Can. J. Math., Vol. XXIII, No. 2, 1971, pp. 345–354

ON MODULES OF SINGULAR SUBMODULE ZERO

VASILY C. CATEFORIS AND FRANCIS L. SANDOMIERSKI

Introduction. In this paper we generalize to modules of singular submodule zero over a ring of singular ideal zero some of the results, which are well known for torsion-free modules over a commutative integral domain, e.g. [**2**, Chapter VII, p. 127], or over a ring, which possesses a classical right quotient ring, e.g. [**13**, § 5].

Let R be an associative ring with 1 and let M be a unitary right R-module, the latter fact denoted by M_R. A submodule N_R of M_R is large in M_R (M_R is an essential extension of N_R) if N_R intersects non-trivially every non-zero submodule of M_R; the notation $N_R \subseteq' M_R$ is used for the statement "N_R is large in M_R". The singular submodule of M_R, denoted $Z(M_R)$, is then defined to be the set $\{m \in M \mid r(m) \subseteq' R_R\}$, where

$$r(m) \equiv \text{r. ann}_R m = \{x \in R \mid mx = 0\}.$$

The module M_R is said to be non-singular (or of singular submodule zero) if $Z(M_R) = (0)$. The ring R is right (left) non-singular according as R_R (${}_RR$) is a non-singular module.

The main tool in proving the results in this paper is the maximal right quotient ring Q of the ring R [**12**, § 4.3, p. 94] and as we deal with a right non-singular ring R, Q is the injective hull of R_R and a von Neumann regular ring, i.e. a ring every finitely generated ideal of which is a direct summand [**12**, § 4.5, p. 106].

As we deal with rings, which are right and left non-singular (this is not an assumption!) we say that a ring S containing a ring R (and sharing the identity of R), is a right (left) quotient ring of R if $R_R \subseteq' S_R$ (${}_RR \subseteq' {}_RS$).

Now let R be a right non-singular ring and let Q be its maximal right quotient ring. The main results of this paper are as follows.

In § 1 the condition *every finitely generated non-singular right R-module is torsionless* is shown to be equivalent to *Q is also a left quotient ring of R*. The condition *every non-singular right module is torsionless* is shown to be equivalent to *Q_R is torsionless*. There are non-singular rings, other than semi-simple artinian, which satisfy the last theorem.

In § 2, with the further assumptions that

(a) Q is also the maximal left quotient ring of R and

(b) both Q_R and ${}_RQ$ are flat modules, the condition *every finitely generated non-singular right (left) R-module is isomorphic to a submodule of a free right*

Received August 5, 1970.

(left) R-module is shown to be equivalent to the condition *the R-(Q-)module* $Q \otimes_R Q$ *is non-singular (as right (left) R-module (Q-module))*.

In § 3 the condition *every non-singular right R-module is projective* is shown to be equivalent to *R is a semi-hereditary right perfect ring and Q is also a left quotient ring of R*. A ring R satisfying either of the last two equivalent conditions is shown to be artinian, hereditary with a two-sided semi-simple artinian maximal quotient ring, and so in particular if one of the conditions above holds, so does the right (left) symmetric of the same, hence the theorem is two-sided.

All rings are assumed to be associative with identity 1 and all modules are assumed to be unitary. For any homological notions used in the following, the reader is referred to [**2**].

1. Torsionless among non-singular modules. Let R be a ring. A module M_R is torsionless if M_R can be embedded in a direct product of copies of the module R_R, equivalently, $\bigcap \ker f = (0)$ where the intersection is taken over all $f \in M^*$, $M^* = \mathrm{Hom}_R(M_R, R_R)$. For more details on the notion of torsionless see (e.g.) [**1**]. The main theorem in this section is the following.

THEOREM 1.1. *For any right non-singular ring R, with maximal right quotient Q, the following statements are equivalent:*

(a) *Every finitely generated non-singular module M_R is torsionless;*

(b) *Q is also a left quotient ring of R.*

We postpone the proof of Theorem 1.1 until several pertinent facts, some of interest in themselves, have been established.

