$ such that $\xi_{jk} = 0$ if $1 \leqq j < k \leqq n$, and $\xi_{jj} = 1$ for $1 \leqq j \leqq n$. M_n will denote the set of real n by n matrices, $\mathbf{R}^{j\times k}$ denotes the j by k real matrices, and E_n will denote the set $\{(\xi_{jk}) \in M_n : \xi_{jk} = 0 \text{ if } j + k \neq n + 1\}$.

The Plancherel formula for G_n is described as follows [**3**, IV]. First G_n is of dimension $\frac{1}{2}n(n - 1)$ and it is necessary to consider separately the cases of n even or odd.

Suppose $n = 2m$, $m \geqq 2$. Then each $x \in G_n$ has the form

$$(5.1) \qquad x = \begin{bmatrix} y & 0 \\ w & z \end{bmatrix}$$

with $y, z \in G_m$ and $w \in M_m$. The set Ω can be taken to be

$$\{e \in E_m : \epsilon_2\epsilon_3 \ldots \epsilon_m \neq 0\}$$

where

$$(5.2) \qquad e = \begin{bmatrix} & 0 & & & \epsilon_m \\ & & & \cdot & \\ & & \cdot & & \\ & \cdot & & & \\ & \epsilon_2 & & 0 & \\ \epsilon_1 & & & & \end{bmatrix}$$

defines $\epsilon_1, \cdots, \epsilon_m$. For each $e \in \Omega$ there is an irreducible unitary representation U_e of G_n on $\mathscr{H} = L^2(G_m \times G_m)$ (Lesbesgue measure) given by

$$(5.3) \qquad (U_e(x)f)(y', z') = f(y'y, z'z) \exp (i \operatorname{tr} (ez'wy^{-1}y'^{-1}))$$

https://doi.org/10.4153/CJM-1976-110-6 Published online by Cambridge University Press

for $x = \begin{bmatrix} y & 0 \\ w & z \end{bmatrix} \in G_n$, $y', z' \in G_m$, $f \in \mathscr{H}$, such that

$$(5.4) \quad \int_{G_n} |\Phi(x)|^2 dx = (2\pi)^{-m^2} \int_\Omega ||U_e(\Phi)||_2^2 \epsilon_2^2 \epsilon_3^4 \ldots \epsilon_m^{2(m-1)} de$$

for all $\Phi \in L^1(G_n) \cap L^2(G_n)$, where $\int_\Omega \ldots de$ denotes $\int \ldots d\epsilon_1 \ldots d\epsilon_m$.

This shows (in the notation of Section 3) that $p = m^2 - m$, $q = m$ (so $p + q = m^2$) and $F(e) = (2\pi)^{-m^3}\epsilon_2^2\epsilon_3^4 \ldots \epsilon_m^{2(m-1)} d\epsilon_1 \ldots d\epsilon_m$. Also (5.3) implies that $U_e(\Phi)$ is an integral operator on $L^2(\mathbf{R}^p)$ (identified with $\mathscr{H}$) with kernel k_e given by

$$(5.5) \quad k_e((y, z), (y', z')) = (2\pi)^{m^2/2}\Phi(y'^{-1}y, z'^{-1}z, \cdot)\hat{}(-y^{-1}ez').$$

A note of explanation might be in order here. The identification of $G_m \times G_m$ with $\mathbf{R}^p$ requires that the argument $-y^{-1}ez'$ in (5.5) be interpreted as a vector in $\mathbf{R}^{m^2}$ whose components are the entries of the matrix $-y^{-1}ez'$.

Suppose now that $n = 2m + 1$, $m \geqq 1$. Then each $x \in G_n$ has the form

$$(5.6) \quad x = \begin{bmatrix} y & 0 & 0 \\ u & 1 & 0 \\ w & v & z \end{bmatrix}$$

with $y, z \in G_m$, $w \in M_m$, $u \in \mathbf{R}^{1\times m}$, $v \in \mathbf{R}^{m\times 1}$. The set Ω can be taken to be $\{e \in E_m: \epsilon_1\epsilon_2 \ldots \epsilon_m \neq 0\}$ (see (5.2)). For each $e \in \Omega$ there is an irreducible unitary representation U_e of G_n on $\mathscr{H} = L^2(G_m \times G_{m+1})$ (Lebesgue measure) given by

$$(5.7) \quad (U_e(x)f)(y', z', v') = f(y'y, z'z, v' + z'v) \exp i \operatorname{tr} (e(v'u + z'w)y^{-1}y'^{-1})$$

for $x \in G_n$ (see (5.6)), $y', z' \in G_m$, $v' \in \mathbf{R}^{m\times 1}$, $f \in \mathscr{H}$; such that

$$(5.8) \quad \int_{G_n} |\Phi(x)|^2 dx = (2\pi)^{-m^2-m} \int_\Omega ||U_e(\Phi)||_2^2 |\epsilon_1\epsilon_2^3 \ldots \epsilon_m^{2m-1}| de$$

for all $\Phi \in L^1(G_n) \cap L^2(G_n)$, where $de = d\epsilon_1 \ldots d\epsilon_m$.

