3

Spacetime spinors

The notion of spinors arises naturally in the construction of a relativistic first-order equation for a quantum wave function – the so-called *Dirac equation*. Spinors are the most basic objects to which one can apply a Lorentz transformation. The seminal work in Penrose (1960) has shown that spinors constitute a powerful tool to analyse the structure of the Einstein field equations and their solutions. Most applications of spinors in general relativity make use not of the *Dirac spinors* but of the so-called *2-spinors*. The latter are more elementary objects, and indeed, the whole theory of the Dirac equation can be reformulated in terms of 2-spinors. In the sequel, 2-spinors will be very often simply called *spinors*.

The purpose of this chapter is to develop the basic formalism of spinors in a spacetime. Accordingly, one speaks of *spacetime spinors*, sometimes also called $SL(2,\mathbb{C})$ spinors; see, for example, Ashtekar (1991). A discussion of spinors in the presence of a singled-out timelike direction, the so-called *space spinor formalism*, is given in Chapter 4. One of the motivations for the use of spinors in general relativity is that they provide a simple representation of null vectors and of several tensorial operations. Although spinors will be used systematically in this book, they are not essential for the analysis. All the key arguments could be carried out in a tensorial way at the expense of lengthier and less transparent computations.

The presentation in this chapter differs sligthly in focus and content from that given in other texts; see, for example, Penrose and Rindler (1984); Stewart (1991); O'Donnell (2003). For reasons to be discussed in the main text, a systematic use of the so-called *Newman-Penrose formalism* will be avoided – although the basic notational conventions of Penrose and Rindler (1984), the authoritative work on the subject, are retained.

3.1 Algebra of 2-spinors

In what follows let $(\mathcal{M}, \boldsymbol{g})$ be a spacetime. The present discussion begins by analysing spinorial structures at a given point p of the spacetime manifold $\mathcal{M}$. The concept of a spinor is closely related to the representation theory of the group $SL(2,\mathbb{C})$. This group has two inequivalent representations in terms of two-dimensional complex vector spaces which are complex conjugates of each other; for a discussion of this aspect of the theory, see, for example, Carmeli (1977); Sexl and Urbantke (2000). Thus, the discussion of this chapter starts with a brief discussion of complex vector spaces.

3.1.1 Complex vector spaces

By a ***complex vector space*** it will be understood a vector space over the field of the complex numbers, $\mathbb{C}$. In what follows let $\mathfrak{S}$ denote a complex vector space, and let $\mathfrak{S}^*$ denote its dual, that is, the complex vector space of all linear maps from $\mathfrak{S}$ to $\mathbb{C}$. As in the case of real vector spaces, given $\boldsymbol{\varsigma} \in \mathfrak{S}$ and $\boldsymbol{\zeta} \in \mathfrak{S}^*$, the application of $\boldsymbol{\zeta}$ on $\boldsymbol{\varsigma}$ will be denoted by $\langle \boldsymbol{\zeta}, \boldsymbol{\varsigma} \rangle$. Notice, however, that in this case $\langle \boldsymbol{\zeta}, \boldsymbol{\varsigma} \rangle \in \mathbb{C}$.

Given $\mathfrak{S}$, it is natural to define an operation of complex conjugation over $\mathfrak{S}$: given $\boldsymbol{\varsigma} \in \mathfrak{S}$, its complex conjugate $\bar{\boldsymbol{\varsigma}}$ is defined via

$$\langle \boldsymbol{\zeta}, \bar{\boldsymbol{\varsigma}} \rangle \equiv \overline{\langle \boldsymbol{\zeta}, \boldsymbol{\varsigma} \rangle}, \qquad \boldsymbol{\zeta} \in \mathfrak{S}^*.$$

The operation of complex conjugation from $\mathfrak{S}$ to $\mathfrak{S}^*$ can be defined in an analogous way: given $\boldsymbol{\zeta} \in \mathfrak{S}^*$, its complex conjugate $\bar{\boldsymbol{\zeta}}$ satisfies

$$\langle \bar{\boldsymbol{\zeta}}, \boldsymbol{\varsigma} \rangle \equiv \overline{\langle \boldsymbol{\zeta}, \boldsymbol{\varsigma} \rangle}, \qquad \boldsymbol{\varsigma} \in \mathfrak{S}.$$

Given $\boldsymbol{\xi}$, $\boldsymbol{\zeta} \in \mathfrak{S}$ and $z \in \mathbb{C}$, the complex conjugate of the linear combination $\boldsymbol{\xi} + z\boldsymbol{\zeta}$ is $\bar{\boldsymbol{\xi}} + \bar{z}\bar{\boldsymbol{\zeta}}$. Thus, the operation of complex conjugation is not an isomorphism between $\mathfrak{S}$ and itself, but an *anti-isomorphism* between $\mathfrak{S}$ and the vector space $\bar{\mathfrak{S}}$, the ***complex conjugate*** of $\mathfrak{S}$. Similarly, the complex conjugation defines an anti-isomorphism between $\mathfrak{S}^*$ and the space, $\overline{\mathfrak{S}^*}$, the complex conjugate of $\mathfrak{S}^*$. If one considers the complex conjugate of the spaces $\overline{\mathfrak{S}}$ and $\overline{\mathfrak{S}^*}$, one recovers the spaces $\mathfrak{S}$ and $\mathfrak{S}^*$, respectively. Moreover, because of the way the complex conjugate operation has been defined, one has that $\overline{\mathfrak{S}^*} = \bar{\mathfrak{S}}^*$, so that $\overline{\mathfrak{S}}$ and $\overline{\mathfrak{S}^*}$ are duals of each other.

The vector spaces $\mathfrak{S}$, $\mathfrak{S}^*$, $\overline{\mathfrak{S}}$ and $\overline{\mathfrak{S}^*}$ will be regarded as the elementary building blocks in the construction of a spinorial formalism. As in the case of real vector spaces one can construct *higher rank* objects by considering arbitrary tensor products of these vector spaces. This will be discussed later in the chapter once further structure and an *abstract index notation* for spinors has been introduced.

3.1.2 Simplectic vector spaces

Key to the notion of spinors is the definition of a ***symplectic vector space***.

Definition 3.1 (***simplectic vector space***) *A* ***simplectic vector space*** *consists of an even-dimensional vector space* $\mathfrak{S}$ *endowed with a function* $[[\cdot,\cdot]] : \mathfrak{S} \times \mathfrak{S} \to \mathbb{C}$ *which is:*

(i) ***antisymmetric (skew)****; that is, given* $\boldsymbol{\xi}$, $\boldsymbol{\eta} \in \mathfrak{S}$

$$[[\boldsymbol{\xi}, \boldsymbol{\eta}]] = -[[\boldsymbol{\eta}, \boldsymbol{\xi}]]$$

(ii) ***bilinear****; that is,*

$$[[\boldsymbol{\xi} + z\boldsymbol{\zeta}, \boldsymbol{\eta}]] = [[\boldsymbol{\xi}, \boldsymbol{\eta}]] + z[[\boldsymbol{\zeta}, \boldsymbol{\eta}]], \qquad [[\boldsymbol{\xi}, \boldsymbol{\eta} + z\boldsymbol{\zeta}]] = [[\boldsymbol{\xi}, \boldsymbol{\eta}]] + z[[\boldsymbol{\xi}, \boldsymbol{\zeta}]]$$

(iii) ***non-degenerate****; that is, if* $[[\boldsymbol{\xi}, \boldsymbol{\eta}]] = 0$ *for all* $\boldsymbol{\eta}$ *then* $\boldsymbol{\xi} = 0$.

The antisymmetric product $[[\cdot,\cdot]]$ defines in a canonical way an isomorphism between $\mathfrak{S}$ and $\mathfrak{S}^*$: to $\boldsymbol{\xi} \in \mathfrak{S}$ one associates $\boldsymbol{\xi}^\flat \equiv [[\boldsymbol{\xi}, \cdot]] \in \mathfrak{S}^*$. A transformation $\mathbf{Q} : \mathfrak{S} \to \mathfrak{S}$ satisfying $[[\mathbf{Q}\boldsymbol{\xi}, \mathbf{Q}\boldsymbol{\eta}]] = [[\boldsymbol{\xi}, \boldsymbol{\eta}]]$ is called a ***symplectic transformation***.

Remark. The rest of this book will be concerned only with the case where the dimension of $\mathfrak{S}$ is 2.

3.1.3 Spin bases

From the definition of a symplectic vector space it follows directly that given non-zero $\boldsymbol{\xi}$, $\boldsymbol{\eta} \in \mathfrak{S}$ such that $[[\boldsymbol{\xi}, \boldsymbol{\eta}]] = 0$, there exists $z \in \mathbb{C}$, $z \neq 0$ such that $\boldsymbol{\xi} = z\boldsymbol{\eta}$. Alternatively, given $\boldsymbol{\xi}$, $\boldsymbol{\eta} \in \mathfrak{S}$, they are linearly independent if and only if $[[\boldsymbol{\xi}, \boldsymbol{\eta}]] \neq 0$. This observation leads to the idea of a ***spin basis***.

Definition 3.2 (***spin basis***) *Given non-zero* $\boldsymbol{o}$, $\boldsymbol{\iota} \in \mathfrak{S}$, *the pair* $\{\boldsymbol{o}, \boldsymbol{\iota}\}$ *is said to be a* ***spin basis*** *for* $\mathfrak{S}$ *if* $[[\boldsymbol{o}, \boldsymbol{\iota}]] = 1$.

Now, given $\boldsymbol{\xi} \in \mathfrak{S}$, the components of $\boldsymbol{\xi}$ with respect to the basis $\{\boldsymbol{o}, \boldsymbol{\iota}\}$ are defined by the equation

$$\boldsymbol{\xi} = \xi^{\mathbf{0}}\boldsymbol{o} + \xi^{\mathbf{1}}\boldsymbol{\iota},$$

where

$$\xi^{\mathbf{0}} \equiv [[\boldsymbol{\xi}, \boldsymbol{\iota}]], \qquad \xi^{\mathbf{1}} \equiv -[[\boldsymbol{\xi}, \boldsymbol{o}]].$$

3.1.4 Abstract index notation for spinors

The discussion of spinors in this book makes use of a combination of index-free and abstract index notations. Following the general discussion on abstract index notation given in Penrose and Rindler (1984), an element $\boldsymbol{\xi} \in \mathfrak{S}$ will also be denoted by ξ^A, where the abstract superindex ^{A} provides information about the vector space to which the object belongs – in this case $\mathfrak{S}$. Similarly given $\boldsymbol{\eta} \in \mathfrak{S}^*$,

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it will often be written as η_A. This notation of abstract sub- and superindices will also be extended to the vector spaces themselves; thus, the symbols $\mathfrak{S}^A$ and $\mathfrak{S}_A$ will be used, respectively, instead of $\mathfrak{S}$ and $\mathfrak{S}^*$. Furthermore, given $\xi^A \in \mathfrak{S}^A$, then ξ_A will denote $\boldsymbol{\xi}^\flat$, the dual of $\boldsymbol{\xi}$ under the antisymmetric product in $\mathfrak{S}$. Following this notation, the product $[[\boldsymbol{\eta}, \boldsymbol{\xi}]] = \langle \boldsymbol{\eta}^\sharp, \boldsymbol{\xi} \rangle$ will be written as $\eta_A \xi^A$.

In order to extend the formalism, one introduces an infinite number of copies (realisations) of the spaces $\mathfrak{S}$ and $\mathfrak{S}^*$: $\mathfrak{S}^A$, $\mathfrak{S}^B, \ldots$ and $\mathfrak{S}_A$, $\mathfrak{S}_B$, $\ldots$. The different realisations are connected to each other by a ***sameness map*** such that ξ^A and ξ^B correspond to two different copies of the same object $\boldsymbol{\xi}$ belonging to different realisations of $\mathfrak{S}$, that is, $\mathfrak{S}^A$ and $\mathfrak{S}^B$. A peculiarity of the abstract index notation is that although ξ^A and ξ^B describe the same object, expressions like $\xi^A = \xi^B$ are not allowed – *the indices in an equation must be balanced.*

Objects like ξ^A and η_B are called ***valence 1 spinors***. Following the terminology used for tensors, ξ^A is said to be ***contravariant***, while η_A is said to be ***covariant***. ***Higher valence spinors*** can be introduced using the tensorial product $\otimes$ of the basic vector spaces $\mathfrak{S}$ and $\mathfrak{S}^*$. The use of the abstract index notation simplifies the underlying discussion of these tensorial products. For example, a valence 3 spinor $\chi_{AB}{}^C$ is defined through a multilinear map $\boldsymbol{\chi} : \mathfrak{S}^A \times \mathfrak{S}^B \times \mathfrak{S}_C \to \mathbb{C}$. As a consequence of the $\mathfrak{S}$-linearity of this mapping, there exists a spinor $\chi_{AB}{}^C \in \mathfrak{S}_{AB}{}^C$. The space $\mathfrak{S}_{AB}{}^C$ is a vector space. This procedure extends in a natural way to higher valence spinors with arbitrary combinations of covariant and contravariant indices. The collection of all the spaces of the form $\mathfrak{S}_{A\cdots C}{}^{D\cdots F}$ is called the ***spin algebra*** and is denoted by $\mathfrak{S}^\bullet$. The spin algebra ensures that the multiplication of spinors renders a spinor. The operation of addition in $\mathfrak{S}^\bullet$ is defined only between spinors of the same type, that is, the same rank and same combination of covariant and contravariant indices.