As usual, if A is a non-empty subset of a module M_R, we set $\mathrm{r.\,ann}_R A = \{x \in R \mid Ax = 0\}$ and we abbreviate this to $r_R(A)$. In an appropriate setting, $l_R(A)$ is similarly defined.

LEMMA 1.2. *Let R be a right non-singular ring with maximal right quotient ring Q and let M_R be a torsionless submodule of Q_R. If $A = \{p \in Q \mid pM \subset R\}$ (a subset of Q_Q), then $r_Q(A) \cap M = (0)$.*

Proof. Every element f of $M^* = \mathrm{Hom}_R(M_R, R_R)$ can be extended to an element f' of $\mathrm{Hom}_R(Q_R, Q_R)$, since Q_R is injective. However, each element of $\mathrm{Hom}_R(Q_R, Q_R)$ is given as left multiplication by some element of Q (because, e.g., $\mathrm{Hom}_Q(Q_Q, Q_Q) = \mathrm{Hom}_R(Q_R, Q_R)$); thus there exists $q \in Q$ such that $f'(p) = qp$ for each $p \in Q$. Since f' extends f, we have $qM = f(M) \subset R$, and so $q \in A$. Now if $x \in r_Q(A) \cap M$, then for any $f \in M^*$ we have $f(x) = qx = 0$, thus $x \in \bigcap_{f \in M^*} \ker f = (0)$; hence $r_Q(A) \cap M = (0)$.

PROPOSITION 1.3. *Let R be a right non-singular ring with maximal right quotient ring Q and suppose that every finitely generated R-submodule of Q_R is torsionless. If M_R is any finitely generated submodule of Q_R and if $A = \{p \in Q \mid pM \subset R\}$, then $r_Q(A) = (0)$.*

Proof. Consider $x \in r_Q(A)$; let $M_R' = xR + M_R$ and $A' = \{p \in Q | pM' \subset R\}$. The module M_R' is a finitely generated submodule of Q_R, and hence torsionless by assumption. Since $A' \subseteq A$, we have $r_Q(A) \subset r_Q(A')$, and so in particular $x \in r_Q(A')$. Since $x \in M_R'$, we have $x \in r_Q(A') \cap M'$; thus $x = 0$ follows from Lemma 1.2; we have $r_Q(A) = (0)$.

COROLLARY 1. (Same assumptions as in Proposition 1.3.) *If ${}_RK$ and ${}_RL$ are left R-submodules of ${}_RQ$, such that $K \cap L = (0)$, then $QK \cap QL = (0)$.*

Proof. Consider $b \in QK \cap QL$; there exist elements $p_i, q_i \in Q$, $k_i \in K$, $l_i \in L$, $i = 1, \ldots, n$, such that $b = \sum_i p_i k_i = \sum_i q_i l_i$. Let $M_R = \sum p_i R + \sum q_i R$ and let $A = \{p \in Q | pM \subset R\} = \{p \in Q | pp_i, pq_i \in R, \text{ all } i\}$. It follows from Proposition 1.3 that $r_Q(A) = (0)$. Now $Ab = (0)$ since $Ab \subset K \cap L = (0)$, hence $b = 0$, and so $QK \cap QL = (0)$.

LEMMA 1.4. *If R is a right non-singular ring with maximal right quotient ring Q, then every finitely generated non-singular module M_R can be embedded in a finitely generated free right Q-module F_Q.*

Proof. This is [**3**, p. 42, Lemma 2.2].

