COMPOSITION THEOREMS ON DIRICHLET SERIES

M. D. Penry

(received October 18, 1962)

1. When two uniform functions $f(z) = \sum_1^\infty a_n z^{-\lambda_n - 1}$, $g(z) = \sum_1^\infty b_\nu z^{-\mu_\nu - 1}$ are given, each with a finite radius of absolute convergence R_1, R_2 respectively, and $\{\lambda_n\}$, $\{\mu_\nu\}$ are real positive increasing sequences tending to infinity, a theorem due to Eggleston [1], which is a generalisation of Hurwitz's composition theorem, gives information about the position of the singularities of a composition function $h(z)$, which is assumed to be uniform, in terms of the position of the singularities of $f(z)$ and $g(z)$. This result can be extended to Dirichlet series with real exponents by use of the transformation $z = e^s$.

If $\{a_n\}$, $\{b_\nu\}$ are sequences of complex numbers and if the functions $F(s)$, $G(s)$ and $H(s)$ are given by the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \quad G(s) = \sum_{\nu=1}^{\infty} b_\nu e^{-\mu_\nu s},$$

(1)

$$H(s) = \sum_{n,\nu=1}^{\infty} a_n b_\nu \frac{\Gamma(\lambda_n + \mu_\nu - 1)}{\Gamma(\lambda_n)\Gamma(\mu_\nu)} e^{-(\lambda_n + \mu_\nu - 1)s},$$

the following result comes immediately from Eggleston's theorem:

https://doi.org/10.4153/CMB-1963-034-1 Published online by Cambridge University Press

THEOREM 1. If the Dirichlet series F(s), G(s) have finite abscissae of absolute convergence σ_F, σ_G respectively, then the composition series H(s) is absolutely convergent in $R(s) > \log_e(e^{\sigma_F} + e^{\sigma_G})$. Further H(s) can be continued analytically to all points of the S-plane except those points γ which are such that (i) $\gamma = \log_e(e^{\alpha} + e^{\beta})$ or (ii) γ is separated from the half plane of regularity of H(s) by a singular line. Here α belongs to the closure of the singularities of F(s) and β to the closure of the singularities of G(s). The expressions $\log_e(e^{\alpha} + e^{\beta})$ are to be taken in all their determinations.

2. Although Theorem 1 has no immediate transformation into a composition theorem on Dirichlet series with complex exponents, there are certain special cases of this type of series for which we can consider the absolute convergence of the composition series H(s) of the form given in (1). We consider here two such special cases.

In the proof of Theorem 2 we will need the following lemma, which includes a result due to Schwengeler [3].

LEMMA 1*. If the sequence of complex numbers $\{\lambda_n\}$ is bounded and the series

$$F(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$$

absolutely convergent at a point s_o, then for any other s there exists a square with centre at s_o within which the series is absolutely and uniformly convergent. Further, the series F(s) represents an integral function

We can assume without loss of generality that the point at which the series is absolutely convergent is the origin. The

---

* I am indebted to the referee for the form taken by this lemma.

https://doi.org/10.4153/CMB-1963-034-1 Published online by Cambridge University Press

series is then absolutely convergent at a point s if $|e^{-\lambda_n s}|$ is bounded for all n.

If $\lambda_n = \alpha_n + i\beta_n$ and $s = \sigma + i\tau$, we have

$$|\exp\{-\lambda_n s\}| = \exp\{-R(\lambda_n s)\} = \exp\{\tau\beta_n - \sigma\alpha_n\}$$
$$\leq \exp\{|\tau||\beta_n| + |\sigma||\alpha_n|\} \leq \exp\{|\sigma| + |\tau|\},$$

since we may assume $|\lambda_n| \leq 1$.

It follows that $|e^{-\lambda_n s}|$ is bounded if there exists a constant K such that

$$\exp\{|\sigma| + |\tau|\} \leq K,$$

or else

$$|\sigma| + |\tau| \leq \log K = K'.$$

It is clear that once $s = \sigma + i\tau$ is known, K can be chosen, but will depend on s. The series is therefore absolutely convergent when s lies within the square

$$|\sigma| + |\tau| \leq K',$$

and hence, according to Hille [2], the series is uniformly convergent inside this square.

