BULL. AUSTRAL. MATH. SOC. 16A76
VOL. 39 (1989) [171-175]

ON NEAR-RINGS IN WHICH THE CONSTANTS FORM AN IDEAL

PETER FUCHS

Let $\mathcal{C}$ denote the class of all near-rings which have the property that the subnear-ring of constants forms an ideal. Prominent examples are abstract affine near-rings and a generalisation of these by Feigelstock [**1**]. In this note we show $\mathcal{C}$ forms a variety and construct a proper sub-class $\bar{\mathcal{C}} \subset \mathcal{C}$ such that every $N \in \mathcal{C}$ can be embedded into some $\bar{N} \in \bar{\mathcal{C}}$. It turns out that near-rings $N \in \bar{\mathcal{C}}$ have an ideal structure which is similar to the ideal structure of abstract affine near-rings, in contrast to the situation for arbitrary elements of $\mathcal{C}$.

The following result which describes the arithmetic in near-rings $N \in \mathcal{C}$ is implicitly in Pilz [**2**, p.318]. The centre of a group G will be denoted by $Z(G)$.

PROPOSITION 1. *Let* $N \in \mathcal{C}$, $n_0, n_0', n_0'' \in N_0$, $n_c, n_c' \in N_c$. *Then:*

(1) $n_0 + n_c = n_c + n_0$;
(2) $N_0 N_c \subseteq Z(N_c)$;
(3) $n_0(n_0' + n_c) = n_0 n_0' + n_0 n_c$;
(4) $n_0(n_c + n_0' n_c') = n_0 n_c + n_0 n_0' n_c'$.

We now show that conditions (1), (3) in Proposition 1 already imply that $N \in \mathcal{C}$.

PROPOSITION 2. *For a near-ring* N *the following are equivalent:*

(1) $N \in \mathcal{C}$;
(2a) $\forall\, n_0 \in N_0 \quad \forall\, n_c \in N_c$, $n_0 + n_c = n_c + n_0$,
(2b) $\forall\, n_0, n_0' \in N_0 \quad \forall\, n_c \in N_c$, $n_0(n_0' + n_c) = n_0 n_0' + n_0 n_c$.

PROOF: (1) $\Rightarrow$ (2): by Proposition 1.

(2) $\Rightarrow$ (1): by (2a), $(N_c, +)$ is a normal subgroup of $(N, +)$ and since N_c is always right invariant it suffices to show that N_c is a left ideal. Let $n \in N$, $n' \in N$, $n = n_0 + n_c$, $n' = n_0' + n_c'$ and $\bar{n}_c \in N_c$. Then $(n_0 + n_c)(n_0' + n_c' + \bar{n}_c) - (n_0 + n_c)(n_0' + n_c') = n_0(n_0' + n_c' + \bar{n}_c) + n_c - (n_0(n_0' + n_c') + n_c) = n_0 n_0' + n_0(n_c' + \bar{n}_c) - n_0 n_c' - n_0 n_0' = n_0(n_c' + \bar{n}_c) - n_0 n_c' \in N_c$. ∎

Received 20 April, 1988

THEOREM 3. *$\mathcal{C}$ is a variety and therefore closed under the formation of subnear-rings, direct products and homomorphic images.*

PROOF: If $n \in N$ and $n = n_0 + n_c$, then $n_c = n0$ and $n_0 = n - n0$. Thus equations (2a) and (2b) in Proposition 2 are equivalent to:

(a') $(\forall\, n, n')(n - n0 + n'0 = n'0 + n - n0)$;

(b') $(\forall\, n, n', n'')((n - n0)(n' - n'0 + n''0) = (n - n0)(n' - n'0) + (n - n0)n''0)$.

The result now follows. ∎

It has already been shown that the class of abstract affine near-rings is a variety [**2**, p.316].

Let G be a group, G_1, G_2 normal subgroups of G such that $G_1 \oplus G_2 = G$ and let G_3 be a subgroup of $Z(G_2)$.

Define $R(G_1, G_2, G_3) = \{m \in M_0(G) \mid m(G_1) \subseteq G_1, m(G_2) \subseteq G_3$ & $(\forall\, g_1 \in G_1, g_2 \in G_2, g_3 \in G_3)\,(m(g_1 + g_2) = m(g_1) + m(g_2), m(g_2 + g_3) = m(g_2) + m(g_3))\}$ and $C(G_2) = \{m \in M_c(G) \mid (\exists\, \bar{g} \in G_2)(\forall\, g \in G)(m(g) = \bar{g})\}$.

