The $SL(2, C)$ Casson Invariant for Knots and the $\widehat{A}$-polynomial

Hans U. Boden and Cynthia L. Curtis

Abstract. In this paper, we extend the definition of the $SL(2, \mathbb{C})$ Casson invariant to arbitrary knots K in integral homology 3-spheres and relate it to the m-degree of the $\widehat{A}$-polynomial of K. We prove a product formula for the $\widehat{A}$-polynomial of the connected sum $K_1 \# K_2$ of two knots in S^3 and deduce additivity of the $SL(2, \mathbb{C})$ Casson knot invariant under connected sums for a large class of knots in S^3. We also present an example of a nontrivial knot K in S^3 with trivial $\widehat{A}$-polynomial and trivial $SL(2, \mathbb{C})$ Casson knot invariant, showing that neither of these invariants detect the unknot.

Introduction

Given a knot $K \subset \Sigma$ in an integral homology 3-sphere, let $M = \Sigma \smallsetminus \tau(K)$ denote the complement of K and let $M_{p/q}$ be the result of p/q surgery on K. In the case where K is a small knot, [12, Theorem 4.8] gives a surgery formula for $\lambda_{SL(2,\mathbb{C})}(M_{p/q})$, and it follows that the difference $\lambda_{SL(2,\mathbb{C})}(M_{p/(q+1)}) - \lambda_{SL(2,\mathbb{C})}(M_{p/q})$ is independent of p, q, provided p and q are relatively prime and q is chosen sufficiently large. The $SL(2, \mathbb{C})$ Casson knot invariant is therefore defined for small knots by setting, for $q \gg 1$,

$$\lambda'_{SL(2,\mathbb{C})}(K) = \lambda_{SL(2,\mathbb{C})}(M_{1/(q+1)}) - \lambda_{SL(2,\mathbb{C})}(M_{1/q}).$$

In this paper, we present a method for defining the invariant $\lambda'_{SL(2,\mathbb{C})}(K)$ more generally for knots in integral homology 3-spheres. Unfortunately, the surgery formula does not hold for non-small knots; the proof breaks down when M contains a closed essential surface. Here, we adopt a different approach and study the asymptotic behavior of $\lambda_{SL(2,\mathbb{C})}(M_{p/q})$ as $q \to \infty$, where the limit is taken over all q relatively prime to p. As a function in q, we prove that $\lambda_{SL(2,\mathbb{C})}(M_{p/q})$ has linear growth, and we define $\lambda'_{SL(2,\mathbb{C})}(K)$ to be the leading coefficient of $\lambda_{SL(2,\mathbb{C})}(M_{p/q})$ as $q \to \infty$.

For small knots there is a close relationship between the knot invariant $\lambda'_{SL(2,\mathbb{C})}(K)$ and the m-degree of the $\widehat{A}$-polynomial of K, which is the A-polynomial with multiplicities as defined by Boyer and Zhang [5]. For instance, in the case of a two-bridge knot K, it is known that $\lambda'_{SL(2,\mathbb{C})}(K) = \frac{1}{2} \deg_m \widehat{A}_K(m, \ell)$; see [4, Section 3.3]. We extend this relationship to the general setting of knots in homology 3-spheres. We show

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that the $\widehat{A}$-polynomial is multiplicative under connected sums in S^3 and deduce additivity of $\lambda'_{SL(2,\mathbb{C})}(K)$ under connected sums for most (conjecturally all) knots in S^3.

We conclude the paper with an example of a nontrivial knot K in S^3 for which the $\widehat{A}$-polynomial and $\lambda'_{SL(2,\mathbb{C})}(K)$ are trivial. Thurston classified knots into three types: torus, hyperbolic, and satellite. For torus and hyperbolic knots, one can show directly that the m-degree of the $\widehat{A}$-polynomial is nontrivial. Using the relationship between $\lambda'_{SL(2,\mathbb{C})}(K)$ and the m-degree of the $\widehat{A}$-polynomial, it then follows that any knot with $\lambda'_{SL(2,\mathbb{C})}(K) = 0$ is necessarily a satellite knot.