COROLLARY 2. (R and Q as in Lemma 1.4.) *Every finitely generated non-singular module M_R can be embedded in a finite direct sum of finitely generated R-submodules of Q_R.*

Proof. Let $M_R = \sum_{i=1}^{t} m_i R$ be a finitely generated non-singular module and let $F_Q = Q^{(1)} \times \ldots \times Q^{(n)}$, where $Q^{(i)} = Q_Q$ for each i, be a free right Q-module such that $M_R \subset F_Q$ (the latter given by Lemma 1.4). For each i, $i = 1, \ldots, t$, there exist elements $q_{ij} \in Q$, $j = 1, \ldots, n$, such that $m_i = (q_{i1}, \ldots, q_{in})$. We see from this that

$$m_i R \subset (q_{i1}, \ldots, q_{in})R \subset q_{i1}R \oplus \ldots \oplus q_{in}R,$$

where q_{ij} is identified with $(0, \ldots, q_{ij}, \ldots, 0)$ in F_Q. Setting $A_j = \sum_{i=1}^{t} q_{ij}R$, we have $M_R \subset A_1 \oplus \ldots \oplus A_n$, with $A_i \subseteq Q_R$ for each i.

Proof of Theorem 1.1. (a) $\Rightarrow$ (b). By definition of essential extension, it suffices to show that if ${}_RK \subset {}_RQ$ satisfies $K \cap R = (0)$, then $K = (0)$. Now (a), together with $Z(Q_R) = (0)$, implies that every finitely generated R-submodule of Q_R is torsionless; thus Corollary 1 yields $QK \cap QR = (0)$. However, $K \subset QK \cap QR$, and so ${}_RK = (0)$ whenever $K \cap R = (0)$; we have (b).

(b) $\Rightarrow$ (a). In view of Corollary 2, it is sufficient to show (a) in case M_R is a finitely generated R-submodule of Q_R. To this end, let $M_R = \sum_{i=1}^{n} q_i R \subset Q_R$ and let $A = \{r \in R | rq_i \in R, i = 1, \ldots, n\}$. Since ${}_RR \subseteq' {}_RQ$, it follows from [**4**, p. 242, Proposition 1.1 (vi)] that ${}_RA \subseteq' {}_RR$. Now using A as an indexing set, define

$$\phi: M_R \rightarrow \prod_{r \in A} R^{(r)}, \qquad R^{(r)} \equiv R_R,$$

https://doi.org/10.4153/CJM-1971-035-0 Published online by Cambridge University Press

by $\phi(m) = [rm| r \in A]$, for each $m \in M$. The map ϕ is clearly a homomorphism of right R-modules and $m \in \ker \phi$ if and only if $Am = 0$. However, ${}_RA \subseteq' {}_RR$ and $Am = 0$, $m \in Q$, implies that $m \in Z({}_RQ)$; also, $Z({}_RQ) = (0)$ since Q is von Neumann regular, and so $m = 0$. Thus $\ker \phi = (0)$ and ϕ is an embedding of R-modules.

This completes the proof of Theorem 1.1.

Remark 1. Right non-singular rings R over which the maximal right quotient ring Q is not also a left quotient ring exist [**5**], and so Theorem 1.1 is not "automatic" for non-singular rings as it is for commutative integral domains; see e.g. [**2**, p. 131, Proposition 2.4].

Wei has shown [**17**, p. 416, Proposition 7] that every non-singular module M_R can be embedded in a direct product of copies of the module Q_R, where Q is the maximal right quotient ring of a right non-singular ring R. Since a submodule of a torsionless module is clearly torsionless and a direct product of torsionless modules is a torsionless module, the following theorem is immediate.

THEOREM 1.5. *For a right non-singular ring R with maximal right quotient ring Q the following statements are equivalent:*

(a) *Every non-singular module M_R is torsionless;*

(b) *Q_R is torsionless.*

Remark 2. (1) In view of Theorem 1.1, if Q_R is torsionless, then Q is also a left quotient ring of R.

(2) Although the class of commutative integral domains that satisfies Theorem 1.5 coincides with the class of fields, among right non-singular rings there exist rings with no finiteness conditions, that satisfy the theorem. An example is the following.

Let F be a field and let Q be the (ring) (full) direct product

$$\prod_{n=1}^{\infty} F^{(n)}, \qquad F^{(n)} = F \text{ for each } n.$$

Let

$$R = \bigoplus_{n=1}^{\infty} F^{(n)} + 1 \cdot F \subset Q,$$

where 1 is the identity of Q. The ring Q is the maximal (two-sided) quotient ring of R and Q_R is torsionless.