Let $N = R(G_1, G_2, G_3) + C(G_2) = \{m_1 + m_2 \mid m_1 \in R(G_1, G_2, G_3)\,\&\, m_2 \in C(G_2)\}$.

We often write simply R, C if it is clear which parameters G_1, G_2, G_3 are meant.

PROPOSITION 4. *N is a subnear-ring of $M(G)$, $N_0 = R$, $N_c = C$ and N_c is an ideal in N.*

PROOF: It is easy to check that R is a subnear-ring of $M_0(G)$ and that C is a subnear-ring of $M_c(G)$. If $m \in R$, then $m(G_2) \subseteq G_3$, hence $RC \subseteq C$. Let m_1, $m_2 \in R$, $m_3 \in C$, $m_3(g) = \bar{g}$ for all $g \in G$ and let $g \in G$, $g = g_1 + g_2$. Then $(m_1 + m_3)(g) = m_1(g) + \bar{g} = m_1(g_1) + m_1(g_2) + \bar{g} = \bar{g} + m_1(g_1 + g_2) = (m_3 + m_1)(g)$, hence $m_1 + m_3 = m_3 + m_1$. Also $m_1(m_2(g) + m_3(g)) = m_1(m_2(g_1) + m_2(g_2) + \bar{g}) = m_1 m_2(g_1) + m_1(m_2(g_2) + \bar{g}) = m_1 m_2(g_1) + m_1 m_2(g_2) + m_1(\bar{g}) = m_1 m_2(g) + m_1 m_3(g)$, thus $m_1(m_2 + m_3) = m_1 m_2 + m_1 m_3$. Now let $n_1 = m_1 + m_1' \in R + C$ and $n_2 = m_2 + m_2' \in R + C$. Combining our results we get $n_1 - n_2 = m_1 - m_2 + m_1' - m_2' \in N$ and $n_1 n_2 = (m_1 + m_1')(m_2 + m_2') = m_1(m_2 + m_2') + m_1' = m_1 m_2 + m_1 m_2' + m_1' \in N$. Thus N is a subnear-ring of $M(G)$ and clearly $N_0 = R$, $N_c = C$. By Proposition 2, N_c is an ideal of N. ∎

Let $\bar{\mathcal{C}}$ denote the class of all near-rings of the form $N = R(G_1, G_2, G_3) + C(G_2)$.

We are now ready to state our main result.

THEOREM 5. *For a near-ring S, the following are equivalent:*

(1) *$S \in \mathcal{C}$;*

(2) *S can be embedded into some near-ring $N \in \bar{\mathcal{C}}$.*

PROOF: Clearly (2) implies (1), by Theorem 3 and Proposition 4.

Conversely, let $S \in \mathcal{C}$. Let $(G,+) = (S,+) \times (\mathbb{Z}_2,+)$, $(G_1,+) = (S_0,+) \times (\mathbb{Z}_2,+)$, $(G_2,+) = (S_c,+) \times \{0\}$ and $(G_3,+) = \left(\sum_{s_c \in S_c} S_0 s_c, +\right) \times \{0\}$. By Proposition 1 each subgroup $S_0 s_c$ is contained in $Z(S_c)$, hence G_3 is a subgroup of $Z(G_2)$. Define a map $\phi_0 \colon S_0 \to M_0(G)$, $\phi_0(s) = f_s$, by $f_s((s',z)) = (ss',0)$ if $s' \notin S_c$ or $z = 0$, $f_s((s',z)) = (s + ss', 0)$ if $s' \in S_c$ and $z = 1$.

Let $s_1, s_2 \in S_0$ and $s' \in S_c$. Then

$$\begin{aligned} f_{s_1+s_2}((s',1)) &= (s_1 + s_2 + (s_1+s_2)s', 0) = (s_1 + s_2 + s_1 s' + s_2 s', 0) \\ &= (s_1 + s_1 s' + s_2 + s_2 s', 0) = f_{s_1}((s',1)) + f_{s_2}((s',1)). \end{aligned}$$

Also $f_{s_1 s_2}((s',1)) = (s_1 s_2 + s_1 s_2 s', 0)$ and

$$\begin{aligned} f_{s_1} \circ f_{s_2}((s',1)) &= f_{s_1}(f_{s_2}(s',1)) = f_{s_1}(s_2 + s_2 s', 0) \\ &= (s_1(s_2 + s_2 s'), 0) = (s_1 s_2 + s_1 s_2 s', 0) \end{aligned}$$

by Proposition 1.