We therefore consider satellite knots K given as Whitehead doubles, and we examine the $SL(2,\mathbb{C})$ character variety $X(M)$ of the complement $M = S^3 \smallsetminus \tau(K)$. We show that for many untwisted doubles, apart from the component of reducibles, every other component X_j of $X(M)$ has dimension $\dim X_j > 1$. This implies that $\widehat{A}_K(m,\ell) = 0$, and it shows that the $\widehat{A}$-polynomial does not detect the unknot, answering a question raised in [1]. Using the relationship between the knot invariant $\lambda'_{SL(2,\mathbb{C})}(K)$ and the m-degree of the $\widehat{A}$-polynomial of K, this implies further that $\lambda'_{SL(2,\mathbb{C})}(K) = 0$ and answers the question raised in [4] as to whether the $SL(2,\mathbb{C})$ Casson knot invariant detects the unknot (*cf.* [4, Theorem 3.3]).

1 Preliminaries

In this section, we begin by introducing notation for the $SL(2,\mathbb{C})$ representation spaces and character varieties. We also review the definition of the $SL(2,\mathbb{C})$ Casson invariant and surgery formula from [12], as well as the A-polynomial of [7] and the $\widehat{A}$-polynomial of [5].

1.1 Representations and the Character Variety

Given a finitely generated group G, we set $R(G)$ to be the space of representations $\rho: G \to SL(2,\mathbb{C})$ and $R^*(G)$ the subspace of irreducible representations. Recall from [11] that $R(G)$ has the structure of a complex affine algebraic set. The *character* of a representation ρ is the function $\chi_\rho: G \to \mathbb{C}$ defined by setting $\chi_\rho(g) = \mathrm{tr}(\rho(g))$ for $g \in G$. The set of characters of $SL(2,\mathbb{C})$ representations admits the structure of a complex affine algebraic set. We denote by $X(G)$ the underlying variety of this algebraic set and by $X^*(G)$ the variety of characters of irreducible representations. Define $t: R(G) \to X(G)$ by $\rho \mapsto \chi_\rho$, and note that t is surjective.

Next, we will define the character scheme $\mathfrak{X}(G)$ in terms of the universal character ring. Since G is finitely generated, there exist elements $g_1, \ldots, g_n \in G$ such that for any $g \in G$, we have a polynomial $P_{G,g} \in \mathbb{C}[x_1, \ldots, x_n]$ with the property that $\chi_\rho(g) = P_{G,g}(x_1, \ldots, x_n)$ under the substitutions $x_i = \mathrm{tr}\,\rho(g_i)$ for all $\rho: \Gamma \to SL(2,\mathbb{C})$. This assertion follows easily from the Cayley–Hamilton theorem if G is the free group F_k of rank k, and in general, using a presentation for G to write it as the quotient of F_k, we define $\mathscr{R}(G) = \mathbb{C}[x_1, \ldots, x_n]/\mathscr{I}(G)$, where $\mathscr{I}(G) = \{P_{F_k,g} \mid g \in \ker(F_k \to G)\}$. The ring $\mathscr{R}(G)$ can be shown to be independent of the choice of presentation of G and is called the *universal character ring*. The character scheme is defined as $\mathfrak{X}(G) =$

Spec $\mathscr{R}(G)$ and is said to be *reduced* if $\mathscr{R}(G)$ contains no nonzero nilpotent elements, or equivalently if $\mathscr{I}(G) = \sqrt{\mathscr{I}(G)}$ is a radical ideal.

For a manifold M, we set $R(M) = R(\pi_1(M))$ and $X(M) = X(\pi_1(M))$ for the spaces of representations and character variety, and $\mathscr{R}(M) = \mathscr{R}(\pi_1(M))$ and $\mathfrak{X}(M) = \mathfrak{X}(\pi_1(M))$ for the universal character ring and character scheme. We will be mainly interested in the case when M is a compact 3-manifold with boundary a torus; typically, M will be the complement $\Sigma \smallsetminus \tau(K)$ of a knot K in an integral homology 3-sphere Σ. In any case, it is well known that every component X_j of $X(M)$ has $\dim X_j \geq 1$, see [7, Proposition 2.4]. In the case where X_j is a curve, there is a smooth projective curve $\widetilde{X}_j$ and a birational equivalence $\widetilde{X}_j \to X_j$, and we refer to points $\widehat{x} \in \widetilde{X}_j$ where $\widetilde{X}_j \to X_j$ has a pole as *ideal points*. Notice that the set of ideal points is Zariski closed and hence finite.