In this case (in the notation of Section 3) $p = m^2$, $q = m$, $p + q = m^2 + m$, $F(e) = (2\pi)^{-m^2-m}\epsilon_1\epsilon_2^3 \ldots \epsilon_m^{2m-1}\, de$, and $U_e(\Phi)$ is an integral operator on $L^2(\mathbf{R}^p)$ (identified with $\mathscr{H}$) with kernel k_e given by

$$(5.9) \quad k_e((y, z, v), (y', z', v')) \\ = (2\pi)^{(m^2+m)/2}\Phi(y'^{-1}y, z'^{-1}z, v - v', \cdot, \cdot)\hat{}(-y^{-1}ev' - y^{-1}ez')$$

with the same interpretation as in (5.5).

LEMMA. (i) *The Jacobian J_T of the transformation $T: \mathbf{R}^{m^2} \to \mathbf{R}^{m^2}$ determined by the correspondence*

$$(5.10) \quad G_m \times E_m \times G_m \ni (y', e, z') \to y'ez' \in M_m$$

satisfies

$$(5.11) \quad |J_T| = \epsilon_2^2\epsilon_3^4 \ldots \epsilon_m^{2(m-1)}$$

(ii) *The Jacobian* J_S *of the transformation* $S : \mathbf{R}^{m^2+m} \to \mathbf{R}^{m^2+m}$ *determined by the correspondence*

$$(5.12)\quad G_m \times E_m \times G_m \times \mathbf{R}^{m\times 1} \ni (y', e, z', v') \to (y'ez', y'ev') \in M_m \times \mathbf{R}^{m\times 1}$$

satisfies

$$(5.13)\quad |J_S| = |\epsilon_1\epsilon_2^3 \ldots \epsilon_m^{2m-1}|.$$

Proof. (i) is the content of [**3**, IV, Lemme 3] and (ii) follows from (i) since $J_S = J_T \cdot \det(y'e)$.

PROPOSITION 5. *Let* G_n *be the group of all real* n *by* n *lower triangular matrices with ones on the diagonal. Then* $||\mathscr{F}_r(G_n)|| \leq ||\mathscr{F}_r(\mathbf{R}^{p+q})||$ *for all* r, $1 \leqq r \leqq 2$. *Hence* $||\mathscr{F}_r(G_n)|| < 1$ *for all* r, $1 < r < 2$ *(*p *is the defect of commutativity of* G_n *and* $2p + q =$ *the dimension of* G_n*).*

Proof. If n is even, $n = 2m$ and Φ is continuous on G_n with compact support, then

$$(5.14)\quad (2\pi)^{m^2}||\hat{\Phi}||_{r'}^{r'} = \int_{E_m} ||U_e(\Phi)||_{r'}^{r'} F(e)de$$

(by (5.4))

$$\leqq \int_{E_m} (||k_e||_{r,r'}^{r'}||k_e^*||_{r,r'}^{r'})^{1/2}F(e)de$$

(by [**17**, Theorem 3])

$$\leqq \left\{\int_{E_m} ||k_e||_{r,r'}^{r'} F(e)de\right\}^{1/2}\left\{\int_{E_m} ||k_e^*||_{r,r'}^{r'} F(e)de\right\}^{1/2}.$$

$$(5.15)\quad \int_{E_m} ||k_e||_{r,r'}^{r'} F(e)de = (2\pi)^{m^2r'/2}\int_{E_m}\int\int \times\left[\int\int|\Phi(y'^{-1}y, z'^{-1}z, \cdot)^\wedge(-y^{-1}ez')|^r dydz\right]^{r'/r} dy'dz'F(e)de$$

(by (5.5))

$$\leqq (2\pi)^{m^2r'/2}\left[\int\int \times\left[\int_{E_m}\int\int|\Phi(y, z, \cdot)^\wedge(-y^{-1}y'^{-1}ez')|^{r'}dy'dz'F(e)de\right]^{r/r'} dydz\right]^{r'/r}$$