In a similar way we can prove that $f_{s_1+s_2}((s',z)) = f_{s_1}((s',z)) + f_{s_2}((s',z))$ and $f_{s_1 s_2}((s',z)) = f_{s_1} \circ f_{s_2}((s',z))$ for all $(s',z) \in S \times \mathbb{Z}_2$ where $s' \notin S_c$ or $z = 0$. Thus ϕ_0 is a homomorphism and ϕ_0 is also injective, since $s_1 \neq s_2$ implies

$$f_{s_1}((0,1)) = (s_1, 0) \neq (s_2, 0) = f_{s_2}((0,1)).$$

If $s \in S_0$ then clearly $f_s(G_1) \subseteq G_1$ and $f_s(G_2) \subseteq G_3$. Let $g_1 = (s',z) \in G_1$, $g_2 = (\bar{s},0) \in G_2$ and $g_3 = (s^\star, 0) \in G_3$. If $g_1 = (0,1)$ then $f_s(g_1 + g_2) = f_s((\bar{s},1)) = (s + s\bar{s}, 0) = (s,0) + (s\bar{s},0) = f_s((0,1)) + f_s((\bar{s},0))$. If $g_1 \neq (0,1)$ then either $z \neq 1$ or $s' + \bar{s} \notin S_c$, hence

$$f_s(g_1 + g_2) = f_s((s' + \bar{s}, z)) = (s(s' + \bar{s}), 0) = (ss', 0) + (s\bar{s}, 0) = f_s(g_1) + f_s(g_2).$$

By Proposition 1 and by induction it is easy to see that $s_1(s_2 + s_3) = s_1 s_2 + s_1 s_3$ for all $s_1 \in S_0$, $s_2 \in S_c$, $s_3 \in \sum_{s_c \in S_c} S_0 s_c$.

Thus $f_s(g_2 + g_3) = (s(\bar{s} + s^\star), 0) = (s\bar{s} + ss^\star, 0) = f_s(g_2) + f_s(g_3)$. We have shown that $\phi_0(S_0) \subseteq R(G_1, G_2, G_3)$.

Let $\phi_c \colon S_c \to M_c(G)$, $\phi_c(s) = m_s$, where $m_s \colon G \to G$, $m_s(g) = (s,0)$ for all $g \in G$. Evidently ϕ_c is an embedding and $\phi_c(S_c) = \{m \in M_c(G) \mid (\exists\, \bar{g} \in G_2)(\forall\, g \in G)\, (m(g) = \bar{g})\} = C(G_2)$.

Finally define $\phi \colon S \to R + C$, $\phi(s_0 + s_c) = \phi_0(s_0) + \phi_c(s_c)$. One can check that f is an embedding. ∎

In [1] Feigelstock generalised the notion of an abstract affine near-ring (a.a.n.r.). It is easy to see that these generalised abstract affine near-rings (g.a.a.n.r.) are just all near-rings N which have the property that N_c is an ideal of N, $(N_c, +)$ is abelian and $n_0(n_c + \bar{n}_c) = n_0 n_c + n_0 \bar{n}_c$ for all $n_0 \in N_0$, $n_c \in N_c$ and $\bar{n}_c \in N_c$. If $N \in \mathcal{C}$ and $N_0 N_c = N_c$ then it readily follows from Proposition 1 that N is a g.a.a.n.r. Feigelstock showed that if N is a g.a.a.n.r., then I is an ideal of N if and only if $I = I_0 + I_c$, where I_0 is an ideal of N_0 and $(I_c, +)$ is a subgroup of $(N_c, +)$ such that $I_0 N_c \subseteq I_c$ and $N_0 I_c \subseteq I_c$. This is well-known for a.a.n.r. For near-rings $N \in \overline{\mathcal{C}}$ we have a similar result.

THEOREM 6. *For a near-ring* $N \in \bar{\mathcal{C}}$, $N = R(G_1, G_2, G_3) + C(G_2)$ *the following are equivalent:*

(1) *I is an ideal of N;*
(2) *$I = I_0 + I_c$, where I_0 is an ideal of N_0 and $(I_c, +)$ is a normal subgroup of $(N_c, +)$ such that $I_0 N_c \subseteq I_c$ and $N_0 I_c \subseteq I_c$.*

PROOF: (1) $\Rightarrow$ (2): similar to the a.a.n.r. case.