To see the latter part of the last statement, observe that given any $0 \neq q \in Q$, there is

$$0 \neq x \in \mathrm{Soc}(R) = \mathrm{Soc}(Q) = \bigoplus_{n=1}^{\infty} F^{(n)}$$

such that $0 \neq xq \in R$ and the map $\boldsymbol{x^*}$, multiplication of elements of Q by x, is an element of $\mathrm{Hom}_R(Q_R, R_R)$.

(3) It is appropriate to mention here the class of rings, satisfying Theorem 1.5, determined by Colby and Rutter in [**7**]; these are the right non-singular, right QF-3 rings among the semi-primary ones.

2. Submodules of free modules among finitely generated non-singular modules. In this section we investigate the following condition:

(NF) Every finitely generated non-singular R-module is isomorphic to a submodule of a free R-module.

We say that a ring R has right (left) NF if the condition (NF) holds for right (left) R-modules.

THEOREM 2.1.† *Over a right non-singular ring R, whose maximal right quotient ring Q is also the maximal left quotient ring and such that the R-modules Q_R and ${}_RQ$ are flat, the following statements are equivalent:*

(a) *R has right* NF*;*

(b) *The singular submodule of $Q \otimes_R Q$ (as an R-right or Q-right or left module) is zero;*

(c) *R has left* NF.

We precede the proof of the theorem by Proposition 2.2. below, which lies in the heart of the matter. A module M_R is essentially finitely generated if M_R is an essential extension of a finitely generated submodule, e.g. [**3** or **4**].

PROPOSITION 2.2. *If R is a right non-singular ring with maximal right quotient ring Q and has the property that $(R:q) = \{x \in R \mid qx \in R\}$ is essentially finitely generated for every $q \in Q$, then every finitely generated left R-submodule of ${}_RQ$ is isomorphic to a submodule of a free left R-module.*

Proof. Let ${}_RA = Rq_1 + \ldots + Rq_n$, $q_i \in Q$. By [**3**, p. 40, Theorem 1.6(c)], ${}_RQ$ is flat, and so it follows from [**4**, p. 426, Theorem 2.1 (Remark (d′))] that $\bigcap_{i=1}^n (R:q_i)$ is essentially finitely generated. Thus there exist elements $u_1, \ldots, u_k$ in $\bigcap (R:q_i)$ such that $I = \sum u_iR \subseteq' \bigcap (R:q_i)$, and hence $I_R \subseteq' R_R$. Let $F = R^{(1)} \times \ldots \times R^{(k)}$, where $R^{(i)} = {}_RR$, $i = 1, \ldots, k$. Define $\phi: {}_RA \to {}_RF$ by $\phi(x) = (xu_1, \ldots, xu_k)$ for each $x \in A$. Clearly, ϕ is a homomorphism of left R-modules and $\phi(x) = 0$ implies $xu_i = 0$ for each i, so that $xI = 0$; since $I_R \subseteq' R_R$, this puts x in $Z(Q_R)$ but $Z(Q_R) = (0)$, and so ϕ is an embedding of ${}_RA$ into ${}_RF$.

Remark 3. Proposition 2.2, together with Corollary 2, yields the well-known (e.g. [**2**, p. 131, Proposition 2.4]) fact that a commutative integral domain R

†*Added in proof.* K. R. Goodearl in his paper *Embedding non-singular modules in free modules* (to appear in J. Pure Appl. Algebra) has established the following theorem, which makes our Theorem 2.1 inadequate:

If R is a ring with zero right singular ideal, then every finitely generated non-singular right R-module can be embedded in a free right R-module if and only if Q_R is flat and $(Q \otimes_R Q)_R$ is non-singular, where Q is the maximal right quotient ring of R.

has NF. It suffices to observe that for each $q \in Q$, Q the field of quotients of R, and any $0 \neq x \in (R{:}q)$ we have $xR \subseteq' (R{:}q)$; thus $(R{:}q)$ is essentially finitely generated for each $q \in Q$.