1.2   The SL(2, ℂ) Casson Invariant

We briefly recall the definition of the $SL(2, \mathbb{C})$ Casson invariant. Suppose Σ is a closed orientable 3-manifold with a Heegaard splitting (W_1, W_2, S). Here, S is a closed orientable surface embedded in Σ, and W_1 and W_2 are handlebodies with boundaries $\partial W_1 = S = \partial W_2$ such that $\Sigma = W_1 \cup_S W_2$. The inclusion maps $S \hookrightarrow W_i$ and $W_i \hookrightarrow \Sigma$ induce surjections of fundamental groups. On the level of character varieties, this identifies $X(\Sigma)$ as the intersection

$$X(\Sigma) = X(W_1) \cap X(W_2) \subset X(S).$$

There are natural orientations on all the character varieties determined by their complex structures. The invariant $\lambda_{SL(2,\mathbb{C})}(\Sigma)$ is defined as an oriented intersection number of $X^*(W_1)$ and $X^*(W_2)$ in $X^*(S)$ that counts only compact, zero-dimensional components of the intersection. Specifically, there exist a compact neighborhood U of the zero-dimensional components of $X^*(W_1) \cap X^*(W_2)$ that is disjoint from the higher dimensional components of the intersection and an isotopy $h : X^*(S) \to X^*(S)$ supported in U such that $h(X^*(W_1))$ and $X^*(W_2)$ intersect transversely in U. Given a zero-dimensional component $\{\chi\}$ of $h(X^*(W_1)) \cap X^*(W_2)$, we set $\varepsilon_\chi = \pm 1$, depending on whether the orientation of $h(X^*(W_1))$ followed by that of $X^*(W_2)$ agrees with or disagrees with the orientation of $X^*(S)$ at χ.

Definition 1.1   Let $\lambda_{SL(2,\mathbb{C})}(\Sigma) = \sum_\chi \varepsilon_\chi$, where the sum is over all zero-dimensional components of the intersection $h(X^*(W_1)) \cap X^*(W_2)$.

1.3   The Surgery Formula for Small Knots

In this subsection, we recall from [12] the surgery formula for the Casson $SL(2, \mathbb{C})$ invariant for Dehn surgeries on small knots in integral homology 3-spheres.

Given a compact, irreducible, orientable 3-manifold M with boundary a torus, an *incompressible surface* in M is a properly embedded surface $(S, \partial S) \hookrightarrow (M, \partial M)$ such that $\pi_1(S) \to \pi_1(M)$ is injective and no component of S is a 2-sphere bounding a 3-ball. The surface S is *essential* if it is incompressible and has no boundary parallel

components. A 3-manifold is called *small* if it does not contain a closed essential surface, and a knot K in Σ is called *small* if its complement $\Sigma \smallsetminus \tau(K)$ is a small manifold.

If γ is a simple closed curve in ∂M, let M_γ be the Dehn filling of M along γ; it is the closed 3-manifold obtained by identifying a solid torus with M along their boundaries so that γ bounds a disk. Note that the homeomorphism type of M_γ depends only on the slope of γ, that is, the unoriented isotopy class of γ. Primitive elements in $H_1(\partial M; \mathbb{Z})$ determine slopes under a two-to-one correspondence.

If S is an essential surface in M with nonempty boundary, then all of its boundary components are parallel and the slope of one (and hence all) of these curves is called the *boundary slope* of S. A slope is called a *strict boundary slope* if it is the boundary slope of an essential surface that is not the fiber of any fibration of M over S^1.