If we allow K to become large, this square covers the whole finite plane and we have Schwengeler's result that the series $F(s)$ represents an integral function.

We now examine the two special cases mentioned above.
(a) Suppose $F(s)$, $G(s)$ represent two Dirichlet series with sequences of complex exponents $\{\lambda_n\}$, $\{\mu_\nu\}$, respectively, such that both sequences are bounded, i.e. $|\lambda_n| < \lambda$, $|\mu_\nu| < \mu$ and that both $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{\nu=1}^{\infty} |b_\nu|$ are convergent. Then if the composition series $H(s)$ is given by (1) we have the following result:

https://doi.org/10.4153/CMB-1963-034-1 Published online by Cambridge University Press

THEOREM 2. <u>The composition series $H(s)$ of (1) represents an integral function, provided only that for all n and ν, $(\lambda_n+\mu_\nu-1)$ is not zero or a negative integer, and does not have any of these values as a limit point.</u>

It will be sufficient for the proof of the theorem if we show that $H(s)$ is of the same form as $F(s)$ and $G(s)$. That is, we will show that the series representing $H(s)$ is absolutely convergent at one point, and the exponents of this series are bounded. The conditions of Lemma 1 thus are satisfied, and $H(s)$ is an integral function.

Since $|\lambda_n| < \lambda$ and $|\mu_\nu| < \mu$, the ratio

$$\left|\frac{\Gamma(\lambda_n+\mu_\nu-1)}{\Gamma(\lambda_n)\cdot\Gamma(\mu_\nu)}\right|$$

will be bounded because of the conditions imposed on $(\lambda_n+\mu_\nu-1)$. Also $\Sigma|a_n b_\nu|$ is convergent because of the convergence of $\Sigma|a_n|$ and $\Sigma|b_\nu|$. Thus the series

$$\Sigma\left|a_n b_\nu \frac{\Gamma(\lambda_n+\mu_\nu-1)}{\Gamma(\lambda_n)\,\Gamma(\mu_\nu)}\right|$$

is convergent.

Since the exponents $\{\lambda_n+\mu_\nu-1\}$ of $H(s)$ are bounded, $H(s)$ is of the same form as $F(s)$ and $G(s)$ and therefore, by Lemma 1, represents an integral function.

It is worth noting that the condition that $(\lambda_n+\mu_\nu-1)$ does not have zero or a negative integer as a limit point is necessary. This may be seen by considering the case where $\lambda_n = \frac{1}{2}+\frac{1}{n}$, $\mu_\nu = \frac{1}{2}+\frac{1}{\nu}$, when $\Gamma(\lambda_n+\mu_\nu-1)$ is not bounded.

https://doi.org/10.4153/CMB-1963-034-1 Published online by Cambridge University Press

(b) Suppose now that the Dirichlet series $F(s) = \sum_{1}^{\infty} a_n e^{-\lambda_n s}$ is of the form considered in (a) (i.e. $\Sigma |a_n|$ convergent and $|\lambda_n| < \lambda$) and that the Dirichlet series $G(s) = \sum_{1}^{\infty} b_\nu e^{-\mu_\nu s}$ is such that $\Sigma |b_\nu|$ is convergent, and $\lim_{\nu \to \infty} |\mu_\nu| = \infty$. We have the following result concerning the region of absolute convergence of the composition series $H(s)$ given by (1).

THEOREM 3. If the Dirichlet series $F(s)$ and $G(s)$ have the property that there exists a $\delta > 0$ such that for all n and ν

$$|\arg(\lambda_n + \mu_\nu - 2)| < \pi - \delta, \quad |\arg(\lambda_n + \mu_\nu - 1)| \leq \pi - \delta,$$

$$|\arg(\mu_\nu - 1)| \leq \pi - \delta, \quad |\arg \mu_\nu| \leq \pi - \delta,$$

and if further $R(\lambda_n - 1) \leq 0$ for all n, then the Dirichlet series $H(s)$ is absolutely convergent at least in the region of absolute convergence S_G of $G(s)$, with perhaps the exception of the point at infinity if this belongs to S_G.