3.1.5 The spinor ϵ_{AB}

As the antisymmetric 2-form $[[\cdot, \cdot]]$ is a function from $\mathfrak{S} \otimes \mathfrak{S}$ to $\mathbb{C}$, it follows that there exists a valence 2 spinor $\epsilon_{AB} \in \mathfrak{S}_{AB}$ such that

$$[[\boldsymbol{\xi}, \boldsymbol{\eta}]] = \epsilon_{AB} \xi^A \eta^B.$$

The spinor ϵ_{AB} is called the ***ϵ-spinor***. Now, as $[[\boldsymbol{\xi}, \boldsymbol{\eta}]] = -[[\boldsymbol{\eta}, \boldsymbol{\xi}]]$, it follows that $\epsilon_{AB} = -\epsilon_{BA}$; that is, ϵ_{AB} is *antisymmetric*. It has already been shown that $[[\boldsymbol{\xi}, \boldsymbol{\eta}]]$ can be written as $\xi_A \eta^A$; thus, it follows that

$$\xi_B = \epsilon_{AB} \xi^A = \xi^A \epsilon_{AB}. \tag{3.1}$$

That is, ϵ_{AB} can be regarded as an *index lowering object*. In other words, the spinor ϵ_{AB} provides a convenient way to express the duality between the spaces $\mathfrak{S}$ and $\mathfrak{S}^*$. This duality is a bijection, so that it follows that there must exist a further spinor, $(\epsilon^{-1})^{AB} \in \mathfrak{S}^{AB}$, by means of which one can raise back the index

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of the spinor ξ_A; that is, $\xi^A = (\epsilon^{-1})^{CA}\xi_C$. In order to simplify the appearance of the above expressions it is convenient to define a further spinor $\epsilon^{AB} \in \mathfrak{S}^{AB}$ via

$$\epsilon^{AB} \equiv -(\epsilon^{-1})^{AB}, \tag{3.2}$$

so that one obtains

$$\xi^A = -\epsilon^{CA}\xi_C. \tag{3.3}$$

Combining Equations (3.1) and (3.3) one obtains $\xi_B = -\epsilon_{AB}\epsilon^{CA}\xi_C$, which together with the requirement that ϵ_{AB} and $(\epsilon^{-1})^{AB}$ represent inverse operations, implies

$$\delta_B{}^C = -\epsilon_{AB}\epsilon^{CA},$$

with $\delta_B{}^C$ the ***two-dimensional Kronecker's delta***. The spinor ϵ^{AB} is also *antisymmetric*. This can be seen from

$$\begin{aligned}[[\boldsymbol{\xi},\boldsymbol{\eta}]] = \xi_B\eta^B = \xi_B\delta_C{}^B\eta^C = -\xi_B(\epsilon_{DC}\epsilon^{BD})\eta^C \\ = \epsilon^{BD}\xi_B(\epsilon_{CD}\eta^C) = \epsilon^{BD}\xi_B\eta_D.\end{aligned}$$

A similar computation shows that $[[\boldsymbol{\eta},\boldsymbol{\xi}]] = \epsilon^{DB}\eta_D\xi_B$. Finally, as $[[\boldsymbol{\xi},\boldsymbol{\eta}]] = -[[\boldsymbol{\eta},\boldsymbol{\xi}]]$ one concludes that $\epsilon^{AB} = -\epsilon^{BA}$ as claimed.

If $\epsilon^A{}_C$ and $\epsilon_A{}^C$ denote the spinors in $\mathfrak{S}^\bullet$ obtained by raising the first and second index of ϵ_{AB}, respectively, it follows from the above calculations that

$$\epsilon_C{}^A = -\epsilon^A{}_C = \delta_C{}^A, \qquad \epsilon_{AB}\epsilon^{AB} = \epsilon_A{}^A = 2.$$

The above formulae lead to the so-called ***see-saw rule***. Given a spinor $\chi^{P\cdots QA}$ one has that

$$\chi^{P\cdots QA} = \epsilon^{AB}\chi^{P\cdots Q}{}_B = -\chi^{P\cdots Q}{}_B\epsilon^{BA} = \chi^{P\cdots QB}\epsilon_B{}^A, \tag{3.4a}$$

$$\chi^{P\cdots Q}{}_A = -\epsilon_{AB}\chi^{P\cdots QB} = \chi^{P\cdots QB}\epsilon_{BA} = -\chi^{P\cdots Q}{}_B\epsilon^B{}_A. \tag{3.4b}$$

Comparing the above expressions one concludes that

$$\chi^{P\cdots Q}{}_A{}^A = -\chi^{P\cdots QA}{}_A.$$

3.1.6 The Jacobi identity and decompositions in irreducible components

As $\mathfrak{S}$ is a vector space of dimension 2, it follows that any antisymmetrisation over a set of three or more spinorial indices must vanish. In particular, one obtains what is known as the ***Jacobi identity***:

$$\epsilon_{A[B}\epsilon_{CD]} = \epsilon_{AB}\epsilon_{CD} + \epsilon_{AC}\epsilon_{DB} + \epsilon_{AD}\epsilon_{BC} = 0. \tag{3.5}$$

A direct consequence of the Jacobi identity is the following lemma:

Lemma 3.1 (***irreducible decomposition of a pair of indices***) *Consider the spinor* $\zeta_{\cdots AB\cdots}$. *Then*

$$\zeta_{\cdots AB\cdots} = \zeta_{\cdots(AB)\cdots} + \frac{1}{2}\epsilon_{AB}\zeta_{\cdots C}{}^{C}{}_{\cdots}.$$

Proof Consider the Jacobi identity rewritten in the form

$$\epsilon_A{}^C\epsilon_B{}^D - \epsilon_B{}^C\epsilon_A{}^D = \epsilon_{AB}\epsilon^{CD},$$

and multiply it by $\zeta_{\cdots CD\cdots}$. One readily obtains

$$2\zeta_{\cdots[AB]\cdots} = \epsilon_{AB}\zeta_{\cdots C}{}^{C}{}_{\cdots}.$$

Finally, combining the latter with the identity

$$\zeta_{\cdots AB\cdots} = \zeta_{\cdots(AB)\cdots} + \zeta_{\cdots[AB]\cdots},$$

one obtains the required result. □

The previous result can be used to interchange the order of two spinorial indices. In this case Lemma 3.1 directly yields

$$\zeta_{\cdots BA\cdots} = \zeta_{\cdots AB\cdots} - \epsilon_{AB}\zeta_{\cdots P}{}^{P}{}_{\cdots}. \tag{3.6}$$

The above lemma leads to the following result:

Proposition 3.1 (***irreducible decomposition of spinors***) *Any spinor* $\zeta_{A\cdots F}$ *can be decomposed as the sum of the spinor* $\zeta_{(A\cdots F)}$ *and products of* $\boldsymbol{\epsilon}$*-spinors with symmetrised contractions of* $\zeta_{A\cdots F}$.

Proof Assume $\zeta_{ABC\cdots F}$ to have valence n. In the following argument, the symbol $\sim$ between two spinors indicates that their difference is a linear combination of the outer product of $\boldsymbol{\epsilon}$-spinors and spinors of lower valence. The key idea of the decomposition is to show that

$$\zeta_{ABC\cdots EF} \sim \zeta_{(ABC\cdots EF)}.$$

To this end, one first notices that

$$n\zeta_{(ABC\cdots EF)} = \zeta_{A(BC\cdots EF)} + \zeta_{B(AC\cdots EF)} + \zeta_{C(AB\cdots EF)} + \cdots + \zeta_{F(AB\cdots E)}. \tag{3.7}$$

Now, one looks at the terms in the right-hand side of the above equation and considers the difference between the first and the second term, the first and the third term and so on. Using Lemma 3.1, these differences can be rewritten as

$$\begin{aligned}
\zeta_{A(BC\cdots EF)} - \zeta_{B(AC\cdots EF)} &= -\zeta^X{}_{(XC\cdots EF)}\epsilon_{AB},\\
\zeta_{A(BC\cdots EF)} - \zeta_{C(AB\cdots EF)} &= -\zeta^X{}_{(XB\cdots EF)}\epsilon_{AC},\\
&\vdots\\
\zeta_{A(BC\cdots EF)} - \zeta_{F(ABC\cdots E)} &= -\zeta^X{}_{(XBC\cdots E)}\epsilon_{AF}.
\end{aligned}$$

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The above expressions can be used in Equation (3.7) to eliminate the terms

$$\zeta_{B(AC\cdots EF)}, \qquad \zeta_{B(AC\cdots EF)}, \qquad \cdots \qquad \zeta_{F(ABC\cdots E)}.$$

One obtains

$$\zeta_{(ABC\cdots EF)} = \zeta_{A(BC\cdots EF)} + \frac{1}{n}\zeta^{X}{}_{(XC\cdots EF)}\epsilon_{AB} + \cdots + \frac{1}{n}\zeta^{X}{}_{(XBC\cdots E)}\epsilon_{AF}.$$

That is,

$$\zeta_{(ABC\cdots EF)} \sim \zeta_{A(BC\cdots EF)}.$$

The procedure described above can be repeated for each of the terms

$$\zeta^{X}{}_{(XC\cdots EF)}, \qquad \cdots \qquad \zeta^{X}{}_{(XB\cdots E)},$$

to obtain

$$\zeta_{(ABC\cdots EF)} \sim \zeta_{A(BC\cdots EF)} \sim \zeta_{AB(C\cdots EF)} \sim \cdots \sim \zeta_{ABC\cdots(EF)} \sim \zeta_{ABC\cdots EF}.$$

□

Remark. If one has a spinor with a set of contravariant indices, these can be lowered so that Proposition 3.1 applies.

The type of decompositions of spinors provided by Proposition 3.1 will be used systematically in the rest of the book. A particularly useful example is given by

$$\begin{aligned}\chi_{ABCD} = \chi_{(ABCD)} + \frac{1}{2}\chi_{(AB)P}{}^{P}\epsilon_{CD} + \frac{1}{2}\chi_{P}{}^{P}{}_{(CD)}\epsilon_{AB} + \frac{1}{4}\chi_{P}{}^{P}{}_{Q}{}^{Q}\epsilon_{AB}\epsilon_{CD}\\ + \frac{1}{2}\epsilon_{A(C}\chi_{D)B} + \frac{1}{2}\epsilon_{B(C}\chi_{D)A} - \frac{1}{3}\epsilon_{A(C}\epsilon_{D)B}\chi,\end{aligned} \tag{3.8}$$

with

$$\chi_{AB} \equiv \chi_{Q(AB)}{}^{Q}, \qquad \chi \equiv \chi_{PQ}{}^{PQ}.$$

A decomposition like the one given in Equation (3.8) will be called a ***decomposition in irreducible components***. The spinors $\chi_{(ABCD)}, \chi_{(AB)P}{}^{P}, \ldots, \chi$ are *independent* in the sense that $\chi_{ABCD} = 0$ if and only if

$$\chi_{(ABCD)} = 0, \quad \chi_{(AB)P}{}^{P} = 0, \quad \cdots \quad \chi = 0.$$

The latter fact will be used repeatedly in the following. Finally, it is observed that the number of independent components an arbitrary symmetric spinor can have is given by the following proposition; see Penrose and Rindler (1984).

Proposition 3.2 (***number of independent components***) *If $\zeta_{A\cdots C} = \zeta_{(A\cdots C)}$ is of valence p, then it has $(p+1)$ independent components.*

In conjunction with Proposition 3.1 the latter result can be used to count the total number of independent components of an arbitrary spinor.

3.1.7 Components with respect to a basis

As in the case of tensors, it is often convenient to discuss spinors in terms of a specific basis. To express this idea, it is convenient to introduce bold indices $\boldsymbol{A}$, $\boldsymbol{B}, \ldots$ ranging over $\mathbf{0}$ and $\mathbf{1}$. Thus, $\xi^{\boldsymbol{A}}$ and $\eta_{\boldsymbol{A}}$ represent the components of ξ^A and η_B with respect to a specific basis. This idea extends in a natural way to higher valence spinors.

Given a spin basis $\{\boldsymbol{o}, \boldsymbol{\iota}\}$, one often requires a notation to describe the basis in a more systematic manner. This will be done by means of the symbol $\epsilon_{\boldsymbol{A}}{}^A$ where

$$\epsilon_{\mathbf{0}}{}^A \equiv o^A, \qquad \epsilon_{\mathbf{1}}{}^A \equiv \iota^A. \tag{3.9}$$

Similarly, the dual cobasis of $\epsilon_{\boldsymbol{A}}{}^A$ will be denoted collectively by $\epsilon^{\boldsymbol{A}}{}_A$. By definition one has that

$$\epsilon_{\boldsymbol{A}}{}^A \epsilon^{\boldsymbol{B}}{}_A = \delta_{\boldsymbol{A}}{}^{\boldsymbol{B}}.$$

It follows from Equation (3.9) and the previous condition that

$$\epsilon^{\mathbf{0}}{}_A = -\iota_A, \qquad \epsilon^{\mathbf{1}}{}_A = o_A.$$

Using this notation and given two spinors ξ^A and η_B, one can write

$$\xi^A = \xi^{\boldsymbol{A}} \epsilon_{\boldsymbol{A}}{}^A, \qquad \eta_B = \eta_{\boldsymbol{B}} \epsilon^{\boldsymbol{B}}{}_B,$$

where

$$\xi^{\boldsymbol{A}} \equiv \xi^A \epsilon^{\boldsymbol{A}}{}_A, \qquad \eta_{\boldsymbol{B}} \equiv \eta_B \epsilon_{\boldsymbol{B}}{}^B.$$

Hence

$$[[\eta, \xi]] = \eta_A \xi^A = \left(\eta_{\boldsymbol{P}} \epsilon^{\boldsymbol{P}}{}_A\right)\left(\xi^{\boldsymbol{Q}} \epsilon_{\boldsymbol{Q}}{}^A\right) = \eta_{\boldsymbol{P}} \xi^{\boldsymbol{P}}.$$

The components $\epsilon_{\boldsymbol{AB}}$ of the antisymmetric spinor ϵ_{AB} with respect to the basis $\epsilon_{\boldsymbol{A}}{}^A$ are given by

$$(\epsilon_{\boldsymbol{AB}}) \equiv \left(\epsilon_{AB} \epsilon_{\boldsymbol{A}}{}^A \epsilon_{\boldsymbol{B}}{}^B\right) = \begin{pmatrix} o_A o^A & o_A \iota^A \\ \iota_A o^A & \iota_A \iota^A \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.10}$$

Now, a direct computation shows that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence, consistent with Equation (3.2) one has that

$$(\epsilon^{\boldsymbol{AB}}) \equiv \left(\epsilon^{AB} \epsilon^{\boldsymbol{A}}{}_A \epsilon^{\boldsymbol{B}}{}_B\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

An alternative way of rewriting the previous discussion is

$$\delta_A{}^B = \epsilon_A{}^{\boldsymbol{A}} \epsilon_{\boldsymbol{A}}{}^B, \qquad \epsilon_{AB} = \epsilon_{\boldsymbol{AB}} \epsilon_A{}^{\boldsymbol{A}} \epsilon_B{}^{\boldsymbol{B}}, \qquad \epsilon^{AB} = \epsilon^{\boldsymbol{AB}} \epsilon_{\boldsymbol{A}}{}^A \epsilon_{\boldsymbol{B}}{}^B.$$

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From the latter it follows that

$$\delta_A{}^B = o_A \iota^B - \iota_A o^B, \tag{3.11a}$$
$$\epsilon_{AB} = o_A \iota_B - \iota_A o_B, \tag{3.11b}$$
$$\epsilon^{AB} = o^A \iota^B - \iota^A o^B. \tag{3.11c}$$

3.1.8 Complex conjugation of spinors

In order to relate spinors with tensors one has to consider the operation of complex conjugation discussed in Section 3.1.1. The convention to denote the operation of complex conjugation in the abstract index notation is to add a bar to the kernel symbol and a prime to each of the indices. For example, one has that

$$\overline{\zeta^A} = \bar{\zeta}^{A'} \in \mathfrak{S}^{A'}.$$

The operation of complex conjugation is *idempotent* – given $\boldsymbol{\zeta} \in \mathfrak{S}$, then $\bar{\bar{\boldsymbol{\zeta}}} = \boldsymbol{\zeta}$. Using abstract index notation one writes the latter as $\overline{\bar{\zeta}^{A'}} = \zeta^A$.

