BULL. AUSTRAL. MATH. SOC. 55M10, 55P20
VOL. 62 (2000) [407–416]

# EDWARDS-WALSH RESOLUTIONS OF COMPLEXES AND ABELIAN GROUPS

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We give a necessary and sufficient condition for the existence of an Edwards-Walsh resolution of a complex. Our theorem is an extension of Dydak-Walsh's theorem to all simplicial complexes of dimension $\geqslant n+2$. We also determine the structure of an Abelian group with the Edwards-Walsh condition, (which was introduced by Koyama and the author).

## 1. INTRODUCTION

We recall that the covering dimension $\dim X$ of a compactum $X$ is the smallest natural number $n$ such that there exists an $(n+1)$-fold covering by arbitrarily fine open sets. The characterisation of dimension in terms of mappings to spheres led to the cohomological characterisation of dimension under the assumption of finite-dimensionality of a space [**8**]. This characterisation was the point of departure for cohomological dimension theory. We give below the definition of cohomological dimension. The cohomological dimension $\text{c-dim}_G X$ of a compactum $X$ with coefficients in an Abelian group $G$ is the largest integer $n$ such that there exists a closed subset $A$ of $X$ with $H^n(X, A; G) \neq 0$, where $H^n(\ ;G)$ means the Čech cohomology with coefficients in $G$. Clearly, $\dim X \leqslant n$ implies that $\text{c-dim}_G X \leqslant n$ for all $G$. Alexandroff formulated the theory in his paper [**1**].

Recent progress in cohomological dimension theory follows from Edwards' theorem [**6**] (details can be found in [**13**]). The theorem is based on an excellent idea, which is the so-called *Edwards-Walsh modification.* An equivalent reformulation below caused the advances: associating to each simplicial complex $L$, a combinatorial resolution $\omega\colon \mathrm{EW}_G(L,n) \to |L|$ (see Definition 2.1 below) specified that $\text{c-dim}_G X \leqslant n$ if and only if for every simplicial complex $L$ and map $f\colon X \to L$, there exists an approximate lift $\tilde{f}\colon X \to \mathrm{EW}_G(L,n)$ of $f$; see [**5**]. Recent analysis of the theory led to a need for those resolutions for general groups. Dydak-Walsh [**5**, Theorem 3.1] stated a necessary and sufficient condition for the existence of an Edwards-Walsh resolution of an $(n+1)$-dimensional simplicial complex. They [**5**, Theorem 4.1] also analysed the modification

Received 14th February, 2000
The author was partially supported by the Ministry of Education, Science, Sports and Culture under Grant-in-Aid for Encouragement of Young Scientists, (No.11740035, 1999-2000).

and investigated a general property of an Abelian group $G$ that admits such a resolution of a complex.

Because of a difficulty, Koyama and the author [**11**] introduced a property of an Abelian group $G$ that induces the existence of an Edwards-Walsh resolution of a simplicial complex: an Abelian group $G$ has *property* (EW) provided that there exists a homomorphism $\alpha\colon \mathbf{Z} \to G$ such that

($\mathrm{EW}_1$) $\alpha \otimes \mathrm{id}\colon \mathbf{Z} \otimes G \to G \otimes G$ is an isomorphism, and
($\mathrm{EW}_2$) $\alpha^*\colon \mathrm{Hom}(G,G) \to \mathrm{Hom}(\mathbf{Z},G)$ is an isomorphism.

In Section 2, we give a necessary and sufficient condition for the existence of such a resolution for all simplicial complexes of dimension $\geqslant n+2$, that is, ($\mathrm{EW}_2$) is the necessary and sufficient condition. The groups $\mathbf{Z}$, $\mathbf{Z}/p$ and $\mathbf{Z}_{(p)}$ satisfy such a condition. As we have previously stated, property (EW) seems very strong to construct a resolution. However, the condition group-theoretically give us an interesting future. In Section 3, we see that the condition characterises the group of integers and the Bockstein groups except quasi-cyclic ones.

Throughout this paper, $\mathbf{Z}$ is the additive group of all integers and $\mathbf{Q}$ is the additive group of all rational numbers. $\mathbf{Z}_{(P)}$ is the ring of integers localised at a subset $P$ of $\mathcal{P} = \{\text{all prime numbers}\}$. We denote by $\mathbf{Z}/p$, $\mathbf{Z}/p^\infty$ and $\widehat{\mathbf{Z}}_p$ the cyclic group of order $p$, the quasi-cyclic group of type $p^\infty$ and the group of $p$-adic integers, respectively.