For $\gamma \in \pi_1(M)$, there is a regular map $I_\gamma: X(M) \to \mathbb{C}$ defined by $I_\gamma(\chi) = \chi(\gamma)$. Let $e: H_1(\partial M; \mathbb{Z}) \to \pi_1(\partial M)$ be the inverse of the Hurewicz isomorphism. Identifying $e(\xi) \in \pi_1(\partial M)$ with its image in $\pi_1(M)$ under $\pi_1(\partial M) \to \pi_1(M)$, we obtain a well-defined function $I_{e(\xi)}$ on $X(M)$ for $\xi \in H_1(\partial M; \mathbb{Z})$. Let $f_\xi: X(M) \to \mathbb{C}$ be the regular function defined by $f_\xi = I_{e(\xi)} - 2$ for $\xi \in H_1(\partial M; \mathbb{Z})$.

For any algebraic component X_j of $X(M)$ with $\dim X_j = 1$, let $f_{j,\xi}: X_j \to \mathbb{C}$ be the regular function obtained by restricting f_ξ to X_j. Let $\widetilde{X}_j$ denote the smooth, projective curve birationally equivalent to X_j. Regular functions on X_j extend to rational functions on $\widetilde{X}_j$, and we abuse notation and use $f_{j,\xi}$ also for the extension $f_{j,\xi}: \widetilde{X}_j \to \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$.

Definition 1.2   Let $r: X(M) \to X(\partial M)$ be the restriction map. Given a one-dimensional component X_j of $X(M)$ containing an irreducible character such that $r(X_j)$ is also one-dimensional, define the seminorm $\|\cdot\|_j$ on $H_1(\partial M; \mathbb{R})$ by setting $\|\xi\|_j = \deg(f_{j,\xi})$ for all $\xi \in H_1(\partial M; \mathbb{Z})$. We refer to $\|\cdot\|_j$ as the *Culler–Shalen seminorm* associated with X_j, and we say X_j is a *norm curve* if $\|\cdot\|_j$ defines a norm on $H_1(\partial M; \mathbb{R})$.

Note that if M is hyperbolic, then any algebraic component X_0 of $X(M)$ containing the character χ_{ρ_0} of a discrete faithful irreducible representation $\rho_0: \pi_1(M) \to SL(2, \mathbb{C})$ is a norm curve, see [10, Section 1.4].

If M is the complement of a small knot K, the $SL(2, \mathbb{C})$ Casson invariant of a Dehn filling is closely related to this semi-norm; however, we must impose certain restrictions on the slope of the Dehn filling.

Definition 1.3   The slope of a simple closed curve γ in ∂M is called *irregular* if there exists an irreducible representation $\rho: \pi_1(M) \to SL(2, \mathbb{C})$ such that

(i)   the character χ_ρ of ρ lies on a one-dimensional component X_j of $X(M)$ such that $r(X_j)$ is one-dimensional,

(ii) $\operatorname{tr} \rho(\alpha) = \pm 2$ for all α in the image of $i^*: \pi_1(\partial M) \to \pi_1(M)$,

(iii) $\ker(\rho \circ i^*)$ is the cyclic group generated by $[\gamma] \in \pi_1(\partial M)$.

A slope is called *regular* if it is not irregular.

If M is the complement of a knot K in an integral homology 3-sphere Σ, then the meridian $\mathscr{M}$ and longitude $\mathscr{L}$ of the knot K provide a preferred basis for $H_1(\partial M; \mathbb{Z})$. We say that the curve $\gamma = p\mathscr{M} + q\mathscr{L}$ has slope p/q, and we denote by $M_{p/q} = M_\gamma$ the 3-manifold obtained by p/q-Dehn surgery along K.

Definition 1.4   A slope p/q is called *admissible* for K if

(i) p/q is a regular slope that is not a strict boundary slope, and
(ii)   no p'-th root of unity is a root of the Alexander polynomial of K, where $p' = p$ if p is odd and $p' = p/2$ if p is even.

The next result is a restatement of [12, Theorem 4.8], as corrected in [13].

Theorem 1.5   *Suppose K is a small knot in an integral homology 3-sphere Σ with complement M. Let $\{X_j\}$ be the collection of all one-dimensional components of the character variety $X(M)$ such that $r(X_j)$ is one-dimensional and such that $X_j \cap X^*(M)$ is nonempty. Define $\sigma: \mathbb{Z} \to \{0, 1\}$ by $\sigma(p) \equiv p \mod 2$.*

Then there exist integral weights $m_j > 0$ depending only on X_j and non-negative numbers $E_0, E_1 \in \frac{1}{2}\mathbb{Z}$ depending only on K such that for every admissible slope p/q, we have

$$\lambda_{SL(2,\mathbb{C})}(M_{p/q}) = \tfrac{1}{2} \sum_j m_j \|p\mathscr{M} + q\mathscr{L}\|_j - E_{\sigma(p)}.$$

We briefly recall some useful properties of the $SL(2, \mathbb{C})$ Casson invariant, and we refer to [2, 12] for further details.