A spinor $\xi^{A\cdots CS'\cdots U'}{}_{D\cdots EW'\cdots Y'}$ with, say, p unprimed contravariant indices, r primed contravariant indices, q unprimed covariant indices and s primed covariant indices describes the most general type of spinors. It is obtained from the $\mathfrak{S}$-linear map

$$\boldsymbol{\xi} : \underbrace{\mathfrak{S}_A \times \cdots \times \mathfrak{S}_C}_{p \text{ times}} \times \underbrace{\mathfrak{S}_{S'} \times \cdots \times \mathfrak{S}_{U'}}_{r \text{ times}} \times \underbrace{\mathfrak{S}^D \times \cdots \times \mathfrak{S}^E}_{q \text{ times}} \times \underbrace{\mathfrak{S}^{W'} \times \cdots \times \mathfrak{S}^{Y'}}_{s \text{ times}} \to \mathbb{C}.$$

The algebra $\mathfrak{S}^\bullet$ is then extended to accommodate this more general type of spinors with unprimed and primed indices.

An important consequence of the fact that the spaces $\mathfrak{S}$ and $\bar{\mathfrak{S}}$ are not isomorphic is that it is not possible to single out 2-spinors which are intrinsically real or imaginary unless one assumes further structure on $\mathfrak{S}^\bullet$. From a notational point of view, as $\mathfrak{S}$ and $\bar{\mathfrak{S}}$ are not isomorphic, the relative position of primed and unprimed indices is irrelevant. Thus, one can write expressions like $\zeta_{AA'} = \zeta_{A'A}$. Notice, in contrast, that the reordering of groups of primed indices or groups of unprimed indices is not allowed unless the spinor possesses special symmetries.

The rules for the raising and lowering of indices of valence 1 spinors are extended to higher valence spinors in a natural way. Primed indices are raised and lowered using the spinors $\epsilon^{A'B'} \in \mathfrak{S}^{A'B'}$ and $\epsilon_{A'B'} \in \mathfrak{S}_{A'B'}$ which are related, respectively, to ϵ^{AB} and ϵ_{AB} by complex conjugation. That is,

$$\bar{\epsilon}_{A'B'} \equiv \overline{\epsilon_{AB}}, \qquad \bar{\epsilon}^{A'B'} \equiv \overline{\epsilon^{AB}}.$$

It is conventional to write $\epsilon_{A'B'}$, $\epsilon^{A'B'}$ instead of $\bar{\epsilon}_{A'B'}$ and $\bar{\epsilon}^{A'B'}$.

Finally, note that the discussion of Section 3.1.6 concerning the decomposition of spinors in irreducible components, and in particular Lemma 3.1 and Proposition 3.1, can be directly extended to the case of spinors containing primed

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indices or combinations of primed or unprimed indices. In particular, one has the following decomposition of a spinor with two unprimed and two primed indices:

$$\eta_{AA'BB'} = \eta_{(AB)(A'B')} + \frac{1}{2}\eta_P{}^P{}_{(A'B')}\epsilon_{AB} + \frac{1}{2}\eta_{(AB)Q'}{}^{Q'}\epsilon_{A'B'} + \frac{1}{4}\epsilon_{AB}\epsilon_{A'B'}\eta_Q{}^Q{}_{Q'}{}^{Q'}. \tag{3.12}$$

A particular case of the above decomposition is when $\zeta_{AA'BB'}$ is the spinorial counterpart of an antisymmetric rank-2 tensor $\zeta_{ab} = -\zeta_{ba}$. In this case one has that

$$\zeta_{AA'BB'} = \zeta_{AB}\epsilon_{A'B'} + \bar{\zeta}_{A'B'}\epsilon_{AB}, \tag{3.13}$$

where $\zeta_{AB} \equiv \frac{1}{2}\zeta_{AP'B}{}^{P'}$, and one has that $\zeta_{AB} = \zeta_{(AB)}$.

3.1.9 The relation between spinors and tensors

Spinors provide a simple representation of several tensorial operations. Although every four-dimensional tensor (***world tensor***) can be represented in terms of spinors, the converse is not true. There are spinors which admit no discussion in terms of tensors. This observation is based on the fact that 2-spinors are related to representations of the group of (2×2) complex matrices with unit determinant, $SL(2,\mathbb{C})$, while tensors are related to the *Lorentz group*. These groups are not isomorphic to each other. The group $SL(2,\mathbb{C})$ covers the Lorentz group in a $2:1$ way; see, for example, Carmeli (1977); Sexl and Urbantke (2000) for further discussions on this issue.

Hermitian spinors

The key property to relate 2-spinors to world tensors is *hermicity*. A spinor $\boldsymbol{\xi} \in \mathfrak{S}^\bullet$ is said to be ***Hermitian*** if and only if $\boldsymbol{\xi} = \bar{\boldsymbol{\xi}}$, that is, if the spinor is equal to its complex conjugate. For this to be the case, $\boldsymbol{\xi}$ needs to have the same number of unprimed and primed indices. By raising and lowering the indices as necessary one can, without loss of generality, assume that the spinor has the same number of unprimed and primed contravariant indices and the same number of unprimed and primed covariant indices, for example, $\xi_{AA'\cdots DD'}{}^{EE'\cdots HH'}$. In this case the hermicity condition reads

$$\xi_{AA'\cdots DD'}{}^{EE'\cdots HH'} = \bar{\xi}_{AA'\cdots DD'}{}^{EE'\cdots HH'},$$

where on the right-hand side it has been used that the position of primed and unprimed indices can be interchanged.

Consider now $\xi^{AA'} \in \mathfrak{S}^{AA'}$. If $\{\boldsymbol{o},\boldsymbol{\iota}\}$ and $\{\bar{\boldsymbol{o}},\bar{\boldsymbol{\iota}}\}$ are, respectively, spin bases of $\mathfrak{S}$ and $\bar{\mathfrak{S}}$, one can write

$$\xi^{AA'} = a o^A\bar{o}^{A'} + b\iota^A\bar{\iota}^{A'} + c o^A\bar{\iota}^{A'} + d\iota^A\bar{o}^{A'}, \tag{3.14}$$

for some $a,\ b,\ c,\ d \in \mathbb{C}$. In other words, a pair ${}^{AA'}$ of indices is associated to four complex components. If one assumes, in addition, $\xi^{AA'}$ to be Hermitian, then it follows that $a,\ b \in \mathbb{R}$ and $c = \bar{d}$. Thus, the hermicity condition reduces the number of independent components to four real ones. Consequently, one can think of the Hermitian spinor $\xi^{AA'} \in \mathfrak{S}^{AA'}$ as describing a four-dimensional vector (***world-vector***) ξ^a.

The argument described in the previous paragraph can be extended in a natural fashion to higher valence Hermitian spinors, $\xi_{AA'\cdots DD'}{}^{EE'\cdots HH'}$, so that one can regard each pair of unprimed-primed indices (i.e. ${}_{AA'}$, ${}^{EE'}, \cdots$) as associated to a tensorial index (i.e. ${}_a$, ${}^e, \cdots$).

In what follows let

$$g_{AA'BB'} \equiv \epsilon_{AB}\epsilon_{A'B'}. \tag{3.15}$$

A computation then shows that $\bar{g}_{AA'BB'} = g_{AA'BB'}$ and, in addition, that

$$\begin{aligned}
&g^{AA'BB'} = \epsilon^{AB}\epsilon^{A'B'}, \\
&g_{AA'BB'}g^{BB'CC'} = g_{AA'}{}^{CC'} \equiv \delta_A{}^C\delta_{A'}{}^{C'}, \\
&g_{AA'BB'}g^{AA'BB'} = 4, \\
&g_{AA'BB'} = g_{BB'AA'}.
\end{aligned}$$

Furthermore, given $v_{AA'} \in \mathfrak{S}_{AA'}$ it can be readily verified that

$$v_{AA'}g^{AA'BB'} = v^{BB'}, \qquad v^{AA'}g_{AA'BB'} = v_{BB'}.$$

Hence, the spinor $g_{AA'BB'}$ has all the properties of a spinorial counterpart of the metric tensor. These ideas will now be put in more precise terms.

The Infeld-van der Waerden symbols

In order to describe explicitly the correspondence between spinors and tensors at a point $p \in \mathcal{M}$, consider a basis $\{\boldsymbol{e_a}\} \subset T|_p(\mathcal{M})$ and let $g_{\boldsymbol{ab}} \equiv \boldsymbol{g}(\boldsymbol{e_a}, \boldsymbol{e_b})$ denote the components of the metric $\boldsymbol{g}$ with respect to this basis. Let also $\{\boldsymbol{\omega^a}\} \subset T^*|_p(\mathcal{M})$ denote the dual basis to $\{\boldsymbol{e_a}\}$ so that $\langle \boldsymbol{\omega^b}, \boldsymbol{e_a}\rangle = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}$. *It is conventional to assume that the basis is $\boldsymbol{g}$-orthogonal*; that is, $g_{\boldsymbol{ab}} = \eta_{\boldsymbol{ab}}$. Finally, let $\{\boldsymbol{\epsilon_A}\} \subset \mathfrak{S}$ denote a spin basis, and let $\epsilon_{\boldsymbol{AB}}$ denote the components of the spinor ϵ_{AB} with respect to the latter basis. The scalars $g_{\boldsymbol{ab}}$ and $\epsilon_{\boldsymbol{AB}}$ can be put in correspondence with each other via an equation of the form

$$\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}} = \sigma^{\boldsymbol{a}}{}_{\boldsymbol{AA'}}\sigma^{\boldsymbol{b}}{}_{\boldsymbol{BB'}}\eta_{\boldsymbol{ab}}, \tag{3.16}$$

where $\sigma^{\boldsymbol{a}}{}_{\boldsymbol{AA'}}$ are the so-called ***Infeld-van der Waerden symbols***. These can be regarded as the entries of four (2×2) matrices $(\sigma^{\boldsymbol{a}}{}_{\boldsymbol{AA'}})$, $\boldsymbol{a} = \boldsymbol{0}, \ldots, \boldsymbol{3}$. Unprimed indices denote the rows and the primed indices the columns of the matrix. Given $\sigma^{\boldsymbol{a}}{}_{\boldsymbol{AA'}}$, one defines the inverse symbol $\sigma_{\boldsymbol{b}}{}^{\boldsymbol{BB'}}$ via the relations

$$\sigma_{\boldsymbol{a}}{}^{\boldsymbol{AA'}}\sigma^{\boldsymbol{b}}{}_{\boldsymbol{AA'}} = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}, \qquad \sigma_{\boldsymbol{a}}{}^{\boldsymbol{AA'}}\sigma^{\boldsymbol{a}}{}_{\boldsymbol{BB'}} = \delta_{\boldsymbol{B}}{}^{\boldsymbol{A}}\delta_{\boldsymbol{B'}}{}^{\boldsymbol{A'}}. \tag{3.17}$$

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From these expressions it follows that the correspondence (3.16) can be inverted to yield

$$\eta_{\boldsymbol{ab}} = \sigma_{\boldsymbol{a}}{}^{\boldsymbol{AA}'}\sigma_{\boldsymbol{b}}{}^{\boldsymbol{BB}'}\epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A}'\boldsymbol{B}'}. \tag{3.18}$$

Using Equation (3.18) and observing that $\eta_{\boldsymbol{ab}} = \overline{\eta_{\boldsymbol{ab}}}$, it follows that

$$\sigma_{\boldsymbol{a}}{}^{\boldsymbol{AA}'} = \overline{\sigma_{\boldsymbol{a}}{}^{\boldsymbol{AA}'}}. \tag{3.19}$$

Hence, $(\sigma_{\boldsymbol{a}}{}^{\boldsymbol{AA}'})$ and $(\sigma^{\boldsymbol{a}}{}_{\boldsymbol{AA}'})$ describe *Hermitian matrices*. An explicit computation shows that the matrices

$$(\sigma_{\boldsymbol{0}}{}^{\boldsymbol{AA}'}) \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (\sigma_{\boldsymbol{1}}{}^{\boldsymbol{AA}'}) \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$(\sigma_{\boldsymbol{2}}{}^{\boldsymbol{AA}'}) \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \quad (\sigma_{\boldsymbol{3}}{}^{\boldsymbol{AA}'}) \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$(\sigma^{\boldsymbol{0}}{}_{\boldsymbol{AA}'}) \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\sigma^{\boldsymbol{1}}{}_{\boldsymbol{AA}'}) \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$(\sigma^{\boldsymbol{2}}{}_{\boldsymbol{AA}'}) \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}, \quad (\sigma^{\boldsymbol{3}}{}_{\boldsymbol{AA}'}) \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

satisfy the relations (3.16), (3.17), (3.18) and (3.19). The above matrices correspond, up to a normalisation factor, to the so-called *Pauli matrices*.

Now, consider arbitrary $\boldsymbol{v} \in T|_p(\mathcal{M})$ and $\boldsymbol{\alpha} \in T^*|_p(\mathcal{M})$. In terms of the bases $\{\boldsymbol{e_a}\}$ and $\{\boldsymbol{\omega^a}\}$, $\boldsymbol{v}$ and $\boldsymbol{\alpha}$ can be written as

$$\boldsymbol{v} = v^{\boldsymbol{a}}\boldsymbol{e_a}, \qquad v^{\boldsymbol{a}} \equiv \langle \boldsymbol{\omega^a}, \boldsymbol{v}\rangle,$$
$$\boldsymbol{\alpha} = \alpha_{\boldsymbol{a}}\boldsymbol{\omega^a}, \qquad \alpha_{\boldsymbol{a}} \equiv \langle \boldsymbol{\alpha}, \boldsymbol{e_a}\rangle.$$

The components $v^{\boldsymbol{a}}$ and $\alpha_{\boldsymbol{a}}$ can be put in correspondence with Hermitian spinors using the Infeld-van der Waerden symbols via the rules

$$v^{\boldsymbol{a}} \mapsto v^{\boldsymbol{AA}'} = v^{\boldsymbol{a}}\sigma_{\boldsymbol{a}}{}^{\boldsymbol{AA}'}, \tag{3.20a}$$
$$\alpha_{\boldsymbol{a}} \mapsto \alpha_{\boldsymbol{AA}'} = \alpha_{\boldsymbol{a}}\sigma^{\boldsymbol{a}}{}_{\boldsymbol{AA}'}. \tag{3.20b}$$

In terms of arrays of explicit components and matrices one has

$$(v^{\boldsymbol{0}}, v^{\boldsymbol{1}}, v^{\boldsymbol{2}}, v^{\boldsymbol{3}}) \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix} v^{\boldsymbol{0}} + v^{\boldsymbol{3}} & v^{\boldsymbol{1}} + \mathrm{i}v^{\boldsymbol{2}} \\ v^{\boldsymbol{1}} - \mathrm{i}v^{\boldsymbol{2}} & v^{\boldsymbol{0}} - v^{\boldsymbol{3}} \end{pmatrix},$$
$$(\alpha_{\boldsymbol{0}}, \alpha_{\boldsymbol{1}}, \alpha_{\boldsymbol{2}}, \alpha_{\boldsymbol{3}}) \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha_{\boldsymbol{0}} + \alpha_{\boldsymbol{3}} & \alpha_{\boldsymbol{1}} - \mathrm{i}\alpha_{\boldsymbol{2}} \\ \alpha_{\boldsymbol{1}} + \mathrm{i}\alpha_{\boldsymbol{2}} & \alpha_{\boldsymbol{0}} - \alpha_{\boldsymbol{3}} \end{pmatrix}.$$

A quick computation shows that

$$\begin{aligned}\langle \boldsymbol{\alpha}, \boldsymbol{v}\rangle &= v^{\boldsymbol{a}}\alpha_{\boldsymbol{a}} = v^{\boldsymbol{AA}'}\alpha_{\boldsymbol{AA}'} \\ &= v^{\boldsymbol{00}'}\alpha_{\boldsymbol{00}'} + v^{\boldsymbol{01}'}\alpha_{\boldsymbol{01}'} + v^{\boldsymbol{10}'}\alpha_{\boldsymbol{10}'} + v^{\boldsymbol{11}'}\alpha_{\boldsymbol{11}'} \\ &= v^{\boldsymbol{0}}\alpha_{\boldsymbol{0}} - v^{\boldsymbol{1}}\alpha_{\boldsymbol{1}} - v^{\boldsymbol{2}}\alpha_{\boldsymbol{2}} - v^{\boldsymbol{3}}\alpha_{\boldsymbol{3}}.\end{aligned}$$

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Thus, one has that the assignments defined in (3.20a) and (3.20b) are consistent with the inner product defined on $T|_p(\mathcal{M})$ by the metric $\boldsymbol{g}$.