For a brief historical view of cohomological dimension theory, we refer the reader to [**2, 4, 9, 10**].

## 2. Edwards-Walsh resolutions of complexes

An important tool for characterising compacta $X$ with finite cohomological dimension with respect to $G$ is an Edwards-Walsh resolution $\omega\colon \mathrm{EW}_G(L,n) \to |L|$ of a simplicial complex $L$. For $G = \mathbf{Z}$, these resolutions were formulated in [**13**]. The relation of Edwards-Walsh resolutions to cohomological dimension theory and their existence for certain other groups were discussed in [**3**] and [**5**].

Definition 2.1: Let $G$ be an Abelian group and $L$ a simplicial complex. An *Edwards-Walsh resolution* of $L$ in the dimension $n$ is a pair $(\mathrm{EW}_G(L,n),\omega)$ consisting of a CW-complex $\mathrm{EW}_G(L,n)$ and a combinatorial map $\omega\colon \mathrm{EW}_G(L,n) \to |L|$ (that is, $\omega^{-1}(|L'|)$ is a subcomplex for each subcomplex $L'$ of $L$) such that

(i) $\omega^{-1}(|L^{(n)}|) = |L^{(n)}|$ and $\omega|_{|L^{(n)}|}$ is the identity map of $|L^{(n)}|$ onto itself,
(ii) for every simplex $\sigma$ of $L$ with $\dim\sigma > n$, the preimage $\omega^{-1}(\sigma)$ is an Eilenberg-MacLane complex of type $(\bigoplus G, n)$, where the sum here is finite, and

(iii) for every simplex $\sigma$ of $L$ with $\dim\sigma > n$, the inclusion $\omega^{-1}(\partial\sigma) \to \omega^{-1}(\sigma)$ induces an epimorphism $H^n(\omega^{-1}(\sigma);G) \to H^n(\omega^{-1}(\partial\sigma);G)$.

Dydak-Walsh established a property of $G$ that characterises those groups for which such resolutions exist for all $(n+1)$-dimensional simplicial complexes.

**THEOREM.** [**5**, Theorem 3.1] *Let $G$ be an Abelian group and $n \geqslant 1$. An Edwards-Walsh resolution $\omega\colon \mathrm{EW}_G(L,n) \to |L|$ exists for all simplicial complexes $L$ with $\dim L \leqslant n+1$ if and only if there exists an integer $m \geqslant 1$ and a homomorphism $\alpha\colon \mathbf{Z} \to G^m$ such that any homomorphism $\beta\colon \mathbf{Z} \to G$ factors as $\beta = \widetilde{\beta}\circ\alpha$ for some $\widetilde{\beta}\colon G^m \to G$.*

We extend the theorem above to all simplicial complexes of dimension $\geqslant n+2$. Before stating our theorem, we recall a proposition in [**11**].

**PROPOSITION 2.2.** *Let $\sigma$ be an $(n+2)$-simplex and $(K(G,n),S^n)$ a pair of an Eilenberg-MacLane complex of type $(G,n)$ and an $n$-dimensional sphere $S^n$ in $K(G,n)$. Let $E$ be the CW-complex obtained by replacing each $(n+1)$-face $\tau$ of $\partial\sigma$ by $(K(G,n),S^n)$ along $\partial\tau \cong S^n$. Then we have*

$$H_n(E) \approx (G/\operatorname{Im}\alpha) \oplus \underbrace{G \oplus \cdots \oplus G}_{n+2}$$

*and an exact sequence*

$$\mathbf{Z} \xrightarrow{\Delta_\alpha} \underbrace{G \oplus \cdots \oplus G}_{n+3} \xrightarrow{q} (G/\operatorname{Im}\alpha) \oplus \underbrace{G \oplus \cdots \oplus G}_{n+2} \longrightarrow 0,$$