On closed 3-manifolds Σ, the invariant $\lambda_{SL(2,\mathbb{C})}(\Sigma) \geq 0$ is nonnegative, satisfies $\lambda_{SL(2,\mathbb{C})}(-\Sigma) = \lambda_{SL(2,\mathbb{C})}(\Sigma)$ under orientation reversal, and is additive under connected sum of $\mathbb{Z}/2$–homology 3-spheres (*cf.* [2, Theorem 3.1]). If Σ is hyperbolic, then $\lambda(\Sigma) > 0$ by [12, Proposition 3.2].

If K is a small knot in an integral homology 3-sphere Σ, then Theorem 1.5 implies that the difference

$$\tfrac{1}{p}\big(\lambda_{SL(2,\mathbb{C})}(M_{p/(p+q)}) - \lambda_{SL(2,\mathbb{C})}(M_{p/q})\big)$$

is independent of p and q, provided p and q are relatively prime and q is chosen sufficiently large. This allows one to define an invariant of small knots K in homology 3-spheres by setting

$$\lambda'_{SL(2,\mathbb{C})}(K) = \lambda_{SL(2,\mathbb{C})}(M_{1/(q+1)}) - \lambda_{SL(2,\mathbb{C})}(M_{1/q})$$

for q sufficiently large.

1.4   The $\widehat{A}$-polynomial

We begin with the definition of the A-polynomial $A_K(m, \ell)$ from [7] (see also [8, 9]). Given a knot K in a homology 3-sphere Σ, let $M = \Sigma \smallsetminus \tau(K)$ be its complement and choose a standard meridian-longitude pair $(\mathscr{M}, \mathscr{L})$ for $\pi_1(\partial M)$. Set

$$\Lambda = \big\{ \rho: \pi_1(\partial M) \longrightarrow SL(2, \mathbb{C}) \mid \rho(\mathscr{M}) \text{ and } \rho(\mathscr{L}) \text{ are diagonal matrices} \big\}$$

and define the eigenvalue map $\Lambda \to \mathbb{C}^* \times \mathbb{C}^*$ by setting $\rho \mapsto (u, v) \in \mathbb{C}^* \times \mathbb{C}^*$, where

$$\rho(\mathscr{M}) = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \quad \text{and} \quad \rho(\mathscr{L}) = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}.$$

This map identifies Λ with $\mathbb{C}^* \times \mathbb{C}^*$, and the natural projection $t \colon \Lambda \to X(\partial M)$ is a degree 2, surjective, regular map.

The natural inclusion $\pi_1(\partial M) \to \pi_1(M)$ induces a map $r \colon X(M) \to X(\partial M)$, which is regular. We define $V \subset X(\partial M)$ to be the Zariski closure of the union of the images $r(X_j)$ over each component $X_j \subset X(M)$ that contains an irreducible character and for which $r(X_j)$ is one-dimensional. We define $D \subset \mathbb{C}^2$ to be the Zariski closure of the algebraic curve $t^{-1}(V) \subset \Lambda$, where we identify Λ and $\mathbb{C}^* \times \mathbb{C}^*$ via the eigenvalue map. The A-polynomial $A_K(m, \ell)$ is just the defining polynomial of $D \subset \mathbb{C}^2$; it is well defined up to sign by requiring it to have integer coefficients with greatest common divisor one and to have no repeated factors. Some authors include the factor $\ell - 1$ of reducible characters in $A_K(m, \ell)$, but our convention is to only include components $X_j \subset X(M)$ that contain irreducible characters. Thus $\ell - 1$ is a factor of $A_K(m, \ell)$ if and only if there is a component $X_j \subset X(M)$ containing an irreducible character whose restriction $r(X_j) \subset X(\partial M)$ is the curve $\ell - 1$.