The assignment given by (3.20a) and (3.20b) can be extended to tensors of arbitrary rank. For example, given the tensor $T_{ab}{}^{c}$, denote its components with respect to $\{\boldsymbol{e_a}\}$ and $\{\boldsymbol{\omega^b}\}$ by $T_{\boldsymbol{ab}}{}^{\boldsymbol{c}}$. One then has the assignment

$$T_{\boldsymbol{ab}}{}^{\boldsymbol{c}} \mapsto T_{\boldsymbol{AA'BB'}}{}^{\boldsymbol{CC'}} \equiv \sigma^{\boldsymbol{a}}{}_{\boldsymbol{AA'}}\sigma^{\boldsymbol{b}}{}_{\boldsymbol{BB'}}\sigma_{\boldsymbol{c}}{}^{\boldsymbol{CC'}}T_{\boldsymbol{ab}}{}^{\boldsymbol{c}}.$$

The object $T_{\boldsymbol{AA'BB'}}{}^{\boldsymbol{CC'}}$ will be called the ***spinorial counterpart*** of the tensor components $T_{\boldsymbol{ab}}{}^{\boldsymbol{c}}$.

3.1.10 The spinorial representation of null vectors

As already mentioned in the introduction to this chapter, one of the key advantages of the use of spinors is the convenient representation of null vectors they provide. More precisely, one has the following result:

Proposition 3.3 (***spinorial counterpart of null vectors***) *The spinorial counterpart of a non-vanishing real null vector k^a can be written as*

$$k^{AA'} = \pm\kappa^A\bar{\kappa}^{A'}, \tag{3.21}$$

for some valence 1 spinor κ^A.

Proof A direct computation shows that $k^{AA'}$ as given by Equation (3.21) is indeed the spinorial counterpart of a null vector. Conversely, a computation yields

$$\begin{aligned}\boldsymbol{g}(\boldsymbol{k},\boldsymbol{k}) &= \epsilon_{AB}\epsilon_{A'B'}k^{AA'}k^{BB'}\\ &= 2(k^{00'}k^{11'} - k^{01'}k^{10'}) = \det(k^{AA'}).\end{aligned}$$

Thus, the requirement $\boldsymbol{g}(\boldsymbol{k},\boldsymbol{k}) = 0$ implies that $k^{AA'}$, regarded as a (2×2) matrix, has rows/columns which are linearly dependent. Accordingly, there exist valence 1 spinors κ^A and λ^B such that $k^{AA'} = \kappa^A\bar{\lambda}^{A'}$. As, $\boldsymbol{k}$ is non-zero, it follows that κ_A, $\lambda_B \neq 0$. From the reality of $\boldsymbol{k}$, it follows that its spinor counterpart $k^{AA'}$ must be Hermitian; that is, $k^{AA'} = \bar{k}^{AA'}$. Hence, $\kappa^A\bar{\lambda}^{A'} = \bar{\kappa}^{A'}\lambda^A$. Contracting the latter with κ_A one has that $\kappa_A\lambda^A = 0$, so that κ^A and λ^A must be proportional to each other. The proportionality factor can be absorbed into κ^A by means of a redefinition of the spinor. The sign in Equation (3.21) is that of the proportionality constant. □

Remark. A null vector constructed using the *positive sign* in Equation (3.21) will be said to be ***future pointing***, while one using the *negative* sign will be called ***past pointing***.

From Proposition 3.3 it follows that every valence 1 spinor κ^A defines a null vector $\boldsymbol{k}$. However, this is not a one-to-one correspondence. More precisely, a spinor differing from κ^A by a complex phase, that is, $e^{i\vartheta}\kappa^A$, with $\vartheta \in \mathbb{R}$ will give rise to the same null vector. The phase change is said to be ***right-handed*** if $\vartheta > 0$. This phase does not affect the construction of the vector $\boldsymbol{k}$. Nevertheless, it contains some geometric information. To see this, consider a further spinor μ^A such that $\kappa_A\mu^A = 1$ so that $\{\kappa^A, \mu^A\}$ constitute a spin basis. Now, one can readily verify that

$$s^{AA'} \equiv \frac{1}{\sqrt{2}}(\kappa^A\bar{\mu}^{A'} + \mu^A\bar{\kappa}^{A'}), \qquad t^{AA'} = \frac{\mathrm{i}}{\sqrt{2}}(\kappa^A\bar{\mu}^{A'} - \mu^A\bar{\kappa}^{A'}),$$

are the spinorial counterparts of two unit spacelike vectors $\boldsymbol{s}$ and $\boldsymbol{t}$ and that they are both orthogonal to $\boldsymbol{k}$. At each point $p \in \mathcal{M}$, $\boldsymbol{s}$ and $\boldsymbol{t}$ span a subspace of $T|_p(\mathcal{M})$ which is orthogonal to $\boldsymbol{k}$. This subspace is called the ***flag*** of the spinor κ^A; the ***pole of the flag*** is the vector $\boldsymbol{k}$.

Now, suppose κ^A is subject to a phase change such that

$$\kappa^A \mapsto e^{i\vartheta}\kappa^A. \tag{3.22}$$

In order to retain the normalisation $\kappa_A\mu^A = 1$, the transformation (3.22) implies the transformation $\mu^A \mapsto e^{-i\vartheta}\mu^A$. Furthermore, one has that

$$\boldsymbol{s} \mapsto \cos 2\vartheta \boldsymbol{s} + \sin 2\vartheta \boldsymbol{t}, \qquad \boldsymbol{t} \mapsto -\sin 2\vartheta \boldsymbol{s} + \cos 2\vartheta \boldsymbol{t},$$

so that a phase change of ϑ in κ^A implies a change of 2ϑ in its flag; the flagpole, however, remains unchanged.

3.1.11 Null tetrads

Inspection of Equation (3.14) shows that every spin basis $\{\boldsymbol{o}, \boldsymbol{\iota}\}$ gives rise to an associated vector basis consisting of null vectors. This ***null tetrad*** has the peculiarity of consisting of two *real* null vectors and two *complex* null vectors which are the complex conjugates of each other. In order to analyse this further, let

$$l^{AA'} \equiv o^A\bar{o}^{A'}, \qquad n^{AA'} \equiv \iota^A\bar{\iota}^{A'}, \qquad m^{AA'} \equiv o^A\bar{\iota}^{A'}, \qquad \bar{m}^{AA'} \equiv \iota^A\bar{o}^{A'}.$$

Furthermore, let l^a, n^a, m^a and $\bar{m}^a$ (or $\boldsymbol{l}$, $\boldsymbol{n}$, $\boldsymbol{m}$, $\bar{\boldsymbol{m}}$) denote the tensorial counterparts of the above spinors. Using the above definitions one can verify that

$$l_a n^a = -m_a\bar{m}^a = 1, \tag{3.23}$$

while all the other remaining contractions vanish. Using relations (3.11a)–(3.11c) it can be readily shown that

$$g_{ab} = 2l_{(a}n_{b)} - 2m_{(a}\bar{m}_{b)}, \qquad g^{ab} = 2l^{(a}n^{b)} - 2m^{(a}\bar{m}^{b)}.$$

An orthonormal tetrad $\{\boldsymbol{e_a}\}$ can be readily obtained from the null tetrad $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$. Namely, let

$$\boldsymbol{e_0} = \frac{1}{\sqrt{2}}(\boldsymbol{l} + \boldsymbol{n}), \tag{3.24a}$$

$$\boldsymbol{e_1} = \frac{1}{\sqrt{2}}(\boldsymbol{m} + \bar{\boldsymbol{m}}), \tag{3.24b}$$

$$\boldsymbol{e_2} = \frac{\mathrm{i}}{\sqrt{2}}(\boldsymbol{m} - \bar{\boldsymbol{m}}), \tag{3.24c}$$

$$\boldsymbol{e_3} = \frac{1}{\sqrt{2}}(\boldsymbol{l} - \boldsymbol{n}). \tag{3.24d}$$

Using the relations in (3.23) it can be verified that the latter vectors indeed constitute an orthonormal tetrad. Furthermore, it can be readily checked that $\boldsymbol{e_0}$ is timelike while $\boldsymbol{e_1}$, $\boldsymbol{e_2}$ and $\boldsymbol{e_3}$ are spacelike. The vector $\boldsymbol{e_0}$ is said to be *future pointing* as both $\boldsymbol{l}$ and $\boldsymbol{n}$ are future pointing in the sense of Section 3.1.10. Moreover, a right-handed phase change (i.e. $\vartheta > 0$) in the spin basis of the form $o^A \mapsto e^{i\vartheta} o^A$, $\iota^A \mapsto e^{-i\vartheta}\iota^A$ leads to the *right-handed* rotations

$$\boldsymbol{e_1} \mapsto \cos 2\vartheta \boldsymbol{e_1} + \sin 2\vartheta \boldsymbol{e_2}, \qquad \boldsymbol{e_2} \mapsto -\sin 2\vartheta \boldsymbol{e_1} + \cos 2\vartheta \boldsymbol{e_2},$$

while at the same time leaving $\boldsymbol{e_0}$ and $\boldsymbol{e_3}$ unchanged. Accordingly, the triad of spacelike vectors $\{\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}\}$ defined by (3.24b)–(3.24d) is said to be *right-handed*. The inverse relations to (3.24a)–(3.24d) are given by

$$\boldsymbol{l} = \frac{1}{\sqrt{2}}(\boldsymbol{e_0} + \boldsymbol{e_3}), \qquad \boldsymbol{n} = \frac{1}{\sqrt{2}}(\boldsymbol{e_0} - \boldsymbol{e_3}),$$
$$\boldsymbol{m} = \frac{1}{\sqrt{2}}(\boldsymbol{e_1} - \mathrm{i}\boldsymbol{e_2}), \qquad \bar{\boldsymbol{m}} = \frac{1}{\sqrt{2}}(\boldsymbol{e_1} + \mathrm{i}\boldsymbol{e_2}).$$

The spinorial counterpart of the volume form

The ***spinorial counterpart of the volume 4-form*** ϵ_{abcd} is given by

$$\epsilon_{AA'BB'CC'DD'} = \mathrm{i}(\epsilon_{AB}\epsilon_{CD}\epsilon_{A'C'}\epsilon_{B'D'} - \epsilon_{AC}\epsilon_{BD}\epsilon_{A'B'}\epsilon_{C'D'}). \tag{3.25}$$

Using the Jacobi identity (3.5) it can be verified that the above expression is indeed totally antisymmetric under interchange of the pairs $_{AA'}$, $_{BB'}$, $_{CC'}$ and $_{DD'}$. Moreover, one has

$$\epsilon_{AA'BB'CC'DD'}\epsilon^{AA'BB'CC'DD'} = 24,$$

and

$$\sigma_{\boldsymbol{0}}{}^{\boldsymbol{AA'}}\sigma_{\boldsymbol{1}}{}^{\boldsymbol{BB'}}\sigma_{\boldsymbol{2}}{}^{\boldsymbol{CC'}}\sigma_{\boldsymbol{3}}{}^{\boldsymbol{DD'}}\epsilon_{\boldsymbol{AA'BB'CC'DD'}} = 1;$$

compare Section 2.5.3. The expression (3.25) can be deduced applying a decomposition in irreducible components to $\epsilon_{AA'BB'CC'DD'}$ and exploiting its antisymmetry properties.

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3.1.12 Changes of basis and $SL(2,\mathbb{C})$ transformations

Let $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$ and $\{\tilde{\epsilon}_{\boldsymbol{A}}{}^{A}\}$ denote two spin bases for $\mathfrak{S}$. The spinors of one basis can be expressed as linear combinations of the spinors of the other basis. This can be conveniently be written as

$$\tilde{\epsilon}_{\boldsymbol{A}}{}^{A} = \Lambda_{\boldsymbol{A}}{}^{\boldsymbol{P}} \epsilon_{\boldsymbol{P}}{}^{A}, \tag{3.26}$$

where $(\Lambda_{\boldsymbol{A}}{}^{\boldsymbol{P}})$ denotes an invertible (2×2) matrix. The associated spinor cobases $\{\epsilon^{\boldsymbol{A}}{}_{A}\}$ and $\{\tilde{\epsilon}^{\boldsymbol{A}}{}_{A}\}$ are related in a similar way:

$$\tilde{\epsilon}^{\boldsymbol{A}}{}_{A} = \Lambda^{\boldsymbol{A}}{}_{\boldsymbol{P}} \epsilon^{\boldsymbol{P}}{}_{A}, \tag{3.27}$$

where $(\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{P}})$ is another invertible (2×2) matrix. Now, one has that

$$\begin{aligned}\delta_{\boldsymbol{A}}{}^{\boldsymbol{B}} &= \tilde{\epsilon}_{\boldsymbol{A}}{}^{P} \tilde{\epsilon}^{\boldsymbol{B}}{}_{P} = \left(\Lambda_{\boldsymbol{A}}{}^{\boldsymbol{P}} \epsilon_{\boldsymbol{P}}{}^{Q}\right)\left(\Lambda^{\boldsymbol{B}}{}_{\boldsymbol{Q}} \epsilon^{\boldsymbol{Q}}{}_{Q}\right) = \left(\Lambda_{\boldsymbol{A}}{}^{\boldsymbol{P}} \Lambda^{\boldsymbol{B}}{}_{\boldsymbol{Q}}\right) \epsilon_{\boldsymbol{P}}{}^{Q} \epsilon^{\boldsymbol{Q}}{}_{Q} \\ &= \Lambda_{\boldsymbol{A}}{}^{\boldsymbol{P}} \Lambda^{\boldsymbol{B}}{}_{\boldsymbol{Q}} \delta_{\boldsymbol{P}}{}^{\boldsymbol{Q}} = \Lambda_{\boldsymbol{A}}{}^{\boldsymbol{P}} \Lambda^{\boldsymbol{B}}{}_{\boldsymbol{P}}.\end{aligned}$$

Hence, the matrices $(\Lambda_{\boldsymbol{A}}{}^{\boldsymbol{P}})$ and $(\Lambda^{\boldsymbol{A}}{}_{\boldsymbol{P}})$ are inverses of each other.