Consider

$$(3) \qquad H(s) = \sum_{n,\nu=1}^{\infty} a_n b_\nu \frac{\Gamma(\lambda'_n + \mu'_\nu + 1)}{\Gamma(\lambda'_n + 1)\,\Gamma(\mu'_\nu + 1)} e^{-(\lambda'_n + \mu'_\nu + 1)s}$$

where we have put

$$\lambda'_n + 1 = \lambda_n, \quad \mu'_\nu + 1 = \mu_\nu .$$

From the asymptotic expansion [4]

$$\log \Gamma(z+a) = (z + a - \tfrac{1}{2}) \log z - z + \tfrac{1}{2} \log 2\pi + o(1),$$

(where the principal value of the logarithm is taken) valid for

$$|\arg z| \leq \pi - \delta, \quad |\arg(z+a)| \leq \pi - \delta \quad (\delta > 0), \quad 0 < a \leq 1$$

https://doi.org/10.4153/CMB-1963-034-1 Published online by Cambridge University Press

and $|z|$ sufficiently large, we have, for sufficiently large $|\mu'_\nu|$,

$$\log \frac{\Gamma(\lambda'_n+\mu'_\nu+1)}{\Gamma(\mu'_\nu+1)} = \lambda'_n \log(\lambda'_n+\mu'_\nu) + (\mu'_\nu+\frac{1}{2}) \log(1+\frac{\lambda'_n}{\mu'_\nu}) - \lambda'_n + o(1).$$

Consider the expression

$$(4) \qquad (\mu'_\nu+\frac{1}{2}) \log(1+\frac{\lambda'_n}{\mu'_\nu}) - \lambda'_n .$$

The function log (1+z) is analytic for $|z| < 1$ and has the Taylor expansion

$$\log(1+z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1} .$$

Thus for sufficiently large $|\mu'_\nu|$, (4) has the form

$$(\mu'_\nu+\frac{1}{2}) \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left\{\frac{\lambda'_n}{\mu'_\nu}\right\}^{n+1} - \lambda'_n = o(1) .$$

Thus

$$(5) \qquad \log \frac{\Gamma(\lambda'_n+\mu'_\nu+1)}{\Gamma(\mu'_\nu+1)} = R(\lambda'_n) \log|\lambda'_n+\mu'_\nu| - \mathcal{I}(\lambda'_n) \arg(\lambda'_n+\mu'_\nu)$$

$$+ i[R(\lambda'_n)\arg(\lambda'_n+\mu'_\nu) + \mathcal{I}(\lambda'_n)\log|\lambda'_n+\mu'_\nu|] + o(1) .$$

We see therefore that the real part of (5) is governed by the term $R(\lambda'_n) \log|\lambda'_n+\mu'_\nu|$, for the second term is bounded since $0 \leq |\arg(\lambda'_n+\mu'_\nu)| < \pi$.

Hence, provided that $R(\lambda'_n) \leq 0$, we see that

https://doi.org/10.4153/CMB-1963-034-1 Published online by Cambridge University Press

$$\frac{\Gamma(\lambda'_n+\mu'_\nu+1)}{\Gamma(\mu'_\nu+1)}$$

is bounded, and so the series (3) converges absolutely for $s = 0$, provided also that $(\lambda'_n+\mu'_\nu+1)$ is not zero or a negative integer. The series $H(s)$ is therefore of the same form as $G(s)$.

The series $H(s)$ obviously converges absolutely in the region of absolute convergence of $G(s)$ except perhaps for the point at infinity.

REFERENCES

1. H. G. Eggleston, A generalization of the Hurwitz composition theorem to irregular power series, Proc. Cambridge Philos. Soc. 47 (1951), pp. 477-482.

2. E. Hille, Note on Dirichlet series with complex exponents, Ann. of Math. (2), 25, (1924) pp. 261-78.

3. E. Schwengeler, Geometrisches über die Verteilung der Nullstellen spezieller ganzer Funktionen, Dissertation, Zurich (1925).

4. E. T. Whittaker and G. N. Watson, A course of modern analysis, Cambridge (1902).

University College of Wales,
Aberystwyth

https://doi.org/10.4153/CMB-1963-034-1 Published online by Cambridge University Press