In [5], Boyer and Zhang define an A-polynomial $\widehat{A}_{K,X_j}(m, \ell)$ for each one-dimensional component X_j of $X(M)$ for which $r(X_j)$ is one-dimensional. (Although Boyer and Zhang assume that X_j is a norm curve in this definition, the approach works for any one-dimensional component X_j of $X(M)$.) Their definition takes the defining polynomial $\widehat{A}$ to have factors with multiplicities given by the degree of the restriction map $r|_{X_j}$ rather than requiring that the polynomial have no repeated factors. Taking the product

$$\widehat{A}_K(m, \ell) = \widehat{A}_{K,X_1}(m, \ell) \cdots \widehat{A}_{K,X_n}(m, \ell)$$

over one-dimensional components X_j of $X(M)$ different from the component of reducibles gives an alternative version of the A-polynomial that includes factors with multiplicities. Note that only one-dimensional components X_j of $X(M)$ with one-dimensional image $r(X_j)$ contribute to $\widehat{A}_K(m, \ell)$. As with the A-polynomial, by convention the component of reducible characters does not contribute to $\widehat{A}_K(m, \ell)$. For small knots, it is not difficult to check that $A_K(m, \ell)$ and $\widehat{A}_K(m, \ell)$ have the same factors, only that $\widehat{A}_K(m, \ell)$ may include some repeated factors.

2   Main Results

In this section, we give a general definition of the $SL(2, \mathbb{C})$ Casson knot invariant for knots and relate it to the m-degree of the $\widehat{A}$-polynomial. We prove product formulas for both invariants under the operation of connected sum, and we use Whitehead doubling to construct examples of knots whose character variety contains only components X_j of dimension $\dim X_j > 1$. These knots are nontrivial but have trivial $\widehat{A}$-polynomial and trivial $SL(2, \mathbb{C})$ Casson knot invariant.

We are particularly interested in knots K in integral homology 3-spheres Σ that satisfy the following property, where $M = \Sigma \smallsetminus \tau(K)$ is the complement of K in Σ:

$(*)$ The character scheme of $\mathfrak{X}(M)$ is reduced.

Note that $(*)$ is equivalent to the condition that the universal character ring $\mathscr{R}(M)$ is reduced, and in [19] the authors explain its relationship to the AJ conjecture. For instance, in [19, Conjecture 2], Le and Tran conjecture that $(*)$ holds for all knots in S^3, and they point out that it has been verified in numerous cases, including two-bridge knots [18], torus knots [20], and many pretzel knots [19,23]. On the other hand, in [16] Kapovich and Millson have proved a kind of Murphy's law for 3-manifold groups that implies that there are 3-manifolds whose character schemes are not reduced; thus, $(*)$ does not hold in general (see also [22]).

2.1 The SL(2, $\mathbb{C}$) Casson Invariant for Knots

In this subsection, we extend the $SL(2, \mathbb{C})$ Casson knot invariant to knots in integral homology 3-spheres satisfying $(*)$.

Theorem 2.1 *For any knot K in an integral homology 3-sphere satisfying $(*)$, the limit*

$$\lim_{q \to \infty} \frac{1}{q} \lambda_{SL(2,\mathbb{C})}(M_{p/q})$$

exists, is independent of p, and equals $\frac{1}{2} \deg_m \widehat{A}_K(m, \ell)$.

We then define the $SL(2, \mathbb{C})$ Casson knot invariant by setting

$$(2.1) \qquad \lambda'_{SL(2,\mathbb{C})}(K) = \lim_{q \to \infty} \frac{1}{q} \lambda_{SL(2,\mathbb{C})}(M_{p/q}).$$

Here, p is fixed, and the limit is taken over all q relatively prime to p. The theorem implies that this gives a well-defined invariant of knots. As a direct consequence of Theorem 2.1, we deduce the following corollary.

Corollary 2.2 *For any knot K in an integral homology 3-sphere satisfying $(*)$, we have $\lambda'_{SL(2,\mathbb{C})}(K) = \frac{1}{2} \deg_m \widehat{A}_K(m, \ell)$.*

The rest of this subsection is devoted to proving Theorem 2.1, and we begin with a definition.