*where $\alpha = \pi_n(S^n \hookrightarrow K(G,n))$ and $\Delta_\alpha$ and $q$ are given by*

$$\Delta_\alpha(j) = (\alpha(j), -\alpha(j), \dots, -\alpha(j))$$

*and*

$$q((g_0, g_1, \dots, g_{n+2})) = ([g_0], g_1 + g_0, \dots, g_{n+2} + g_0).$$

PROOF: We write $\partial\sigma$ as the union $\tau_0 \cup \tau_1 \cup \cdots \cup \tau_{n+2}$, where each $\tau_i$ is an $(n+1)$-face of $\sigma$. Then by the construction,

$$E = K(G_0,n) \cup K(G_1,n) \cup \cdots \cup K(G_{n+2},n)$$

and

$$K(G_i,n) \cap K(G_j,n) = \tau_i \cap \tau_j \text{ for each pair } i,j \in \{0,1,\dots,n+2\},$$

where $G_i = G$. We note by use of Mayer-Vietoris exact sequences that

$$\begin{aligned} H_n(K(G_1,n) \cup \cdots \cup K(G_{n+2},n)) &\approx H_n(K(G_1,n)) \oplus \cdots \oplus H_n(K(G_{n+2},n)) \\ &\approx G_1 \oplus \cdots \oplus G_{n+2}. \end{aligned}$$

Next, let us take the following Mayer-Vietoris sequence of the couple $\{K(G_0,n), K(G_1,n) \cup \cdots \cup K(G_{n+2},n)\}$:

$$\begin{aligned} H_n\Big(K(G_0,n) \cap \big(K(G_1,n) \cup \cdots \cup K(G_{n+2},n)\big)\Big) \\ \to H_n\big(K(G_0,n)\big) \oplus H_n\big(K(G_1,n) \cup \cdots \cup K(G_{n+2},n)\big) \\ \to H_n(E) \xrightarrow{\partial} H_{n-1}\Big(K(G_0,n) \cap \big(K(G_1,n) \cup \cdots \cup K(G_{n+2},n)\big)\Big) \to \cdots . \end{aligned}$$

Since $\partial\tau_0 = K(G_0,n) \cap \big(K(G_1,n) \cup \cdots \cup K(G_{n+2},n)\big)$, the sequence above can be reduced to the exact one:

$$\mathbf{Z} \xrightarrow{\Delta\alpha} G_0 \oplus \cdots \oplus G_{n+2} \to H_n(E) \to 0.$$

The homomorphism $q\colon G_0 \oplus G_1 \oplus \cdots \oplus G_{n+2} \to (G_0/\operatorname{Im}\alpha) \oplus G_1 \oplus \cdots \oplus G_{n+2}$ given by

$$q(g_0, g_1, \ldots, g_{n+2}) = \big([g_0], g_1 + g_0, \ldots, g_{n+2} + g_0\big)$$

clearly induces a homomorphism $\widetilde{q}\colon (G \oplus \cdots \oplus G)/\operatorname{Im}\Delta_\alpha \to (G/\operatorname{Im}\alpha) \oplus G \oplus \cdots \oplus G$, where $[g]$, $g \in G$, is the equivalence class of $g$ in $G/\operatorname{Im}\alpha$. Then we have easily that $\widetilde{q}$ is an isomorphism. □

**Proposition 2.3.** *Let $\alpha\colon \mathbf{Z} \to G$ be a homomorphism from the group of integers to an Abelian group $G$. Then the homomorphism $\alpha^*$: $\operatorname{Hom}(G,G) \to \operatorname{Hom}(\mathbf{Z},G)$ induced by $\alpha$ is a monomorphism if and only if $\operatorname{Hom}(G/\operatorname{Im}\alpha, G) = 0$.*

**Theorem 2.4.** *Let $\alpha\colon \mathbf{Z} \to G$ be a homomorphism from the group of integers to an Abelian group $G$. Then the following are equivalent:*

(1) *there exists an Edwards-Walsh resolution $\omega\colon \mathrm{EW}_G(L,n) \to |L|$ of each simplicial complex $L$ with $\dim L \geqslant n+2$ such that*
   - (iv) *the inclusion-induced homomorphism $\pi_n\big(\omega^{-1}(\partial\tau)\big) \to \pi_n\big(\omega^{-1}(\tau)\big)$ is $\alpha$ for each $(n+1)$-simplex $\tau$ of $L$, and*
   - (v) *the inclusion-induced homomorphism $\pi_n\big(\omega^{-1}(\partial\sigma)\big) \to \pi_n\big(\omega^{-1}(\sigma)\big)$ maps the subgroup $G/\operatorname{Im}\alpha$ to zero for any $(n+2)$-simplex $\sigma$ of $L$ (where if $n = 1$, we consider the Abelianisation of the fundamental groups),*

(2) *the homomorphism $\alpha^*$: $\operatorname{Hom}(G,G) \to \operatorname{Hom}(\mathbf{Z},G)$ induced by $\alpha$ is an isomorphism.*

Remark 2.5. The subgroup $G/\operatorname{Im}\alpha$ in condition (v) above depends upon the enumeration of $(n+1)$-faces of each $(n+2)$-simplex, since we calculate the group by Proposition 2.2. We also note that (v) is natural for constructing our desired resolution.