Now, given a contravariant valence 1 spinor κ^{A}, one can expand it in terms of the bases $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$ and $\{\tilde{\epsilon}_{\boldsymbol{A}}{}^{A}\}$ as

$$\kappa^{A} = \kappa^{\boldsymbol{A}} \epsilon_{\boldsymbol{A}}{}^{A} = \tilde{\kappa}^{\boldsymbol{A}} \tilde{\epsilon}_{\boldsymbol{A}}{}^{A}.$$

As a consequence of the change of basis (3.26), the coefficients $\kappa_{\boldsymbol{A}}$ and $\tilde{\kappa}_{\boldsymbol{A}}$ are related to each other via

$$\tilde{\kappa}^{\boldsymbol{A}} = \Lambda^{\boldsymbol{A}}{}_{\boldsymbol{P}} \kappa^{\boldsymbol{P}}.$$

Similarly, from the transformation rule (3.27), the components $\mu_{\boldsymbol{A}}$ and $\tilde{\mu}_{\boldsymbol{A}}$ of a valence 1 covariant spinor μ_{A} with respect to the spin cobasis $\{\epsilon^{\boldsymbol{A}}{}_{A}\}$ and $\{\tilde{\epsilon}^{\boldsymbol{A}}{}_{A}\}$ can be found to be related via

$$\tilde{\mu}_{\boldsymbol{A}} = \Lambda_{\boldsymbol{A}}{}^{\boldsymbol{P}} \mu_{\boldsymbol{P}}.$$

The transformation rules given in the previous paragraph can be extended in a natural way to higher valence spinors and to spinors with primed indices. For example, if $v^{\boldsymbol{A}\boldsymbol{A}'}$ and $\tilde{v}^{\boldsymbol{A}\boldsymbol{A}'}$ denote the components of the spinor $v^{AA'}$ with respect to the two different sets of bases, one has that

$$\tilde{v}^{\boldsymbol{A}\boldsymbol{A}'} = \Lambda^{\boldsymbol{A}}{}_{\boldsymbol{P}} \bar{\Lambda}^{\boldsymbol{A}'}{}_{\boldsymbol{P}'} v^{\boldsymbol{P}\boldsymbol{P}'}.$$

A case of special importance is that of the antisymmetric spinor ϵ_{AB} for which the transformation rule between bases is given by

$$\tilde{\epsilon}_{\boldsymbol{A}\boldsymbol{B}} = \Lambda_{\boldsymbol{A}}{}^{\boldsymbol{P}} \Lambda_{\boldsymbol{B}}{}^{\boldsymbol{Q}} \epsilon_{\boldsymbol{P}\boldsymbol{Q}}. \tag{3.28}$$

Earlier in the chapter, the notion of simplectic transformations was introduced. The properties of these transformations can be investigated from Equation (3.28). As a consequence of the discussion of Section 3.1.7 the matrices

$(\epsilon_{\boldsymbol{AB}})$ and $(\tilde{\epsilon}_{\boldsymbol{AB}})$ both have the form given by Equation (3.10). It follows from Equation (3.28) that

$$\det(\tilde{\epsilon}_{\boldsymbol{AB}}) = \left(\det\left(\Lambda_{\boldsymbol{A}}{}^{\boldsymbol{B}}\right)\right)^2 \det(\epsilon_{\boldsymbol{AB}}).$$

Furthermore as $\det(\tilde{\epsilon}_{\boldsymbol{AB}}) = \det(\epsilon_{\boldsymbol{AB}}) = 1$, one concludes that $\det\left(\Lambda_{\boldsymbol{A}}{}^{\boldsymbol{B}}\right) = \pm 1$. Hence, if one restricts attention to the transformations with positive determinant, one finds that *the set of transformations that preserve the antisymmetric product* $[[\cdot,\cdot]]$ *is given by the group* $SL(2,\mathbb{C})$.

Relation to the Lorentz transformations

Following the discussion of the previous paragraphs, the components $g_{\boldsymbol{AA'BB'}}$ of the spinorial counterpart of the metric transform under a change of spin basis as

$$\tilde{g}_{\boldsymbol{AA'BB'}} \equiv \tilde{\epsilon}_{\boldsymbol{AB}}\tilde{\epsilon}_{\boldsymbol{A'B'}} = \Lambda_{\boldsymbol{A}}{}^{\boldsymbol{P}}\bar{\Lambda}_{\boldsymbol{A'}}{}^{\boldsymbol{P'}}\Lambda_{\boldsymbol{B}}{}^{\boldsymbol{Q}}\bar{\Lambda}_{\boldsymbol{B'}}{}^{\boldsymbol{Q'}}\epsilon_{\boldsymbol{PQ}}\epsilon_{\boldsymbol{P'Q'}}.$$

Using the Infeld-van der Waerden symbols, the latter can be rewritten as

$$\tilde{\eta}_{\boldsymbol{ab}} = \Lambda_{\boldsymbol{a}}{}^{\boldsymbol{c}}\Lambda_{\boldsymbol{b}}{}^{\boldsymbol{d}}\eta_{\boldsymbol{cd}},$$

with

$$\Lambda_{\boldsymbol{a}}{}^{\boldsymbol{c}} \equiv \sigma_{\boldsymbol{a}}{}^{\boldsymbol{AA'}}\sigma^{\boldsymbol{c}}{}_{\boldsymbol{PP'}}\Lambda_{\boldsymbol{A}}{}^{\boldsymbol{P}}\bar{\Lambda}_{\boldsymbol{A'}}{}^{\boldsymbol{P'}}.$$

The above expression provides the relation between $SL(2,\mathbb{C})$ and Lorentz transformations; see, for example, Sexl and Urbantke (2000) for more details.

3.1.13 Soldering forms

The connection between spinors and world tensors has been implemented in terms of the components with respect to some vector and spin bases. There is a different perspective of this translation in terms of so-called soldering forms.

The metric tensor $\boldsymbol{g}$ can be written in terms of the orthonormal cobasis $\{\boldsymbol{\omega}^{\boldsymbol{a}}\}$ as

$$\boldsymbol{g} = \eta_{\boldsymbol{ab}}\boldsymbol{\omega}^{\boldsymbol{a}} \otimes \boldsymbol{\omega}^{\boldsymbol{b}}.$$

This last expression can be rewritten, using the correspondence (3.18), as

$$\boldsymbol{g} = \epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}}\sigma_{\boldsymbol{a}}{}^{\boldsymbol{AA'}}\sigma_{\boldsymbol{b}}{}^{\boldsymbol{BB'}}\boldsymbol{\omega}^{\boldsymbol{a}} \otimes \boldsymbol{\omega}^{\boldsymbol{b}} = \epsilon_{\boldsymbol{AB}}\epsilon_{\boldsymbol{A'B'}}\boldsymbol{\omega}^{\boldsymbol{AA'}} \otimes \boldsymbol{\omega}^{\boldsymbol{BB'}}, \tag{3.29}$$

where $\boldsymbol{\omega}^{\boldsymbol{AA'}} \equiv \sigma_{\boldsymbol{a}}{}^{\boldsymbol{AA'}}\boldsymbol{\omega}^{\boldsymbol{a}}$. The four covectors $\{\boldsymbol{\omega}^{\boldsymbol{AA'}}\}$ are called the ***soldering forms*** . In terms of abstract index notation one writes the soldering form as $\omega^{\boldsymbol{AA'}}{}_a$. A similar discussion can be made with the contravariant metric $\boldsymbol{g}^\sharp$. From $\boldsymbol{g}^\sharp = \eta^{\boldsymbol{ab}}\boldsymbol{e}_{\boldsymbol{a}} \otimes \boldsymbol{e}_{\boldsymbol{b}}$, together with (3.16), one can write

$$\boldsymbol{g}^\sharp = \epsilon^{\boldsymbol{AB}}\epsilon^{\boldsymbol{A'B'}}\boldsymbol{e}_{\boldsymbol{AA'}} \otimes \boldsymbol{e}_{\boldsymbol{BB'}}, \tag{3.30}$$

where $\boldsymbol{e_{AA'}} \equiv \sigma^{\boldsymbol{a}}{}_{\boldsymbol{AA'}}\boldsymbol{e_a}$. In abstract index notation one would write $e_{\boldsymbol{AA'}}{}^a$ instead of $\boldsymbol{e_{AA'}}$. In view of the above, given a vector $\boldsymbol{v} \in T|_p(\mathcal{M})$ and a covector $\boldsymbol{\alpha} \in T^*|_p(\mathcal{M})$, one can write

$$\boldsymbol{v} = v^{\boldsymbol{AA'}}\boldsymbol{e_{AA'}}, \qquad \boldsymbol{\alpha} = \alpha_{\boldsymbol{AA'}}\boldsymbol{\omega}^{\boldsymbol{AA'}}.$$

As a final remark concerning the connection between spinors and world tensors, it is observed that $\boldsymbol{e_a} = \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}\boldsymbol{e_b}$. Thus, $\delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}$ can be interpreted as the components $e_{\boldsymbol{a}}{}^{\boldsymbol{b}}$ of the frame vector $\boldsymbol{e_a}$ with respect to the frame $\{\boldsymbol{e_a}\}$. Contracting $e_{\boldsymbol{a}}{}^{\boldsymbol{b}}$ with $\sigma_{\boldsymbol{b}}{}^{\boldsymbol{BB'}}$ one finds

$$e_{\boldsymbol{a}}{}^{\boldsymbol{BB'}} \equiv e_{\boldsymbol{a}}{}^{\boldsymbol{b}}\sigma_{\boldsymbol{b}}{}^{\boldsymbol{BB'}} = \sigma_{\boldsymbol{a}}{}^{\boldsymbol{BB'}}.$$

3.2 Calculus of spacetime spinors

The discussion of the previous section has been restricted to spinors at a given point of the spacetime manifold $\mathcal{M}$. It is now assumed that a spinorial structure can be constructed in a consistent way on the whole of $\mathcal{M}$ – the conditions ensuring this are discussed in Section 3.3, and essentially amount to requiring the spacetime to be orientable. The spinorial structure over $\mathcal{M}$ (also called a ***spin bundle***) will be denoted by $\mathfrak{S}(\mathcal{M})$. Consistent with this notation, the spinorial structure at a point $p \in \mathcal{M}$ will be denoted by $\mathfrak{S}|_p(\mathcal{M})$.

As is the case with tensors, the idea of relating spinors defined at different points of the spacetime manifold requires the use of the notion of a *connection* and its associated *covariant derivative*. Thus, it is necessary to extend the notion of a connection in such a way that it applies to ***spinor fields***. In what follows, by a spinor field it is understood a *smooth* assignment of a spinor, say, $\xi_{A\cdots CD'\cdots F'}{}^{G\cdots LP'\cdots N'}$, to each point of the spacetime manifold. The sets of spinorial fields over $\mathcal{M}$ will be denoted in a similar manner to the sets of spinors at a point, that is, $\mathfrak{S}^\bullet(\mathcal{M})$, $\mathfrak{S}_A(\mathcal{M})$, $\mathfrak{S}^A(\mathcal{M})$, $\mathfrak{S}_{AA'}{}^B(\mathcal{M})$, and so on.

3.2.1 The spinorial covariant derivative

A ***spinor covariant derivative*** $\nabla_{AA'}$ is a map

$$\nabla_{AA'} : \mathfrak{S}^{B\cdots C'}{}_{D\cdots E'}(\mathcal{M}) \to \mathfrak{S}^{B\cdots C'}{}_{AD\cdots A'E'}(\mathcal{M}).$$

Given an arbitrary spinor $\zeta^{B\cdots C'}{}_{D\cdots E'}$, its spinorial covariant derivative will be denoted by $\nabla_{AA'}\zeta^{B\cdots C'}{}_{D\cdots E'}$. The mapping defined by $\nabla_{AA'}$ is required to satisfy the following properties:

(i) ***Linearity.*** Given $\zeta^{B\cdots C'}{}_{D\cdots E'}$, $\eta^{B\cdots C'}{}_{D\cdots E'} \in \mathfrak{S}^{B\cdots C'}{}_{D\cdots E'}(\mathcal{M})$,

$$\nabla_{AA'}(\zeta^{B\cdots C'}{}_{D\cdots E'} + \eta^{B\cdots C'}{}_{D\cdots E'}) = \nabla_{AA'}\zeta^{B\cdots C'}{}_{D\cdots E'} + \nabla_{AA'}\eta^{B\cdots C'}{}_{D\cdots E'}.$$

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(ii) ***Leibnitz rule.*** Given fields $\zeta^{B\cdots C'}{}_{D\cdots E'} \in \mathfrak{S}^{B\cdots C'}{}_{D\cdots E'}(\mathcal{M})$ and $\xi^{F\cdots G'}{}_{H\cdots I'} \in \mathfrak{S}^{F\cdots G'}{}_{H\cdots I'}(\mathcal{M})$,

$$\nabla_{AA'}(\zeta^{B\cdots C'}{}_{D\cdots E'}\xi^{F\cdots G'}{}_{H\cdots I'}) = \xi^{F\cdots G'}{}_{H\cdots I'}\nabla_{AA'}\zeta^{B\cdots C'}{}_{D\cdots E'} + \zeta^{B\cdots C'}{}_{D\cdots E'}\nabla_{AA'}\xi^{F\cdots G'}{}_{H\cdots I'}.$$

(iii) ***Hermicity.*** Given $\zeta^{B\cdots C'}{}_{D\cdots E'} \in \mathfrak{S}^{B\cdots C'}{}_{D\cdots E'}(\mathcal{M})$,

$$\overline{\nabla_{AA'}\zeta^{B\cdots C'}{}_{D\cdots E'}} = \nabla_{AA'}\bar{\zeta}^{B'\cdots C}{}_{D'\cdots E}.$$

(iv) ***Action on scalars.*** Given a scalar ϕ, then $\nabla_{AA'}\phi$ is the spinorial counterpart of $\nabla_a\phi$.