(2) $\Rightarrow$(1): it follows readily from Proposition 1 that $(I, +)$ is a normal subgroup of $(N, +)$ and that I is right invariant. Let $i \in I$, $m, n \in N$, $i = i_0 + i_c$, $n = n_0 + n_c$, $m = m_0 + m_c$. Using Proposition 1 we get

$$(n_0 + n_c)(m_0 + m_c + i_0 + i_c) - (n_0 + n_c)(m_0 + m_c) = n_0(m_0 + i_0) - n_0 m_0 + n_0(m_c + i_c) - n_0 m_c.$$

We need to show that $j = n_0(m_c + i_c) - n_0 m_c \in I_c$. Let $C(G_3)$ denote the set of all $m \in N_c$ which map into G_3. Clearly $j \in C(G_3)$. If $i_c \in C(G_3)$, then

$$n_0(m_c + i_c) - n_0 m_c = n_0 m_c + n_0 i_c - n_0 m_c = n_0 i_c \in I_c.$$

Suppose that $i_c \notin C(G_3)$ and let $h \in G_3$. Define a function $f \in M_0(G)$ by $f(g_1 + g_2) = h$ if $g_1 \in G_1$, $g_2 \in G_2 \backslash G_3$ and $f(g) = 0$ otherwise. One checks that $f \in R(G_1, G_2, G_3)$ and that $f i_c(g) = h$ for all $g \in G$. Since $f i_c \in I_c$ we have $C(G_3) \subseteq I_c$. ∎

The following example however shows that Theorem 6 does not remain true in general for near-rings $N \in \mathcal{C}$ if $N_0 N_c \neq N_c$, not even if $(N, +)$ is abelian.

Example 7. Let $\mathbf{R}$, $\mathbf{Q}$, $\mathbf{Z}$ denote the sets of all reals, rationals and integers respectively. Let $G_1 = \{0\}$, $G_2 = \mathbf{R}$, $G_3 = \mathbf{Q}$ and $R = \{m \in M_0(\mathbf{R}) \mid m(\mathbf{R}) \subseteq \mathbf{Q} \;\&\; (\forall\, x_1 \in \mathbf{R}, x_2 \in \mathbf{Q})(m(x_1 + x_2) = m(x_1) + m(x_2))\}$.

By Proposition 4, $N = R + M_c(\mathbf{R})$ is a subnear-ring of $M(\mathbf{R})$ and $M_c(\mathbf{R})$ is an ideal of N.

Let $x \in \mathbf{R}\backslash\mathbf{Q}$ and let H denote the subgroup generated by $\{x\} \cup \mathbf{Z}$. Define $R' = \{m \in R \mid m(H) \subseteq \mathbf{Z}\}$. It is easy to see that $N' = R' + M_c(\mathbf{R})$ is a subnear-ring of N, hence, by Theorem 3, $M_c(\mathbf{R})$ is an ideal of N'. For each $c \in \mathbf{R}$ let m_c denote the function $m_c \colon \mathbf{R} \to \mathbf{R}$, $m_c(y) = c$ for all $y \in \mathbf{R}$. Define $I_0 = \{0\}$ and $I_c = \{m_c \mid c \in H\}$. Evidently I_c is a subgroup of $M_c(\mathbf{R})$ and $R'I_c \subseteq I_c$. Denote by $\bar{H}$ the subgroup generated by $H \cup \mathbf{Q}$ and let $\bar{x} \in \mathbf{R}\backslash\mathbf{Q}$, $\bar{x} \notin \bar{H}$. Let $\bar{q} \in \mathbf{Q}\backslash H$ and define a function $m \colon \mathbf{R} \to \mathbf{R}$ by $m(\bar{x} + q) = \bar{q}$ for all $q \in \mathbf{Q}$, $m(y) = 0$ otherwise. One can check that $m \in R'$.

Since $\bar{x} + x \notin \bar{x} + \mathbf{Q}$, $m(m_{\bar{x}} + m_x) - mm_{\bar{x}} = 0 - m_{\bar{q}} \notin I_c$, thus I_c is not an ideal of N'.

From the proof of Theorem 6 we get:

THEOREM 8. *Let $N \in \mathcal{C}$. Then I is an ideal of N if and only if $I = I_0 + I_c$, where I_0 is an ideal of N_0, I_c is an ideal of the N_0-group N_c and $I_0 N_c \subseteq I_c$.*

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Institut für Mathematik,
Johannes Kepler Universität Linz,
A-4040 Linz,
Austria.