PROOF: We first establish the necessity of the group condition. Suppose that there exists an Edwards-Walsh resolution $\omega\colon \mathrm{EW}_G(\sigma,n) \to \sigma$ of an $(n+2)$-simplex $\sigma$ with (iv) and (v). By (iii) of Definition 2.1 and (iv), $\alpha^*$ is an epimorphism. To show that $\alpha^*$ is a monomorphism, it suffices to prove $\mathrm{Hom}(G/\mathrm{Im}\,\alpha, G) = 0$ by Proposition 2.3.

Let $\gamma \in \mathrm{Hom}(G,G)$ with $\gamma(\mathrm{Im}\,\alpha) = 0$.

Let $\tau_0,\ldots,\tau_{n+2}$ be all $(n+1)$-faces of $\sigma$ and $\omega^{-1}(\tau_k) = K(G_k,n)$, where $G_k = G$. We can suppose, if necessary by changing the enumeration of $\tau_i$, that the subgroup $G_0/\mathrm{Im}\,\alpha$ maps to zero in $\pi_n(\omega^{-1}(\sigma))$ by condition (v) and Proposition 2.2.

Choose a continuous map $f_\gamma\colon (K(G_0,n),\partial\tau_0) \to (K(G,n),*)$ which represents the homotopy class $\gamma$ with $\gamma(\mathrm{Im}\,\alpha) = 0$ [**14**, p.244, Theorem 7.2].

Extend the composite $T\circ\omega|_{\omega^{-1}(\sigma^{(n)})}\colon \omega^{-1}(\sigma^{(n)}) \to K(G,n)$ to the map $F\colon \omega^{-1}(\partial\sigma) \to K(G,n)$, where $T\colon \sigma^{(n)} \to K(G,n)$ is the constant map to $*$, defined by

$$F|_{K(G_k,n)} \text{ is the constant map to } * \text{ for } k = 1,\ldots,n+2,$$

and

$$F|_{K(G_0,n)} = f_\gamma.$$

Let $\widetilde{F}\colon \omega^{-1}(\sigma) \to K(G,n)$ be an extension of $F$ by (iii) of Definition 2.1. We note by (v) and the Hurewicz theorem that for each $g \in G$, $i_*\big(([g],0,\ldots,0)\big) = 0$ on the $n$-dimensional homology groups, where $i\colon \omega^{-1}(\partial\sigma) \hookrightarrow \omega^{-1}(\sigma)$.

$$\begin{array}{ccc} & G_0 \oplus G_1 \oplus \cdots \oplus G_{n+2} & \\ & \Big\downarrow{\scriptstyle q} & \\ H_n(\omega^{-1}(\partial\sigma)) \approx & G_0/\mathrm{Im}\,\alpha \oplus G_1 \oplus \cdots \oplus G_{n+2} & \xrightarrow{F_*} G \\ & \Big\downarrow{\scriptstyle i_*} & \\ & H_n(\omega^{-1}(\sigma)) & \end{array}$$

Therefore, for $g \in G_0 = G$, we have

$$\begin{aligned} 0 = \widetilde{F}_* \circ i_*\big(([g],0,\ldots,0)\big) &= F_*\big(([g],0,\ldots,0)\big) \\ &= F_* \circ q\big((g,-g,\ldots,-g)\big) \quad \text{by Proposition 2.2} \\ &= (f_\gamma)_*(g) + 0 \\ &= \gamma(g). \end{aligned}$$

This means that $\gamma$ is trivial. Therefore $\mathrm{Hom}(G/\mathrm{Im}\,\alpha, G) = 0$.

Conversely, we suppose that $\alpha^*$ is an isomorphism. The construction is similar to that in previous works [**5, 3, 11**], that is, our task is only to state the Fact below without condition ($EW_1$) in the Introduction. However, we again give a detailed proof for completeness.