(v) ***Representation of derivations.*** Given a derivation $\mathcal{D}$ on spinor fields, there exists a spinor $\xi^{AA'}$ such that

$$\mathcal{D}\zeta^{B\cdots C'}{}_{D\cdots E'} = \xi^{AA'}\nabla_{AA'}\zeta^{B\cdots C'}{}_{D\cdots E'},$$

for all $\zeta^{B\cdots C'}{}_{D\cdots E'} \in \mathfrak{S}^{\bullet}(\mathcal{M})$.

Remark. The above list of properties is more general than the ones given in, say, Penrose and Rindler (1984) and Stewart (1991), as the present discussion does not assume that the spinor covariant derivative is compatible with the $\boldsymbol{\epsilon}$-spinor; that is, $\nabla_{AA'}\epsilon_{BC} = 0$.

For completeness, the following result proved in Penrose and Rindler (1984) is recalled:

Theorem 3.1 (***existence of the spinorial covariant derivative***) *Every covariant derivative* $\boldsymbol{\nabla}$ *over* $\mathcal{M}$ *has a spinorial counterpart* $\nabla_{AA'}$.

3.2.2 Spin connection coefficients

In specific computations, given a spin basis $\{\epsilon_{\boldsymbol{A}}{}^{A}\}$, it is convenient to introduce the notion of the ***spin connection coefficients*** associated to a certain connection. The direct spinorial counterparts of the connection coefficients $\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}}$ are given after suitable contraction with the Infeld-van der Waerden symbols by the spinor components

$$\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}} \equiv \omega^{\boldsymbol{BB'}}{}_{BB'}\nabla_{\boldsymbol{AA'}}e_{\boldsymbol{CC'}}{}^{BB'}, \tag{3.31}$$

where $\nabla_{\boldsymbol{AA'}} \equiv e_{\boldsymbol{AA'}}{}^{AA'}\nabla_{AA'}$ denotes the ***directional covariant derivative in the direction of*** $\boldsymbol{e_{AA'}}$. Now, using that

$$\omega^{\boldsymbol{BB'}}{}_{BB'} = \epsilon^{\boldsymbol{B}}{}_{B}\bar{\epsilon}^{\boldsymbol{B'}}{}_{B'}, \qquad e_{\boldsymbol{CC'}}{}^{CC'} = \epsilon_{\boldsymbol{C}}{}^{C}\bar{\epsilon}_{\boldsymbol{C'}}{}^{C'},$$

it follows that

$$\begin{aligned}\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}} &= \epsilon^{\boldsymbol{B}}{}_{B}\bar{\epsilon}^{\boldsymbol{B'}}{}_{B'}\bar{\epsilon}_{\boldsymbol{C'}}{}^{B'}\nabla_{\boldsymbol{AA'}}\epsilon_{\boldsymbol{C}}{}^{B} + \epsilon^{\boldsymbol{B}}{}_{B}\bar{\epsilon}^{\boldsymbol{B'}}{}_{B'}\epsilon_{\boldsymbol{C}}{}^{B}\nabla_{\boldsymbol{AA'}}\bar{\epsilon}_{\boldsymbol{C'}}{}^{B'} \\ &= \epsilon^{\boldsymbol{B}}{}_{B}\delta_{\boldsymbol{C'}}{}^{\boldsymbol{B'}}\nabla_{\boldsymbol{AA'}}\epsilon_{\boldsymbol{C}}{}^{B} + \bar{\epsilon}^{\boldsymbol{B'}}{}_{B'}\delta_{\boldsymbol{C}}{}^{\boldsymbol{B}}\nabla_{\boldsymbol{AA'}}\bar{\epsilon}_{\boldsymbol{C'}}{}^{B'}.\end{aligned}$$

Hence, defining the ***spin connection coefficients***

$$\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} \equiv \epsilon^{\boldsymbol{B}}{}_{B} \nabla_{\boldsymbol{AA'}} \epsilon_{\boldsymbol{C}}{}^{B}, \tag{3.32}$$

one obtains

$$\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}} = \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} \delta_{\boldsymbol{C'}}{}^{\boldsymbol{B'}} + \bar{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B'}}{}_{\boldsymbol{C'}} \delta_{\boldsymbol{C}}{}^{\boldsymbol{B}}. \tag{3.33}$$

Using $\delta_{\boldsymbol{C}}{}^{\boldsymbol{B}} = \epsilon_{\boldsymbol{C}}{}^{Q} \epsilon^{\boldsymbol{B}}{}_{Q}$, the definition of $\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}}$ and requiring that

$$\nabla_{\boldsymbol{AA'}} \delta_{\boldsymbol{C}}{}^{\boldsymbol{B}} = 0$$

one also has that

$$\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} = -\epsilon_{\boldsymbol{C}}{}^{Q} \nabla_{\boldsymbol{AA'}} \epsilon^{\boldsymbol{B}}{}_{Q}.$$

The spin connection coefficients provide a way of *computing the covariant derivative of spinors without a tensorial counterpart*. Given $\kappa_A = \kappa_{\boldsymbol{A}} \epsilon_A{}^{\boldsymbol{A}} \in \mathfrak{S}_A(\mathcal{M})$ one has that

$$\begin{aligned}
\nabla_{\boldsymbol{AA'}} \kappa_{\boldsymbol{B}} &\equiv \epsilon_{\boldsymbol{B}}{}^{Q} \nabla_{\boldsymbol{AA'}} \kappa_{Q} \\
&= \epsilon_{\boldsymbol{B}}{}^{Q} \nabla_{\boldsymbol{AA'}} (\kappa_{\boldsymbol{P}} \epsilon^{\boldsymbol{P}}{}_{Q}) \\
&= \epsilon_{\boldsymbol{B}}{}^{Q} \left(\boldsymbol{e}_{\boldsymbol{AA'}}(\kappa_{\boldsymbol{P}}) \epsilon^{\boldsymbol{P}}{}_{Q} + \kappa_{\boldsymbol{P}} \nabla_{\boldsymbol{AA'}} \epsilon^{\boldsymbol{P}}{}_{Q} \right) \\
&= \boldsymbol{e}_{\boldsymbol{AA'}}(\kappa_{\boldsymbol{B}}) - \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{P}}{}_{\boldsymbol{B}} \kappa_{\boldsymbol{P}}.
\end{aligned}$$

Similar computations show, for example, that

$$\begin{aligned}
\nabla_{\boldsymbol{AA'}} \zeta^{\boldsymbol{B}} &= \boldsymbol{e}_{\boldsymbol{AA'}}(\zeta^{\boldsymbol{B}}) + \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{P}} \zeta^{\boldsymbol{P}}, \\
\nabla_{\boldsymbol{AA'}} \xi_{\boldsymbol{B'}}{}^{\boldsymbol{CC'}} &= \boldsymbol{e}_{\boldsymbol{AA'}}(\xi_{\boldsymbol{B'}}{}^{\boldsymbol{CC'}}) - \bar{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{Q'}}{}_{\boldsymbol{B'}} \xi_{\boldsymbol{Q'}}{}^{\boldsymbol{CC'}} \\
&\qquad + \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{C}}{}_{\boldsymbol{Q}} \xi_{\boldsymbol{B'}}{}^{\boldsymbol{QC'}} + \bar{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{C'}}{}_{\boldsymbol{Q'}} \xi_{\boldsymbol{B'}}{}^{\boldsymbol{CQ'}}.
\end{aligned}$$

The generalisation to spinors of arbitrary valence and number of primed indices can be readily obtained from the above examples.

Metric and Levi-Civita spin connection coefficients

So far, the discussion of the spin connection coefficients has been completely general. *In the present section it is assumed that the connection is metric.*

The spinorial counterpart of the metric compatibility condition $\nabla_a g_{bc} = 0$ is given by

$$\nabla_{AA'}(\epsilon_{BC} \epsilon_{B'C'}) = \epsilon_{B'C'} \nabla_{AA'} \epsilon_{BC} + \epsilon_{BC} \nabla_{AA'} \epsilon_{B'C'} = 0.$$

Regarding the second equality as a (partial) decomposition in irreducible terms, one has that

$$\nabla_{AA'} \epsilon_{BC} = 0, \qquad \nabla_{AA'} \epsilon_{B'C'} = 0.$$

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In order to investigate the implications of a metric connection on its associated spin connection coefficients, it is convenient to compute

$$\begin{aligned}\nabla_{\boldsymbol{AA'}}\epsilon_{\boldsymbol{BC}} &= \boldsymbol{e}_{\boldsymbol{AA'}}(\epsilon_{\boldsymbol{BC}}) - \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{B}}\epsilon_{\boldsymbol{QC}} - \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{C}}\epsilon_{\boldsymbol{BQ}}\\ &= -\Gamma_{\boldsymbol{A'A'CB}} + \Gamma_{\boldsymbol{AA'BC}} = 0\end{aligned}$$

as $\boldsymbol{e}_{\boldsymbol{AA'}}(\epsilon_{\boldsymbol{BC}}) = 0$; again, the components $\epsilon_{\boldsymbol{BC}}$ are constants. Hence, one concludes that

$$\Gamma_{\boldsymbol{AA'BC}} = \Gamma_{\boldsymbol{AA'(BC)}}.$$

3.2.3 Spinorial curvature

The spinorial counterpart of the curvature tensors can be introduced in a natural way by looking at the commutator of spinorial covariant derivatives. More precisely, one can write

$$[\![\nabla_{AA'}, \nabla_{BB'}]\!]\xi^{CC'} = R^{CC'}{}_{PP'AA'BB'}\xi^{PP'} \tag{3.34}$$

with

$$[\![\nabla_{AA'}, \nabla_{BB'}]\!] \equiv \nabla_{AA'}\nabla_{BB'} - \nabla_{BB'}\nabla_{AA'} - \Sigma_{AA'}{}^{PP'}{}_{BB'}\nabla_{PP'},$$

consistent with the notation of Section 2.4.3 and with $\Sigma_{AA'}{}^{CC'}{}_{BB'}$ representing the ***spinorial counterpart of the torsion tensor*** of $\boldsymbol{\nabla}$. The spinor $R^{CC'}{}_{DD'AA'BB'}$ is the ***spinorial counterpart of the Riemann curvature tensor*** $R^{c}{}_{dab}$. In the following discussion it is assumed that the connection $\boldsymbol{\nabla}$ is completely general – in particular, it could have torsion and be non-metric, so that $\nabla_{AA'}\epsilon_{BC} \neq 0$. As a consequence, the curvature spinor has only the symmetry

$$R^{CC'}{}_{DD'AA'BB'} = -R^{CC'}{}_{DD'BB'AA'}.$$

The curvature spinor in terms of the spin connection coefficients

In order to obtain a simpler representation of the curvature spinor it is convenient to look first at its expression in terms of spin connection coefficients. To this end, one can consider the frame expression (2.31) for the Riemann tensor, and contract it with the Infeld-van der Waerden symbols. One readily obtains

$$\begin{aligned}R^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'AA'BB'}} &= \boldsymbol{e}_{\boldsymbol{AA'}}(\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'}}) - \boldsymbol{e}_{\boldsymbol{BB'}}(\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'}})\\ &\quad +\Gamma_{\boldsymbol{FF'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'}}\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{FF'}}{}_{\boldsymbol{AA'}} - \Gamma_{\boldsymbol{FF'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'}}\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{FF'}}{}_{\boldsymbol{BB'}}\\ &\quad +\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{FF'}}{}_{\boldsymbol{DD'}}\Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{FF'}} - \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{FF'}}{}_{\boldsymbol{DD'}}\Gamma_{\boldsymbol{BB'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{FF'}}\\ &\quad -\Sigma_{\boldsymbol{AA'}}{}^{\boldsymbol{FF'}}{}_{\boldsymbol{BB'}}\Gamma_{\boldsymbol{FF'}}{}^{\boldsymbol{CC'}}{}_{\boldsymbol{DD'}}.\end{aligned}$$

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Now, making use of the decomposition (3.32) for the spin connection coefficients, one obtains after a lengthy, but straightforward calculation that

$$R^{\boldsymbol{C}\boldsymbol{C}'}{}_{\boldsymbol{D}\boldsymbol{D}'\boldsymbol{A}\boldsymbol{A}'\boldsymbol{B}\boldsymbol{B}'} = R^{\boldsymbol{C}}{}_{\boldsymbol{D}\boldsymbol{A}\boldsymbol{A}'\boldsymbol{B}\boldsymbol{B}'}\delta_{\boldsymbol{D}'}{}^{\boldsymbol{C}'} + \bar{R}^{\boldsymbol{C}'}{}_{\boldsymbol{D}'\boldsymbol{A}\boldsymbol{A}'\boldsymbol{B}\boldsymbol{B}'}\delta_{\boldsymbol{D}}{}^{\boldsymbol{C}}, \tag{3.35}$$

where

$$\begin{aligned} R^{\boldsymbol{C}}{}_{\boldsymbol{D}\boldsymbol{A}\boldsymbol{A}'\boldsymbol{B}\boldsymbol{B}'} \equiv\ & \boldsymbol{e}_{\boldsymbol{A}\boldsymbol{A}'}(\Gamma_{\boldsymbol{B}\boldsymbol{B}'}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}) - \boldsymbol{e}_{\boldsymbol{B}\boldsymbol{B}'}(\Gamma_{\boldsymbol{A}\boldsymbol{A}'}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}) \\ & -\Gamma_{\boldsymbol{F}\boldsymbol{B}'}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\Gamma_{\boldsymbol{A}\boldsymbol{A}'}{}^{\boldsymbol{F}}{}_{\boldsymbol{B}} - \Gamma_{\boldsymbol{B}\boldsymbol{F}'}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\bar{\Gamma}_{\boldsymbol{A}\boldsymbol{A}'}{}^{\boldsymbol{F}'}{}_{\boldsymbol{B}'} + \Gamma_{\boldsymbol{F}\boldsymbol{A}'}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\Gamma_{\boldsymbol{B}\boldsymbol{B}'}{}^{\boldsymbol{F}}{}_{\boldsymbol{A}} \\ & +\Gamma_{\boldsymbol{A}\boldsymbol{F}'}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}\bar{\Gamma}_{\boldsymbol{B}\boldsymbol{B}'}{}^{\boldsymbol{F}'}{}_{\boldsymbol{A}'} + \Gamma_{\boldsymbol{A}\boldsymbol{A}'}{}^{\boldsymbol{C}}{}_{\boldsymbol{F}}\Gamma_{\boldsymbol{B}\boldsymbol{B}'}{}^{\boldsymbol{F}}{}_{\boldsymbol{D}} - \Gamma_{\boldsymbol{B}\boldsymbol{B}'}{}^{\boldsymbol{C}}{}_{\boldsymbol{F}}\Gamma_{\boldsymbol{A}\boldsymbol{A}'}{}^{\boldsymbol{F}}{}_{\boldsymbol{D}} \\ & -\Sigma_{\boldsymbol{A}\boldsymbol{A}'}{}^{\boldsymbol{F}\boldsymbol{F}'}{}_{\boldsymbol{B}\boldsymbol{B}'}\Gamma_{\boldsymbol{F}\boldsymbol{F}'}{}^{\boldsymbol{C}}{}_{\boldsymbol{D}}. \end{aligned}$$

This last expression can be regarded as the *spinorial counterpart of the first Cartan structure equation*; see Equation (2.31).