(by Minkowski's integral inequality and translations)

$$\leqq (2\pi)^{m^2(r'+1)/2}||F_r(\mathbf{R}^{m^2})||^{r'}\left[\int\int||\Phi(y, z, \cdot)||_r^r dydz\right]^{r'/r}$$

(by (5.11) and Hausdorff Young for $\mathbf{R}^{m^2}$)

$$= (2\pi)^{m^2(r'+1-r'/r)/2}||F_r(\mathbf{R}^{m^2})||\,||\Phi||_r^{r'} = (2\pi)^{m^2}||\mathscr{F}_r(\mathbf{R}^{m^2})||^{r'}||\Phi||_r^{r'}.$$

Similarly,

$$\int_{E_m} ||k_e^*||_{r,r'}{}^{r'} F(e)de \leqq (2\pi)^{m^2} ||\mathscr{F}_r(\mathbf{R}^{m^2})||^{r'} ||\Phi||_r{}^{r'}$$

so by (5.14) $||\hat{\Phi}||_{r'} \leqq ||\mathscr{F}_r(\mathbf{R}^{m^2})|| \, ||\Phi||_r$.

The proof for odd n uses (5.8), (5.9) and (5.13) is exactly the same way that (5.4), (5.5) and (5.11) were used above. We omit the details.

References

1. L. Auslander and B. Kostant, *Polarization and unitary representations of solvable Lie groups*, Invent. Math. *14* (1971), 255–354.

2. K. I. Babenko, *On an inequality in the theory of Fourier integrals*, Izv. Akad. Nauk SSSR, Ser. Mat. *25* (1961), 531–542 (Russian).

3. J. Dixmier, *Sur les representations unitaires des groupes de Lie nilpotents, I*, Amer. J. Math. *10* (1958), 160–170. *II*. Bull. Soc. Math. France *85* (1957), 325–388. *III*. Can. J. Math. *10* (1958), 321–348. *IV*. Can. J. Math. *11* (1959), 321–344. *V*. Bull. Soc. Math. France *87* (1959), 65–79. *VI*. Can. J. Math. *12* (1960), 342–352.

4. ——— *Les C*-algebres et leur representations* (Gauthier-Villars, Paris, 1964, Second Edition, 1969).

5. F. W. Keene, *Square integrable representations of Iwasawa subgroups of a semi-simple Lie group*, Thesis, University of California, Berkeley, 1974.

6. A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspekhi Matem. Nauk *106* (1962), 57–110 (Russian).

7. A. Kleppner and R. Lipsman, *The Plancherel formula for group extensions, I*, Ann. Scient. Ec. Norm. Sup., 4th series, *5* (1972), 459–516. *II*, same journal, *6* (1973), 103–132.

8. R. A. Kunze, *L_p Fourier transforms on locally compact unimodular groups*, Trans. Amer. Math. Soc. *89* (1958), 519–540.

9. R. L. Lipsman, *Non-abelian Fourier analysis*, Bull. Sci. Math. *98* (1974), 209–233.

10. J. J. F. Fournier, *Sharpness in the Hausdorff Young theorem on unimodular groups*, Notices Amer. Math. Soc. *22* (1975), A-486.

11. C. C. Moore, *Representations of solvable and nilpotent groups and harmonic analysis on nil and solvmanifolds*, 1972 Summer Institute on Harmonic Analysis on Homogeneous spaces, A.M.S.

12. C. C. Moore and J. A. Wolf, *Square integrable representations of nilpotent groups*, Trans. Amer. Math. Soc. *185* (1973), 445–462.

13. M. A. Naimark, *Linear representations of the Lorentz group*, (Macmillan, 1964).

14. L. Pukanszky, *Lecons sur les representations des groups* (Dunod, 1967).

15. ——— *Unitary representations of solvable Lie groups*, Ann. scient. Ec. Norm. Sup., 4th series, *4* (1971), 457–608.

16. S. R. Quint, *Representations of Lie groups*, Lecture notes, University of California, Berkeley, 1972.

17. B. Russo, *The norm of the L^p-Fourier transform on unimodular groups*, Trans. Amer. Math. Soc. *192* (1974), 293–305.

17a. ——— *On the Hausdorff Young theorem for integral operators*, to appear, Pacific J. Math.

18. I. E. Segal, *An extension of Plancherel's formula to separable unimodular groups*, Ann. of Math. *5* (1950), 272–292.

19. G. Warner, *Harmonic analysis on semisimple Lie groups I, II* (Springer Verlag, 1972).

University of California,
Irvine, California