We first consider the case $n > 1$ and $\dim L < \infty$. Proceed by induction on $m = \dim L$. If $m \leqslant n$, we define $\mathrm{EW}_G(L, n) = |L|$ and $\omega = \mathrm{id}_{|L|}$.

Suppose that $m = n+1$. Attaching via the identity map the mapping cylinder $M(\sigma)$ of the map $\partial\sigma \to K(G, n)$ induced by $\alpha$ on the subcomplex $\partial\sigma$ of $|L^{(n)}|$ for each $(n+1)$-simplex $\sigma$ of $L$, we have the CW-complex $\mathrm{EW}_G(L^{(n+1)}, n)$. The map $\omega$ is chosen so that $\omega(M(\sigma) \setminus \partial\sigma) \subseteq \sigma \setminus \partial\sigma$ and $\omega$ is an extension of the identity map $\mathrm{id}_{|L^{(n)}|}$. Conditions (i) and (ii) of Definition 2.1 and (iv) are trivial. Condition (iii) of 2.1 follows from the surjectiveness of $\alpha^*$ and the universal coefficient theorem for cohomology.

We next consider the case $m = n+2$. Suppose inductively that we have constructed the Edwards-Walsh resolution $\omega\colon \mathrm{EW}_G(L^{(n+1)}, n) \to |L^{(n+1)}|$ with condition (iv). Then we have the homology group $H_n(\omega^{-1}(\partial\sigma)) \approx (G/\operatorname{Im}\alpha) \oplus \underbrace{G \oplus \cdots \oplus G}_{n+2}$ by Proposition 2.2.

Since $\omega^{-1}(\partial\sigma)$ is simply connected, and $\widetilde{H}_k(\omega^{-1}(\partial\sigma))$ is trivial for $k \leqslant n-1$,

$$\pi_n(\omega^{-1}(\partial\sigma)) \approx (G/\operatorname{Im}\alpha) \oplus \left(\bigoplus_1^{n+2} G\right)$$

by the Hurewicz isomorphism theorem. Construct an Eilenberg-MacLane space of type $\left(\bigoplus_1^{n+2} G, n\right)$ from $\omega^{-1}(\partial\sigma)$ by attaching $(n+1)$-cells to kill the subgroup $G/\operatorname{Im}\alpha$, and next attaching cells of dimension $\geqslant n+2$ to kill higher dimensional homotopy groups. Moreover, extend the map $\omega$ such that the interior of each cell used to construct the Eilenberg-MacLane space is mapped into $\sigma \setminus \partial\sigma$. We use the same notation $\omega$ for the extension.

Conditions (i) and (ii) of Definition 2.1 have been built in. For checking condition (iii) of 2.1, we show that for every $(n+2)$-simplex $\sigma \in L$, each map $f\colon \omega^{-1}(\partial\sigma) \to K(G, n)$ extends over $\omega^{-1}(\sigma)$. By the construction,

$$\omega^{-1}(\sigma)^{(n+1)} = \omega^{-1}(\partial\sigma)^{(n+1)} \cup \bigcup_{\beta_i} B^{n+1},$$

where $\beta_i$ represents an element of $G/\operatorname{Im}\alpha$ in $\pi_n(\omega^{-1}(\partial\sigma))$. So, we have $f_*([\beta_i]) = 0$ in $\pi_n(K(G, n))$ by Proposition 2.3. Hence $f$ can be extended over $\omega^{-1}(\sigma)^{(n+1)}$. Therefore we have an extension of $f$ over $\omega^{-1}(\sigma)$ by the triviality of the higher homotopy groups of $K(G, n)$. Condition (v) is satisfied by the construction.

Finally we consider the case $m \geqslant n+3$. Suppose that we have constructed the Edwards-Walsh resolution $\omega\colon \mathrm{EW}_G(L^{(m-1)}, n) \to |L^{(m-1)}|$ with conditions (iv) and (v). Furthermore we assume that for $n+1 \leqslant \dim \tau = k \leqslant m-1$, $\omega^{-1}(\tau)$ is an Eilenberg-MacLane space of type $\left(\bigoplus_{1}^{{}_kC_{n+1}} G, n\right)$, where ${}_rC_s = \left(r!/s!(r-s)!\right)$. Then we can state the following:

FACT. $H_n\left(\omega^{-1}(\partial\sigma)\right) \approx \underbrace{G \oplus \cdots \oplus G}_{{}_mC_{n+1}}$ for any $m$-simplex $\sigma$ of $L$.