The commutator of covariant derivatives on arbitrary spinors

The commutator expression (3.34) applies only to spinors arising from a tensorial counterpart. In this section this commutator expression is applied to arbitrary valence spinors. In order to do this, observe that Equation (3.35) also holds if expressed in terms of abstract spinorial indices. More precisely, one has that

$$R^{CC'}{}_{DD'AA'BB'} = R^{C}{}_{DAA'BB'}\delta_{D'}{}^{C'} + \bar{R}^{C'}{}_{D'AA'BB'}\delta_{D}{}^{C}, \tag{3.36}$$

where, in general $R_{CDAA'BB'} \neq R_{(CD)AA'BB'}$.

Applying the commutator (3.34) to the particular case when $\xi^{CC'} = \epsilon_{\boldsymbol{D}}{}^{C}\epsilon_{\boldsymbol{D}'}{}^{C'}$ one obtains, after taking into account the split (3.36), that

$$\begin{aligned} &\epsilon_{\boldsymbol{D}'}{}^{C'}[\![\nabla_{AA'},\nabla_{BB'}]\!]\epsilon_{\boldsymbol{D}}{}^{C} + \epsilon_{\boldsymbol{D}}{}^{C}[\![\nabla_{AA'},\nabla_{BB'}]\!]\epsilon_{\boldsymbol{D}'}{}^{C'} \\ &\qquad = \epsilon_{\boldsymbol{D}'}{}^{C'}R^{C}{}_{DAA'BB'}\epsilon_{\boldsymbol{D}}{}^{D} + \epsilon_{\boldsymbol{D}}{}^{C}\bar{R}^{C'}{}_{D'AA'BB'}\epsilon_{\boldsymbol{D}'}{}^{D'}. \end{aligned}$$

From the latter one can conclude that

$$\begin{aligned} [\![\nabla_{AA'},\nabla_{BB'}]\!]\epsilon_{\boldsymbol{D}}{}^{C} &= R^{C}{}_{QAA'BB'}\epsilon_{\boldsymbol{D}}{}^{Q}, \\ [\![\nabla_{AA'},\nabla_{BB'}]\!]\epsilon_{\boldsymbol{D}'}{}^{C'} &= \bar{R}^{C'}{}_{Q'AA'BB'}\epsilon_{\boldsymbol{D}'}{}^{Q'}. \end{aligned}$$

Now, using that $\epsilon_{\boldsymbol{P}}{}^{C}\epsilon^{\boldsymbol{Q}}{}_{C} = \delta_{\boldsymbol{P}}{}^{\boldsymbol{Q}}$, and that $[\![\nabla_{AA'},\nabla_{BB'}]\!]\delta_{\boldsymbol{P}}{}^{\boldsymbol{Q}} = 0$, one finds that

$$\epsilon_{\boldsymbol{P}}{}^{C}[\![\nabla_{AA'},\nabla_{BB'}]\!]\epsilon^{\boldsymbol{Q}}{}_{C} = -\epsilon^{\boldsymbol{Q}}{}_{C}[\![\nabla_{AA'},\nabla_{BB'}]\!]\epsilon_{\boldsymbol{P}}{}^{C} \tag{3.37a}$$

$$= -\epsilon^{\boldsymbol{Q}}{}_{C}R^{C}{}_{DAA'BB'}\epsilon_{\boldsymbol{P}}{}^{D}. \tag{3.37b}$$

Multiplying the previous expression by $\epsilon^{\boldsymbol{P}}{}_{D}$ and using that $\epsilon^{\boldsymbol{P}}{}_{D}\epsilon_{\boldsymbol{P}}{}^{C} = \delta_{D}{}^{C}$ one obtains

$$\delta_{D}{}^{C}[\![\nabla_{AA'},\nabla_{BB'}]\!]\epsilon^{\boldsymbol{Q}}{}_{C} = -\epsilon^{\boldsymbol{Q}}{}_{C}R^{C}{}_{QAA'BB'}\delta_{D}{}^{Q}.$$

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Finally, using that $[\![\nabla_{AA'},\nabla_{BB'}]\!]\delta_D{}^C=0$, one concludes that

$$[\![\nabla_{AA'},\nabla_{BB'}]\!]\epsilon^{\boldsymbol{Q}}{}_D=-R^C{}_{DAA'BB'}\epsilon^{\boldsymbol{Q}}{}_C. \tag{3.38}$$

A similar argument applied to primed basis spinors yields

$$[\![\nabla_{AA'},\nabla_{BB'}]\!]\epsilon^{\boldsymbol{Q}'}{}_{D'}=-\bar{R}^{C'}{}_{D'AA'BB'}\epsilon^{\boldsymbol{Q}'}{}_{C'}. \tag{3.39}$$

Now, using that $[\![\nabla_{AA'},\nabla_{BB'}]\!]$ applied to a scalar is zero, one has that Equations (3.37a), (3.37b), (3.38) and (3.39) render the following formulae for arbitrary valence 1 spinors:

$$[\![\nabla_{AA'},\nabla_{BB'}]\!]\mu^C=R^C{}_{QAA'BB'}\mu^Q, \tag{3.40a}$$

$$[\![\nabla_{AA'},\nabla_{BB'}]\!]\bar{\lambda}^{C'}=\bar{R}^{C'}{}_{Q'AA'BB'}\bar{\lambda}^{Q'}, \tag{3.40b}$$

$$[\![\nabla_{AA'},\nabla_{BB'}]\!]\kappa_C=-R^Q{}_{CAA'BB'}\kappa_Q, \tag{3.40c}$$

$$[\![\nabla_{AA'},\nabla_{BB'}]\!]\bar{\nu}_{C'}=-\bar{R}^{Q'}{}_{C'AA'BB'}\bar{\nu}_{Q'}. \tag{3.40d}$$

The extension to higher valence spinors follows from the Leibnitz rule. For example, one has that

$$\begin{aligned}[\![\nabla_{AA'},\nabla_{BB'}]\!]\xi_{CD}{}^{E'} &= -R^Q{}_{CAA'BB'}\xi_{QD}{}^{E'}-R^Q{}_{DAA'BB'}\xi_{CQ}{}^{E'}\\ &\quad+\bar{R}^{E'}{}_{Q'AA'BB'}\xi_{CD}{}^{Q'}.\end{aligned}$$

3.2.4 Decomposition of a general curvature spinor

Expression (3.36) is a convenient starting point to analyse the decomposition of the curvature spinor in terms of irreducible components. Lowering the index pair ${}_{CC'}$ using the $\boldsymbol{\epsilon}$-spinor one obtains:

$$\begin{aligned}R_{CC'DD'AA'BB'} &= R_{CDAA'BB'}\epsilon_{D'C'}+\bar{R}_{C'D'AA'BB'}\epsilon_{DC}\\ &= -R_{CDAA'BB'}\epsilon_{C'D'}-\bar{R}_{C'D'AA'BB'}\epsilon_{CD}.\end{aligned} \tag{3.41}$$

For the curvature spinor of a general connection one has that $R_{CDAA'BB'}\neq R_{(CD)AA'BB'}$. However, one still has that

$$R_{CDAA'BB'}=-R_{CDBB'AA'}.$$

This antisymmetry can be exploited using the split (3.13) in such a way that the indices ${}_{CD}$ are not touched. Accordingly, one obtains

$$R_{CDAA'BB'}=X_{CDAB}\epsilon_{A'B'}+Y_{CDA'B'}\epsilon_{AB}, \tag{3.42}$$

where

$$X_{CDAB}=X_{CD(AB)}\equiv\frac{1}{2}R_{CDAQ'B}{}^{Q'},$$

$$Y_{CDA'B'}=Y_{CD(A'B')}\equiv\frac{1}{2}R_{CDA'QB'}{}^{Q}.$$

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To complete the decomposition of the curvature spinor in irreducible components one can apply the decomposition formulae (3.8) and (3.12) for valence 4 spinors to X_{CDAB} and $Y_{CDA'B'}$. This idea will not be pursued any further here. However, it will be convenient to single out certain components of the decomposition in irreducible terms of X_{CDAB}. It is conventional to set

$$\Psi_{ABCD} \equiv X_{(ABCD)}, \qquad \Lambda \equiv \frac{1}{6} X_{PQ}{}^{PQ}.$$

Let $C_{CC'DD'AA'BB'}$ denote the spinor obtained from the split (3.41) of the curvature spinor by setting $X_{ABCD} = X_{(ABCD)}$ and $Y_{CDA'B'} = 0$. One has that

$$C_{CC'DD'AA'BB'} = -\Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} - \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}. \tag{3.43}$$

As a consequence of the total symmetry of Ψ_{ABCD} it can be readily verified that $C_{CC'DD'AA'BB'}$ is the spinorial counterpart of a trace-free tensor. Following the discussion in Section 2.5.2, it must be the ***spinorial counterpart of the Weyl tensor*** C_{cdab}.

Decomposition of the curvature spinor of a torsion-free connection

The decomposition of the curvature spinor is now particularised to the case of a torsion-free connection. In this case, the Riemann curvature tensor has the cyclic symmetry of the Bianchi identity. The latter is best exploited using the alternative expression of the identity given by Equation (2.23) involving the right-dual of the Riemann tensor. Using the spinorial counterpart of the volume form given by Equation (3.25) one has that

$$\begin{aligned} R^*_{AA'BB'CC'DD'} &= \frac{\mathrm{i}}{2}(\delta_C{}^E\delta_D{}^F\delta_{C'}{}^{F'}\delta_{D'}{}^{E'} - \delta_C{}^F\delta_D{}^E\delta_{C'}{}^{E'}\delta_{D'}{}^{F'})R_{AA'BB'EE'FF'} \\ &= \mathrm{i}R_{AA'BB'CD'DC'}, \end{aligned}$$

so that the spinorial counterpart of Equation (2.23) is given by

$$R_{CC'QQ'A}{}^{Q'Q}{}_{A'} = 0.$$

A direct evaluation of the above condition using the splits (3.41) and (3.42) shows that

$$X_{CQA}{}^{Q}\epsilon_{C'A'} - \bar{X}_{C'Q'A'}{}^{Q'}\epsilon_{CA} + Y_{CAC'A'} - \bar{Y}_{C'A'AC} = 0,$$

so that

$$X_{PQ}{}^{PQ} = \bar{X}_{PQ}{}^{PQ}, \qquad \bar{Y}_{C'A'AC} = Y_{CAC'A'}.$$

Hence, one has that $X_{PQ}{}^{PQ}$ (i.e. Λ) is a real scalar, while $Y_{ABA'B'}$ is a Hermitian tensor, and, thus, it is the spinorial counterpart of a rank 2 tensor.

Decomposition on the curvature spinor of a metric connection

As already seen, a connection which is compatible with a metric $\boldsymbol{g}$ satisfies $\nabla_{AA'}\epsilon_{CD} = 0$. It follows then that $[\![\nabla_{AA'}, \nabla_{BB'}]\!]\epsilon_{CD} = 0$. However, one also has that

$$[\![\nabla_{AA'}, \nabla_{BB'}]\!]\epsilon_{CD} = -R^{Q}{}_{CAA'BB'}\epsilon_{QD} + R^{Q}{}_{DAA'BB'}\epsilon_{CQ},$$

from which one concludes that

$$R_{CDAA'BB'} = R_{(CD)AA'BB'}.$$

The latter can be reexpressed in terms of the following symmetries of the spinors X_{ABCD} and $Y_{ABA'B'}$:

$$X_{ABCD} = X_{(AB)CD}, \qquad Y_{ABA'B'} = Y_{(AB)A'B'}.$$

Decomposition of the curvature spinor of a Levi-Civita connection

Finally, one can collect the results of the previous subsections to obtain the well-known irreducible decomposition of the spinorial counterpart of the Riemann tensor of a Levi-Civita connection. As the Levi-Civita connection associated to the metric $\boldsymbol{g}$ is both torsion-free and metric, it follows then that

$$X_{ABCD} = X_{(AB)(CD)}, \qquad X_{CQA}{}^{Q} = 0.$$

It follows from (3.8) that $X_{ABCD} = X_{CDAB}$ and that

$$\begin{aligned} X_{ABCD} &= X_{(ABCD)} - \frac{1}{3}\epsilon_{A(C}\epsilon_{D)B}X_{PQ}{}^{PQ} \\ &= \Psi_{ABCD} + \Lambda(\epsilon_{DB}\epsilon_{CA} + \epsilon_{CB}\epsilon_{DA}). \end{aligned}$$

Similarly, for $Y_{ABA'B'}$ one has that

$$Y_{ABA'B'} = Y_{(AB)(A'B')},$$

so that according to the general split (3.12) $Y_{ABA'B'}$ corresponds to a trace-free rank 2 tensor.

To conclude the analysis, it is convenient to compute the Ricci tensor and scalar in terms of the spinors X_{ABCD} and $Y_{ABA'B'}$. From Equations (3.41) and (3.42) it follows directly that

$$\begin{aligned} &R_{AA'BB'} = -X_{QA}{}^{Q}{}_{B}\epsilon_{A'B'} - \bar{X}_{Q'A'}{}^{Q'}{}_{B'}\epsilon_{AB} + 2Y_{ABA'B'} \\ &R = -4X_{PQ}{}^{PQ}, \end{aligned}$$

where $R_{AA'BB'}$ denotes the ***spinorial counterpart of the Ricci tensor*** R_{ab} and it has been used that for a Levi-Civita connection $Y_{ABA'B'} = \bar{Y}_{A'B'AB}$ and $X_{PQ}{}^{PQ} = \bar{X}_{P'Q'}{}^{P'Q'}$. In particular, one has that

$$R = -24\Lambda.$$

As $Y_{ABA'B'}$ is trace-free, it has to be related to Φ_{ab}, the symmetric trace-free part of the Ricci tensor. Indeed, a calculation for its spinorial counterpart shows that

$$\begin{aligned} 2\Phi_{ABA'B'} &\equiv R_{AA'BB'} - \frac{1}{4}R\epsilon_{AB}\epsilon_{A'B'} \\ &= 2Y_{ABA'B'}. \end{aligned}$$

It can be verified that $\Phi_{ABA'B'}$ satisfies the symmetries

$$\Phi_{ABA'B'} = \Phi_{BAA'B'} = \Phi_{ABB'A'} = \Phi_{BAB'A'}. \tag{3.44}$$

Putting together the discussion of this section, one finds that the spinor counterpart of the Riemann curvature tensor of a Levi-Civita connection can be decomposed as

$$\begin{aligned} R_{AA'BB'CC'DD'} = &-\epsilon_{A'B'}\epsilon_{C'D'}(\Psi_{ABCD} + 2\Lambda\epsilon_{A(C}\epsilon_{D)B}) \\ &-\epsilon_{AB}\epsilon_{CD}(\bar{\Psi}_{A'B'C'D'} + 2\Lambda\epsilon_{A'(C'}\epsilon_{D')B'}) \\ &+\epsilon_{A'B'}\epsilon_{CD}\Phi_{ABC'D'} + \epsilon_{AB}\epsilon_{C'D'}\Phi_{CDA'B'}. \end{aligned}$$

Working back from this expression one can recover the decomposition of the Riemann tensor in terms of the Weyl and Schouten tensor given in Equations (2.21a) and (2.21b).