Proof of Theorem 2.1. (a) $\Rightarrow$ (b). Let M_R be a finitely generated non-singular module and let F_R be a free module such that $M_R \subset F_R$. Since $_RQ$ is flat, the sequence $(0) \to M \otimes Q \to F \otimes Q$ (tensor product over R), induced by the inclusion $M_R \subset F_R$, is exact; now $F \otimes Q$ is Q-isomorphic to the free Q-module F_Q, and so $Z((F \otimes Q)_R) = (0)$ as $Z(Q_R) = (0)$. It follows that $Z((M \otimes Q)_R) = (0)$, and so (b) follows from [**3**, p. 40, Theorem 1.6] and the fact that non-singularity of $Q \otimes_R Q$, say, as a right R-module is equivalent to the canonical map $\sum p_i \otimes q_i \to \sum p_i q_i$ from $Q \otimes_R Q$ to Q being an isomorphism of right R-modules (or Q-modules) and remains such as one of left modules; then note that $Z(_QQ) = (0)$.

(b) $\Rightarrow$ (c). In view of Corollary 2, it is sufficient to show property (NF) for finitely generated submodules of $_RQ$. However, this follows from Proposition 2.2, as condition (b), together with the fact that $_RQ$ is flat, implies that $(R{:}q) = \{x \in R \mid qx \in R\}$ is essentially finitely generated for every $q \in Q$, [**3**, p. 40, Theorem 1.6(c)].

Arguments symmetric to the ones given above establish that (c) $\Rightarrow$ (b) and (b) $\Rightarrow$ (a).

This completes the proof of Theorem 2.1.

Remark 4. The statement of Theorem 2.1 is admittedly cumbersome; there is some evidence that, perhaps, it cannot be improved.

(1) Any right self-injective ring R, i.e. a ring R such that R_R is injective, has right NF [**16**, p. 227, Theorem 2.7]. In particular, the maximal right quotient ring Q of a right non-singular ring R has right NF since Q_Q is injective [**12**, p. 107]. On the other hand, the ring R described in Remark 2 (2) has a two-sided maximal quotient ring Q such that Q_R and $_RQ$ are flat modules (R is von Neumann regular) but R does not have NF since $Z(Q \otimes_R Q) \neq (0)$ [**3**, p. 42, Remark 1].

(2) A ring R which has right NF but not left NF exists. It suffices to take R to be a right non-singular right but not left self-injective ring, e.g. [**14**]. Such a ring cannot have left NF, since if it did it would have to be a left self-injective ring as well; to see the last assertion consider the following lemma.

LEMMA 2.3. *If R is a right non-singular ring with the property that the maximal right quotient ring Q is also a left quotient ring of R and the maximal left quotient ring S is also a right quotient ring of R, then Q and S coincide up to a ring isomorphism, which extends the identity on R.*

Proof. By uniqueness of injective hull there exists a monomorphism $f{:}\ _RQ \to {_RS}$ such that $f(r) = r$ for all $r \in R$. To show the assertion of the lemma, it is sufficient to show that f is also a homomorphism of right R-modules

(f will remain a monomorphism, of course). To this end, let $q \in Q$ and $r \in R$ and let $s = f(q)r - f(qr) \in S$. The left ideal $I = \{x \in R | \, xq \in R\}$ is large in ${}_RR$, since ${}_RR \subseteq' {}_RQ$, and for any $x \in I$ we have: $xs = x(f(q)r) - xf(qr) = (xf(q))r - f(x(qr)) = f(xq)r - f((xq)r) = (xq)r - (xq)r = 0$, since $xqr \in R$. Thus $s \in Z({}_RS) = (0)$ or $f(q)r = f(qr)$ for all $q \in Q$, $r \in R$.

We note that a ring R such as we are discussing here has left modules, which are finitely generated non-singular but not torsionless; thus Theorem **1.1** is not two-sided.

(3) A ring R which has an artinian semi-simple maximal right quotient ring Q will have right NF if and only if ${}_RR \subseteq' {}_RQ$.