PROOF: For our purpose we show the statement for any face $\tau \preceq \sigma$ with $\dim \tau \geqslant n+3$.

Let $\dim \tau = n+3$. We write $\partial\tau$ as the union $\tau_0 \cup \tau_1 \cup \cdots \cup \tau_{n+3}$, where each $\tau_i$ is an $(n+2)$-face of $\tau$. Then we have the following Mayer-Vietoris exact sequence:

$$(*) \quad H_n\left(\omega^{-1}(\partial\tau_0)\right) \longrightarrow H_n\left(\omega^{-1}(\tau_0)\right) \oplus H_n\left(\omega^{-1}(\tau_1 \cup \cdots \cup \tau_{n+3})\right) \longrightarrow H_n\left(\omega^{-1}(\partial\tau)\right) \longrightarrow 0.$$

By $\operatorname{Hom}(G/\operatorname{Im}\alpha, G) = 0$ and algebraic calculations based on Proposition 2.2, the sequence can be easily reduced to the exact sequence:

$$G/\operatorname{Im}\alpha \oplus \underbrace{(G \oplus \cdots \oplus G)}_{n+2} \overset{(i,-j)}{\longrightarrow} \underbrace{(G \oplus \cdots \oplus G)}_{n+2} \oplus \underbrace{(G \oplus \cdots \oplus G)}_{{}_{n+3}C_{n+1}} \longrightarrow H_n\left(\omega^{-1}(\partial\tau)\right) \longrightarrow 0,$$

where homomorphisms $i$ and $j$ are defined by

$$i\left(([g_0], g_1, \ldots, g_{n+2})\right) = (g_1, \ldots, g_{n+2})$$

and

$$j\left(([g_0], g_1, \ldots, g_{n+2})\right) = (g_1, \ldots, g_{n+2}, 0, \ldots, 0).$$

Thus the exact sequence means that the statement is true for $\dim \tau = n+3$.

For $\dim \tau = n+k \leqslant m$ $(k \geqslant 3)$, we can easily show the following by double induction starting from the case above, using Mayer-Vietoris exact sequences: Let $\tau_0, \tau_1, \ldots, \tau_{n+k}$ be all $(n+k-1)$-faces of $\tau$. Then for $i \leqslant n+2$,

$$H_n\left(\omega^{-1}(\tau_1 \cup \cdots \cup \tau_i)\right) \approx \bigoplus_{1}^{{}_{n+k-1}C_{n+1}} G \oplus \cdots \oplus \bigoplus_{1}^{{}_{n+k-1-(i-1)}C_{n+1-(i-1)}} G,$$

and for $n+3 \leqslant j \leqslant n+k$,

$$H_n\left(\omega^{-1}(\tau_1 \cup \cdots \cup \tau_j)\right) \approx \bigoplus_{1}^{{}_{n+k-1}C_{n+1}} G \oplus \cdots \oplus \bigoplus_{1}^{{}_{k-2}C_0} G.$$

https://doi.org/10.1017/S000497270001892X Published online by Cambridge University Press

Furthermore, we state that the inclusion $\omega^{-1}(\partial\tau_0) \to \omega^{-1}(\tau_1 \cup \cdots \cup \tau_{n+k})$ induces the next homomorphism on the $n$-dimensional homology groups up to automorphisms:

$$(g_1, \ldots, g_{{}_{n+k-1}C_{n+1}}) \longmapsto (g_1, \ldots, g_{{}_{n+k-1}C_{n+1}}, 0, \ldots, 0).$$

Then we have, by the Mayer-Vietoris exact sequence $(*)$ in case of $\dim\tau = m$,

$$\begin{aligned} H_n(\omega^{-1}(\partial\tau)) &\approx \bigoplus_{1}^{{}_{n+k-1}C_{n+1}} G \oplus \cdots \oplus \bigoplus_{1}^{{}_{k-2}C_0} G \\ &\approx \bigoplus_{1}^{{}_{n+k}C_{n+1}} G. \end{aligned}$$