For any representation $\rho: \pi_1(M) \to SL(2, \mathbb{C})$ that extends over p/q-Dehn surgery, the eigenvalues m, ℓ of $\rho(\mathscr{M}), \rho(\mathscr{L})$ satisfy $m^p \ell^q = 1$. So for p, q relatively prime, we define $F_{p/q}$ to be the plane curve given by $m^p \ell^q - 1$ and call $F_{p/q}$ the *surgery curve*.

Lemma 2.3 *For any slope p/q, the surgery curve $F_{p/q}$ is non-singular. If p/q and p'/q' are distinct slopes, then $F_{p/q}$ and $F_{p'/q'}$ are transverse.*

Proof We will show that every point on $F_{p/q}$ is simple, and from this it will follow that $F_{p/q}$ is non-singular.

Let $F = F_{p/q}$ be the polynomial $m^p \ell^q - 1$. Any solution to $F = 0$ must have $m \neq 0$ and $\ell \neq 0$, and it will be a simple point so long as one of the partial derivatives $\partial F/\partial m$ or $\partial F/\partial \ell$ is non-zero at that point. But $\partial F/\partial m = pm^{p-1}\ell^q$ and $\partial F/\partial \ell = qm^p \ell^{q-1}$ are both non-zero at each point on F. Thus, every point on F is simple and consequently F is non-singular.

Now suppose p/q and p'/q' are distinct slopes and set $F = F_{p/q}$ and $F' = F_{p'/q'}$. Suppose (m_0, ℓ_0) is common solution to $F = 0$ and $F' = 0$. The equations of the tangent lines to F and F' at (m_0, ℓ_0) are given by

$$(2.2) \qquad p m_0^{p-1} \ell_0^q (m - m_0) + q m_0^p \ell_0^{q-1} (\ell - \ell_0) = 0,$$
$$p' m_0^{p'-1} \ell_0^{q'} (m - m_0) + q' m_0^{p'} \ell_0^{q'-1} (\ell - \ell_0) = 0.$$

Since (m_0, ℓ_0) lies on both surgery curves, we see that

$$m_0^{-1} \ell_0^{-1} = m_0^{p-1} \ell_0^{q-1} = m_0^{p'-1} \ell_0^{q'-1},$$

and dividing both equations in (2.2) by this common factor, we obtain the tangent lines $p\ell_0(m - m_0) + q m_0(\ell - \ell_0) = 0$ and $p'\ell_0(m - m_0) + q' m_0(\ell - \ell_0) = 0$, which have distinct slopes, since p/q and p'/q' are distinct. This shows that F and F' intersect transversely at (m_0, ℓ_0). $\blacksquare$

We now give an outline of the proof of Theorem 2.1. It is established by comparing the $SL(2, \mathbb{C})$ Casson invariant $\lambda_{SL(2,\mathbb{C})}(M_{p/q})$ for large q with the algebro-geometric intersection number $\sum_x I_x(\widehat{A}_K \cap F_{p/q})$ of the $\widehat{A}$-polynomial with the surgery curve $F_{p/q}$, where the sum is taken over all points in the intersection. A critical step in proving the theorem is to show the following claim.

Claim There exists a real number B such that, for any slope p/q such that $F_{p/q}$ does not divide $\widehat{A}_K$, we have

$$(2.3) \qquad \tfrac{1}{2} \sum_x I_x(\widehat{A}_K \cap F_{p/q}) - B \le \lambda_{SL(2,\mathbb{C})}(M_{p/q}) \le \tfrac{1}{2} \sum_x I_x(\widehat{A}_K \cap F_{p/q}).$$

Before proving the claim, we explain how to use it to deduce Theorem 2.1. The following lemma evaluates

$$\lim_{q \to \infty} \tfrac{1}{q} \sum_x I_x(\widehat{A}_K \cap F_{p/q}),$$

where the limit is taken over all q relatively prime to p. Since $\widehat{A}_K$ has finitely many factors, the claim excludes finitely many slopes p/q. Thus, the theorem follows from the lemma by dividing (2.3) by q, taking the limit as $q \to \infty$, q relatively prime to p, and squeezing.