The direct product $R = \prod R_\alpha$ of a countably infinite collection of commutative integral domains $\{R_\alpha\}$ has a maximal (two-sided) quotient ring $Q = \prod Q_\alpha$, where each Q_α is the field of quotients of R_α; Q is also a classical quotient ring for R, and so the assumptions of Theorem 2.1 and condition (b) are satisfied. Q has no chain conditions.

3. The coincidence of the non-singular property with the projective property. Among commutative integral domains R, those over which torsion-free (in the classical sense) modules are projective, are fields, since, in particular, Q, the field of quotients of R, would be projective. Among right non-singular rings, the condition that every non-singular module be projective characterizes a class of rings which properly contains the class of semi-simple artinian rings. Theorem 3.1, the main result of this section, contains this characterization and the tool in establishing it is Chase's theorem [**6**, p. 467, Theorem 3.3] recorded below for easy reference.

A ring R is left (right) coherent if every finitely generated left (right) ideal of R is finitely related [**6**, p. 459].

THEOREM (Chase [**6**, p. 467, Theorem 3.3]). *For any ring R, the following statements are equivalent:*

(a) *The direct product of any family of projective right R-modules is projective;*

(b) *The direct product of any family of copies of R_R is a projective right R-module;*

(c) *R is right perfect and left coherent.*

THEOREM 3.1. *For any ring R, the following statements are equivalent:*

(a) $Z(R_R) = (0)$, *and every non-singular right R-module is projective;*

(b) *R is right perfect, right semi-hereditary, left coherent, and Q, the maximal right quotient ring, is also a left quotient ring of R;*

(a*) $Z({}_RR) = (0)$ *and every non-singular left R-module is projective;*

(b*) *R is left perfect, left semi-hereditary, right coherent, and S, the maximal left quotient ring, is also a right quotient ring of R.*

Proof. (a) $\Rightarrow$ (b). An arbitrary direct product of copies of R_R is a non-singular right R-module, and so projective by (a). That R is right perfect

https://doi.org/10.4153/CJM-1971-035-0 Published online by Cambridge University Press

and left coherent follows from Chase's theorem; R is right semi-hereditary (in fact right hereditary) since right ideals are non-singular right R-modules; ${}_RR \subseteq' {}_RQ$ follows, e.g., from Theorem 1.1.

(b) $\Rightarrow$ (a). $Z(R_R) = (0)$ since R is right semi-hereditary and $Z(R_R)$ contains no idempotents $\neq 0$. From Theorem 1.1 and ${}_RR \subseteq' {}_RQ$, it follows that a finitely generated non-singular module M_R is torsionless, hence projective from Theorem (C) (b) and the fact that R is right semi-hereditary [**2**, p. 15, Proposition 6.2]. In particular, Q_R is flat, being the direct limit of its finitely generated (non-singular) R-submodules, hence Q_R is projective since R is right perfect [**1**, p. 467, Theorem P(3)]. It follows (Theorem 1.5) that every non-singular right R-module is torsionless, hence projective since in fact R is right hereditary (every right ideal is flat and R is right perfect).

The equivalence of (a*) and (b*) is obtained by a symmetric argument and the equivalence of either (a) or (b) to either (a*) or (b*) is contained in the following proposition.

PROPOSITION 3.2. *For a ring R that satisfies either of the equivalent conditions* (a) *or* (b) *of Theorem* 3.1, *the following statements are true:*

(i) *The maximal right quotient ring Q of R is semi-simple artinian* (*hence also the maximal left quotient ring of R*);

(ii) *R is artinian and hereditary* (*on both sides*).

Proof. (i) Condition (a) implies that $Z(Q \otimes_R Q) = (0)$, e.g. [**3**, p. 43, Theorem 2.3] and since R is right hereditary, [**3**, p. 44, Theorem 2.5] shows that Q is semi-simple artinian. Since Q is left self-injective, the condition ${}_RR \subseteq' {}_RQ$ shows that Q is the maximal left quotient ring as well.