This completes the proof of the fact. ∎

Let us return to the construction. Recall that $m \geqslant n+3$. We have $\pi_n(\omega^{-1}(\partial\sigma)) \approx \underbrace{G \oplus \cdots \oplus G}_{{}_mC_{n+1}}$ for every $m$-simplex $\sigma$ of $L$ by the Fact and the Hurewicz isomorphism theorem. Hence construct an Edwards-Walsh resolution of $L$ by attaching cells of dimension greater than $n+1$ to $\omega^{-1}(\partial\sigma)$ for $\dim\sigma = m$, and extending the map $\omega$ such that the interior of new cell is mapped into $\sigma \setminus \partial\sigma$. The extending map satisfies the property:

$$\omega\Big(\mathrm{EW}_G(L^{(m)}, n) \setminus \mathrm{EW}_G(L^{(m-1)}, n)\Big) \subseteq |L^{(m)}| \setminus |L^{(m-1)}|.$$

Here we note that

$$(\star) \qquad \omega^{-1}(\partial\sigma)^{(n+1)} = \omega^{-1}(\sigma)^{(n+1)}$$

for any $m$-simplex $\sigma$ of $L$. Then conditions (i) and (ii) of Definition 2.1 for $L = L^{(m)}$ are easily seen to be true. Condition (iii) of 2.1 follows from $(\star)$ and properties of $K(G,n)$. Conditions (iv) and (v) are our inductive assumption.

If $\dim L = \infty$, by applying the previous construction inductively, we can have our desired Edwards-Walsh resolution.

In case $n = 1$, it suffices to apply an argument of the Abelianisation (for details, see [**5**], [**11**,Theorem 2.3]). ∎

REMARK. In works [**5, 11**], condition ($\mathrm{EW}_1$), which appeared in the Introduction, was essentially used to show the Fact above.

The groups $\mathbf{Z}$, $\mathbf{Z}/p$ and $\mathbf{Z}_{(p)}$ satisfy such a condition, that is, there are such resolutions with respect to the groups. (These are well-known, [**13, 5**] and [**2, 3**].)

EXAMPLE. If $G = \mathbf{Z}/p \oplus \mathbf{Z}_{(q)}$ or $\widehat{\mathbf{Z}}_p$, where $p \neq q$, then Edwards-Walsh resolutions $\omega\colon \mathrm{EW}_G(L, n) \to |L|$ exist for all $n$ and all simplicial complexes.

## 3. Property (EW) and Abelian groups

**Theorem 3.1.** *Let $G$ be an Abelian group with property* (EW). *Then the group is precisely either a cyclic group or a localisation of the integer group at some prime numbers.*

Remark. We note that if $G$ is either a cyclic group or a localisation of the integer group at some prime numbers, then $G$ has property (EW).

The following fact essentially comes from our previous paper [**11**]. We give a proof for completeness.

**Proposition 3.2.** *Let $G$ be an Abelian group with property* (EW). *Then we have the following properties.*

(i) *The group $G/\operatorname{Im}\alpha$ is a torsion group.*
(ii) *If $\operatorname{Im}\alpha$ is an infinite cyclic group, $G$ is torsion-free.*
(iii) *If $\operatorname{Im}\alpha$ is a finite cyclic group, $G = \operatorname{Im}\alpha$.*

Proof: (i) If $G/\operatorname{Im}\alpha$ has an element of infinite order, so does $G$. However, this is a contradiction by the triviality of $(G/\operatorname{Im}\alpha)\otimes G$, which follows by the surjectiveness of $(\mathrm{EW}_1)$.

(ii) Suppose that the group $G$ has an element of order $p$. Then the group has a direct summand $\mathbf{Z}/p^k$ for some $1 \leqslant k \leqslant \infty$ by [**12**, Corollary 3]. Since $(\operatorname{Im}\alpha)\cap G_p = \{0\}$ by the assumption of (ii), $G/\operatorname{Im}\alpha$ also contains $\mathbf{Z}/p^k$ as a direct summand. Therefore $\operatorname{Hom}(G/\operatorname{Im}\alpha, G)$ has a copy of the non-trivial group $\operatorname{Hom}(\mathbf{Z}/p^k, \mathbf{Z}/p^k)$. This is a contradiction by the injectiveness of $(\mathrm{EW}_2)$. It follows that $G$ is torsion free.

(iii) We note by $(\mathrm{EW}_1)$ that $\alpha$ induces an isomorphism $G \approx \mathbf{Z}\otimes G \approx G\otimes G$.