3.2.5 The $\square_{AB}$-operator

In some applications it is convenient to have a more explicit expression for the commutator of spinorial covariant derivatives. *In the remainder of this section it is assumed that $\nabla_{AA'}$ is the spinorial counterpart of a Levi-Civita connection.*

Exploiting the antisymmetry of Equation (3.40a) with respect to the pairs ${}_{AA'}$ and ${}_{BB'}$ one can rewrite it as

$$\big(\epsilon_{A'B'}\square_{AB} + \epsilon_{AB}\square_{A'B'}\big)\mu^C = R^C{}_{QAA'BB'}\mu^Q, \tag{3.45}$$

where

$$\square_{AB} \equiv \nabla_{Q'(A}\nabla_{B)}{}^{Q'}, \qquad \square_{A'B'} \equiv \nabla_{Q(A'}\nabla^Q{}_{B')}.$$

It can be verified that both $\square_{AB}$ and $\square_{A'B'}$ are linear and satisfy the Leibnitz rule – one has, for example, that

$$\square_{AB}(\mu_C\lambda^D) = (\square_{AB}\mu_C)\lambda^D + \mu_C(\square_{AB}\lambda^D).$$

Defining the ***D'Alembertian operator*** as $\square \equiv \nabla_{PP'}\nabla^{PP'}$, one obtains the decomposition

$$\nabla_{AQ'}\nabla_B{}^{Q'} = \frac{1}{2}\epsilon_{AB}\square + \square_{AB}.$$

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Now, contracting indices suitably in Equation (3.45) one readily obtains

$$\square_{AB}\mu^C = X^C{}_{QAB}\mu^Q, \qquad \square_{A'B'}\mu^C = Y^C{}_{QA'B'}\mu^Q.$$

Using the explicit expressions for the curvature spinors X_{ABCD} and $Y_{ABA'B'}$ for a Levi-Civita connection, as given in Section 3.2.4 one concludes that

$$\square_{AB}\mu_C = \Psi_{ABCD}\mu^D - 2\Lambda\mu_{(A}\epsilon_{B)C}, \qquad \square_{A'B'}\mu_C = \Phi_{CDA'B'}\mu^D. \tag{3.46}$$

The above expressions can be extended to higher order valence spinors by means of the Leibnitz rule.

The expressions in (3.46) can be extended to the case of connections with torsion; see Penrose (1983) for the general theory and Gasperín and Valiente Kroon (2015) for explicit expressions and applications.

3.3 Global considerations

The discussion on null vectors and their flagpoles in Section 3.1.10 makes a natural connection with the notion of orientability and the assumptions needed to ensure the existence of spinorial structures on a region of spacetime.

As seen in Proposition 3.3, every non-vanishing null vector is either future pointing or past pointing, in accordance with the choice of sign made in Equation (3.21). Thus, the existence of spinors on a region of spacetime provides a way to define a time orientation. In a similar way, the idea of a right-handed phase change of a triad of orthonormal vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, as discussed in Section 3.1.10, can be used to define a notion of space orientation. Thus, at least at an intuitive level, the existence of a spinorial structure over a spacetime seems to imply that the spacetime is time orientable and space orientable. It turns out that the converse is also true: time and space orientability ensure the existence of a spinorial structure. More precisely, one has the following result proved in Geroch (1968):

Theorem 3.2 (***orientability and the existence of a spinor structure***) *A non-compact spacetime $(\mathcal{M}, \boldsymbol{g})$ has a spinor structure if and only if there exists on $\mathcal{M}$ a global system of orthonormal tetrads.*

Part IV of this book will be concerned with the construction of spacetimes from suitably posed initial value problems. Thus, it is convenient to have a criterion to encode the existence of a spinorial structure in an initial value problem. An example of this is the following result in Geroch (1970c):

Proposition 3.4 (***global hyperbolicity and the existence of a spinor structure***) *Every globally hyperbolic spacetime has a spinor structure.*

The notion of global hyperbolicity is discussed in Section 14.1.

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An orientable spacetime may have several spinorial structures. One can ensure uniqueness of the spinorial structure if one restricts further the topology of the spacetime. More precisely, one has that (see Geroch (1968)):

Proposition 3.5 (***uniqueness of the spinorial structure***) *The spinorial structure of a spacetime is unique if and only if $\mathcal{M}$ is simply connected.*

3.4 Further reading

Further details on the various topics covered in the present chapter can be found in Penrose and Rindler (1984), Stewart (1991) and O'Donnell (2003). The discussion in these references leads, in a natural way, to the *Newman-Penrose formalism* and applications like the *Petrov algebraic classification of the Weyl tensor*. Some discussion on the use of spinors in the construction and analysis of exact solutions to the Einstein field equations can be found in Stephani et al. (2003) and Griffiths and Podolský (2009). The relation between Dirac spinors and 2-spinors is presented in Penrose and Rindler (1984) and Stewart (1991). A pure mathematics perspective can be found, for example, in Petersen (1991); see also Choquet-Bruhat et al. (1982).

A more general perspective of the discussion of the present chapter can be obtained by making use of the notion of *fibre bundles*; see, for example, Ashtekar et al. (1982). In terms of this language, the spinorial structure arises as a *principal fibre bundle* over the spacetime manifold $\mathcal{M}$ with structure group $SL(2,\mathbb{C})$. This point of view is convenient for computer algebra implementations; see, for example, Martín-García (2014). The fibre bundles are useful in analyses that require the *blowing up* of particular points of spacetime – as in the analysis of caustics in Friedrich and Stewart (1983) or the so-called problem of spatial infinity of Friedrich (1998c).

Appendix: the Newman-Penrose formalism

The idea of a spinor-based null tetrad formalism was introduced in the seminal article by Newman and Penrose (1962); see also Newman and Penrose (1963). This so-called ***Newman-Penrose (NP) formalism*** was first used as a way of analysing the asymptotics of gravitational radiation. The potential of the formalism to obtain exact solutions to the Einstein field equations, in particular, ones having an algebraically special Weyl tensor, was quickly realised; see, for example, Stephani et al. (2003) for an entry point to the literature of exact solutions. Refinements of the formalism which are adapted to specific configurations or types of problems are available in the literature, most noticeably Geroch et al. (1973); see also Machado and Vickers (1995, 1996).

The key aspects of a generic spinor-based null tetrad formalism have already been covered in this book. One of the peculiarities of the formalism, as introduced in Newman and Penrose (1962), is the use of specific symbols to denote directional derivatives and the spin coefficients. This notation will not be used in this book as the Newman-Penrose (NP) formalism assumes, from the onset, a Levi-Civita connection. However, the discussion in this book will very often use more general connections. Hence, one has more independent spin coefficients. Moreover, the labelling of spin coefficients through indices lends itself better for a systematic analysis of the properties of the relevant equations. Additional difficulties with the NP formalism arise with the space spinor formalism; see the next chapter.

The purpose of this appendix is to provide a guide to the translation, whenever possible, between NP objects and the ones used in this book.

The directional derivatives

Let $\{\boldsymbol{o}, \boldsymbol{\iota}\}$ denote, as usual, a spin basis. Also, let $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ denote the null tetrad constructed from the spin basis, as described in Section 3.1.9. The NP convention for the directional derivatives along the directions given by the null tetrad is

$$
\begin{aligned}
D &\equiv l^a \nabla_a = o^A \bar{o}^{A'} \nabla_{AA'} = \nabla_{\mathbf{00'}},\\
\Delta &\equiv n^a \nabla_a = \iota^A \bar{\iota}^{A'} \nabla_{AA'} = \nabla_{\mathbf{11'}},\\
\delta &\equiv m^a \nabla_a = o^A \bar{\iota}^{A'} \nabla_{AA'} = \nabla_{\mathbf{01'}},\\
\bar{\delta} &\equiv \bar{m}^a \nabla_a = \iota^A \bar{o}^{A'} \nabla_{AA'} = \nabla_{\mathbf{10'}}.
\end{aligned}
$$

The spin coefficients

In what follows, it is assumed that the connection $\boldsymbol{\nabla}$ is Levi-Civita so that $\nabla_{AA'}\epsilon_{BC} = 0$. The NP convention for the spin coefficients of $\boldsymbol{\nabla}$ is given by:

$$
\begin{aligned}
\epsilon &= \Gamma_{\mathbf{00'}}{}^{\mathbf{0}}{}_{\mathbf{0}} = -\Gamma_{\mathbf{00'}}{}^{\mathbf{1}}{}_{\mathbf{1}} = \Gamma_{\mathbf{00'10}},\\
\alpha &= \Gamma_{\mathbf{10'}}{}^{\mathbf{0}}{}_{\mathbf{0}} = -\Gamma_{\mathbf{10'}}{}^{\mathbf{1}}{}_{\mathbf{1}} = \Gamma_{\mathbf{10'10}},\\
\beta &= \Gamma_{\mathbf{01'}}{}^{\mathbf{0}}{}_{\mathbf{0}} = -\Gamma_{\mathbf{01'}}{}^{\mathbf{1}}{}_{\mathbf{1}} = \Gamma_{\mathbf{01'10}},\\
\gamma &= \Gamma_{\mathbf{11'}}{}^{\mathbf{0}}{}_{\mathbf{0}} = -\Gamma_{\mathbf{11'}}{}^{\mathbf{1}}{}_{\mathbf{1}} = \Gamma_{\mathbf{11'10}},
\end{aligned}
$$

$$
\begin{aligned}
\pi &= \Gamma_{\mathbf{00'}}{}^{\mathbf{0}}{}_{\mathbf{1}} = \Gamma_{\mathbf{00'11}}, & \kappa &= -\Gamma_{\mathbf{00'}}{}^{\mathbf{1}}{}_{\mathbf{0}} = \Gamma_{\mathbf{00'00}},\\
\lambda &= \Gamma_{\mathbf{10'}}{}^{\mathbf{0}}{}_{\mathbf{1}} = \Gamma_{\mathbf{10'11}}, & \rho &= -\Gamma_{\mathbf{10'}}{}^{\mathbf{1}}{}_{\mathbf{0}} = \Gamma_{\mathbf{10'00}},\\
\mu &= \Gamma_{\mathbf{01'}}{}^{\mathbf{0}}{}_{\mathbf{1}} = \Gamma_{\mathbf{01'11}}, & \sigma &= -\Gamma_{\mathbf{01'}}{}^{\mathbf{1}}{}_{\mathbf{0}} = \Gamma_{\mathbf{01'00}},\\
\nu &= \Gamma_{\mathbf{11'}}{}^{\mathbf{0}}{}_{\mathbf{1}} = \Gamma_{\mathbf{11'11}}, & \tau &= -\Gamma_{\mathbf{11'}}{}^{\mathbf{1}}{}_{\mathbf{0}} = \Gamma_{\mathbf{11'00}}.
\end{aligned}
$$

The above spin coefficients can be expressed entirely in terms of the directional derivatives D, Δ, δ, $\bar{\delta}$ applied to the null frame vectors or, alternatively, applied to the spin basis $\{\boldsymbol{o}, \boldsymbol{\iota}\}$. See O'Donnell (2003) and Stewart (1991) for details on this. Explicit expressions of the spin coefficients in terms of *curls* (antisymmetrised derivatives) have been worked out in Cocke (1989).

The Ricci and Weyl tensors

The NP conventions to denote the components of the Weyl spinor Ψ_{ABCD} with respect to $\{\boldsymbol{o}, \boldsymbol{\iota}\}$ are:

$$\Psi_0 \equiv \Psi_{ABCD} o^A o^B o^C o^D, \quad \Psi_1 \equiv \Psi_{ABCD} o^A o^B o^C \iota^D, \quad \Psi_2 \equiv \Psi_{ABCD} o^A o^B \iota^C \iota^D,$$
$$\Psi_3 \equiv \Psi_{ABCD} o^A \iota^B \iota^C \iota^D, \quad \Psi_4 \equiv \Psi_{ABCD} \iota^A \iota^B \iota^C \iota^D.$$

The conventions for the components of the trace-free Ricci spinor $\Phi_{AA'BB'}$ are:

$$\Phi_{00} \equiv \Phi_{AA'BB'} o^A o^B \bar{o}^{A'} \bar{o}^{B'}, \qquad \Phi_{01} \equiv \Phi_{AA'BB'} o^A o^B \bar{o}^{A'} \bar{\iota}^{B'},$$
$$\Phi_{02} \equiv \Phi_{AA'BB'} o^A o^B \bar{\iota}^{A'} \bar{\iota}^{B'}, \qquad \Phi_{10} \equiv \Phi_{AA'BB'} o^A \iota^B \bar{o}^{A'} \bar{o}^{B'},$$
$$\Phi_{11} \equiv \Phi_{AA'BB'} o^A \iota^B \bar{o}^{A'} \bar{\iota}^{B'}, \qquad \Phi_{12} \equiv \Phi_{AA'BB'} o^A \iota^B \bar{\iota}^{A'} \bar{\iota}^{B'},$$
$$\Phi_{20} \equiv \Phi_{AA'BB'} \iota^A \iota^B \bar{o}^{A'} \bar{o}^{B'}, \qquad \Phi_{21} \equiv \Phi_{AA'BB'} \iota^A \iota^B \bar{o}^{A'} \bar{\iota}^{B'},$$
$$\Phi_{22} \equiv \Phi_{AA'BB'} \iota^A \iota^B \bar{\iota}^{A'} \bar{\iota}^{B'}.$$

Notice that in both lists of definitions the value of the index denotes the number of contractions with the spinor $\boldsymbol{\iota}$.

The NP formalism makes use of the symbol Λ to denote a multiple of the trace of the Ricci tensor. The relation to the Ricci scalar is

$$R = -24\Lambda.$$

The Newman-Penrose field equations

Newman and Penrose (1962) provided explicit expressions of the Ricci and the Bianchi identities in terms of their notation for the spin connection coefficients and the components of Ψ_{ABCD} and $\Phi_{AA'BB'}$. These equations are collectively called the *Newman-Penrose field equations*. Explicit expressions are available in O'Donnell (2003), Penrose and Rindler (1986) and Stewart (1991). Besides the NP field equations, the formalism consists also of explicit expressions for the commutators of the directional derivatives D, Δ, δ, $\bar{\delta}$. Expressions for the source-free Maxwell equations are available in the literature as well; see, for example, the appendix in Stewart (1991).

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