(ii) It follows from (i) and [**15**, p. 115, Theorem 1.6] that the Goldie dimensions $d(R_R)$, $d({}_RR)$ (e.g. [**15**]) are finite; since R is right hereditary, it follows from [**16**, p. 226, Corollary 2] that R is right noetherian. Thus the (Jacobson) radical J of R is nilpotent since it is nil, e.g. [**1**, p. 467, Theorem P]. Now a ring R with

(1) R/J artinian semi-simple,

(2) J nilpotent, and

(3) J finitely generated as a right ideal,

is easily shown to have a right R-module composition series, by the argument used to prove Hopkin's theorem, e.g. [**12**, p. 69, Corollary to Proposition 3] (it suffices to observe that J^k is finitely generated as a right ideal for every positive integer k).

To complete the proof of (ii), we show that R is left artinian as follows: R is left semi-hereditary by [**16**, p. 227, Corollary to Theorem 2.6], and so every left ideal is flat hence projective since R is clearly left perfect also (e.g. R is right artinian). Thus R is left hereditary and $d({}_RR) < \infty$, and so R is left noetherian [**16**, p. 226], hence left artinian.

This completes the proof of the proposition and also the proof of Theorem 3.1.

Remark 5. (1) In the language of QF-3 rings, Theorem 3.1 characterizes the right hereditary, right artinian, right QF-3 rings. Various definitions for "right QF-3" exist in the literature, e.g. [**8**; **9**; **7**]; since they are all equivalent over a right artinian ring [**8**, p. 345], the one immediately relevant here is that the injective hull of R_R be a projective module.

The structure of the right QF-3 rings of Theorem 3.1 has been completely determined by Harada [**9**; **10**; **11**].

It is appropriate to mention here that E. P. Armendariz (oral communication) has independently characterized the rings satisfying condition (a) of Theorem 3.1 as the right hereditary, right artinian, right QF-3 rings.

(2) The class of rings that satisfy Theorem 3.1 is properly contained in the class of hereditary artinian rings.

Let Δ be a division ring and let R be the subring of $M_3(\Delta)$, the ring of 3×3 matrices over Δ, consisting of the matrices of the form

$$\begin{pmatrix} a & 0 & x \\ 0 & a & y \\ 0 & 0 & z \end{pmatrix}, \qquad a, x, y, z \in \Delta.$$

We have the following facts about R:

(i) R is right and left artinian since it is a finite-dimensional (right and left) vector space over $\Delta \equiv \Delta \cdot 1$, $1 \in R$;

(ii) $Z(R_R) = Z({}_RR) = (0)$ and $Q = M_3(\Delta)$ is the maximal right quotient ring of R, not a left quotient ring of R [**5**, Theorem 3.4];

(iii) R is hereditary.

The easiest way to establish the last fact is to observe the following.

(1) A ring R is right (left) hereditary if and only if every large right (left) ideal of R is projective.

(2) Over any ring R, a simple right R-module is projective if and only if it is non-singular.

Now the ring R above has right socle $\mathrm{Soc}(R_R) = \mathrm{l.\,ann}_R J$, a maximal right ideal hence the only proper ($\neq R$) large right ideal; $\mathrm{Soc}(R_R)$ is projective by (2) and (ii), and so R is right hereditary by (1). Similarly, R is left hereditary.

The condition that every finitely generated non-singular module be projective has been dealt with in [**3**] and the theorem obtained there [**3**, p. 43, Theorem 2.3] is the following.

THEOREM 3.3. *For any ring R, the following statements are equivalent:*

(a) $Z(R_R) = (0)$ *and every finitely generated non-singular right R-module is projective;*

(b) *R is right semi-hereditary, Q_R is flat, and $Z(Q \otimes_R Q) = (0)$, where Q is the maximal right quotient ring of R.*

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University of Kentucky,
Lexington, Kentucky;
Kent State University,
Kent, Ohio

https://doi.org/10.4153/CJM-1971-035-0 Published online by Cambridge University Press