The hypothesis of (iii) means that $\operatorname{Im}\alpha = \mathbf{Z}/q$ for some positive integer $q$. Then we have

$$G \approx \mathbf{Z}\otimes G \approx (\operatorname{Im}\alpha)\otimes G \approx G/qG.$$

Namely, $G = G_q$. Furthermore the group $G$ is the direct sum of finite cyclic groups by [**7**, Theorem 61.3]. If $\operatorname{Im}\alpha \neq G$, then so is $G/\operatorname{Im}\alpha$.

Suppose $(G/\operatorname{Im}\alpha)_p$ is non-trivial. Then $G$ and $G/\operatorname{Im}\alpha$ contain $p^k$ and $p^l$ cyclic groups as direct summands, respectively. But this is a contradiction by $\operatorname{Hom}(G/\operatorname{Im}\alpha, G) = 0$. Therefore $\operatorname{Im}\alpha = G$. ∎

**Lemma 3.3.** *The group $G$ is divisible by a prime number $p$ from $\widetilde{P} = \{p : (G/\operatorname{Im}\alpha)_p \neq 0\}$.*

Proof: Let $p \in \widetilde{P}$. Then $G/\operatorname{Im}\alpha$ has a direct summand $\mathbf{Z}/p^k$ for some $1 \leqslant k \leqslant \infty$ by [**12**, Corollary 3]. It follows from the surjectiveness of $(\mathrm{EW}_1)$ that $\mathbf{Z}/p^k\otimes G = 0$. Thus $G = pG$. If $k = \infty$, use that $G$ is torsion free by Proposition 3.2 (iii) and (ii). ∎

https://doi.org/10.1017/S000497270001892X Published online by Cambridge University Press

PROOF OF THEOREM 3.1: If $G/\operatorname{Im}\alpha = 0$, the group $G$ is a cyclic group.

Let $G/\operatorname{Im}\alpha \neq 0$. Put $P = \mathcal{P}\setminus\widetilde{P}$, where $\mathcal{P}$ is the set of all primes. Then we define a function $f\colon \mathbf{Z}_{(P)} \to G$ by $f(n/m) = n\alpha(1)/m$. Here, we note that for each product $q$ of numbers from $\widetilde{P}$ and $g \in G$, there exists a unique element $z \in G$ such that $qz = g$ by Lemma 3.3 and Proposition 3.2 (iii) and (ii). We easily see that the function is an isomorphism. □

## REFERENCES


[1] P.S. Alexandroff, 'Dimensionstheorie, ein Beitrag zur Geometrie der abgesehlossenen Mengen', *Math. Ann.* **106** (1932), 161–238.

[2] A.N. Dranishnikov, 'Homological dimension theory', *Russian Math. Surveys* **43** (1988), 11–63.

[3] A.N. Dranishnikov, 'K-theory of Eilenberg-MacLane spaces and cell-like mapping problem', *Trans. Amer. Math. Soc.* **335** (1993), 91–103.

[4] J. Dydak, 'Cohomological dimension theory', in *Handbook of Geometric Topology*, 1997 (to appear).

[5] J. Dydak and J. Walsh, 'Complexes that arise in cohomological dimension theory: A unified approach', *J. London Math. Soc.* **48** (1993), 329–347.

[6] R.D. Edwards, 'A theorem and a question related to cohomological dimension and cell-like map', *Notices Amer. Math. Soc.* **25** (1978), A-259.

[7] L. Fuchs, *Infinite abelian groups* (Academic Press, New York, 1970).

[8] W. Hurewicz and H. Wallman, *Dimension theory* (Princeton University Press, Princeton, N.J., 1941).

[9] Y. Kodama, 'Cohomological dimension theory', in *Appendix to* K. Nagami, *Dimension theory* (Academic Press, New York, 1970).

[10] W.I. Kuzminov, 'Homological dimension theory', *Russian Math. Surveys* **23** (1968), 1–45.

[11] A. Koyama and K. Yokoi, 'Cohomological dimension and acyclic resolutions', *Topology Appl.* (to appear).

[12] T. Szele, 'On direct decompositions of abelian groups', *J. London Math. Soc.* **28** (1953), 247–250.

[13] J.J. Walsh, 'Dimension, cohomological dimension, and cell-like mappings', in *Shape theory and geometric topology (Dabrovnik, 1981)*, Lecture Notes in Math. **870** (Springer-Verlag, Berlin, Heidelberg, New York, 1981), **pp.** 105-118.

[14] G.W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Mathematics **61** (Springer-Verlag, Berlin, Heidelberg, New York, 1978).



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