Lemma 2.4 *For any p, we have*

$$\lim_{q \to \infty} \tfrac{1}{q} \sum_x I_x(\widehat{A}_K \cap F_{p/q}) = \deg_m \widehat{A}_K(m, \ell).$$

Proof If p and q are relatively prime, then we have integers r, s with $pr + qs = 1$. We parameterize solutions to $F_{p/q}(m, \ell) = 0$ by setting $m = t^q$ and $\ell = t^{-p}$, where $t \in \mathbb{C}^*$. Clearly, $(m, \ell) = (t^q, t^{-p})$ lies on the curve $F_{p/q}$, and $\ell^{-r} m^s = t^{pr} t^{qs} = t^{pr+qs} = t$, thus any point on $F_{p/q}$ lies on this parameterization.

Suppose $\deg_m \widehat{A}_K(m, \ell) = n$. Then we can write

$$\widehat{A}_K(m, \ell) = m^n \alpha_n(\ell) + m^{n-1} \alpha_{n-1}(\ell) + \cdots + m \alpha_1(\ell) + \alpha_0(\ell),$$

where each $\alpha_i(\ell)$ is a polynomial in ℓ. Now substitute $m = t^q$ and $\ell = t^{-p}$ to obtain

$$\widehat{A}_K(t^q, t^{-p}) = (t^q)^n \alpha_n(t^{-p}) + (t^q)^{n-1}\alpha_{n-1}(t^{-p}) + \cdots + t^q \alpha_1(t^{-p}) + \alpha_0(t^{-p}).$$

Further, since $t \neq 0$, the roots of this Laurent polynomial are identical to the roots of the polynomial obtained by multiplying by t^d, where d is chosen so that $t^d \widehat{A}_K(t^q, t^{-p})$ is a polynomial with nonzero constant term. Note that, for large q, we can take $d = p \deg \alpha_0$, which is clearly independent of q.

The fundamental theorem of algebra implies that $\sum_x I_x(\widehat{A}_K \cap F_{p/q})$ equals the degree of $t^d \widehat{A}_K(t^q, t^{-p})$. For q sufficiently large, we have

$$\deg t^d \widehat{A}_K(t^q, t^{-p}) = d + nq - p \deg \alpha_n = nq + p(\deg \alpha_0 - \deg \alpha_n).$$

Thus, fixing p and letting $q \to \infty$, we see that the intersection $\sum_x I_x(\widehat{A}_K \cap F_{p/q})$ grows linearly in q with leading coefficient $n = \deg_m \widehat{A}_K(m, \ell)$. This completes the proof of the lemma. $\blacksquare$

Proof of Claim In order to establish the bound (2.3), we compare the $SL(2, \mathbb{C})$ Casson invariant $\lambda_{SL(2,\mathbb{C})}(M_{p/q})$ with the intersection number $\frac{1}{2}\sum_x I_x(\widehat{A}_K \cap F_{p/q})$. We will show that most points in $\mathbb{C}^* \times \mathbb{C}^*$ contribute equally to $\lambda_{SL(2,\mathbb{C})}(M_{p/q})$ and $\frac{1}{2}\sum_x I_x(\widehat{A}_K \cap F_{p/q})$, with the sole exceptions being points of the following types:

(1) Solutions $x = (m, \ell)$ to both $\widehat{A}_K(m, \ell) = 0$ and $F_{p/q}(m, \ell) = 0$ with $m, \ell = \pm 1$.
(2) Solutions $x = (m, \ell)$ to both $\widehat{A}_K(m, \ell) = 0$ and $F_{p/q}(m, \ell) = 0$ with $\ell = 1$ and m^2 equal to a root of the Alexander polynomial $\Delta_K(t)$.
(3) Solutions $x = (m, \ell)$ to both $\widehat{A}_K(m, \ell) = 0$ and $F_{p/q}(m, \ell) = 0$ with $t(x) = r(\widehat{x})$ for an ideal point $\widehat{x} \in \widetilde{X}_j$.

Suppose $F_{p/q}$ does not divide $\widehat{A}_K$ and that $x \in \widehat{A}_K \cap F_{p/q}$ is not a point of type (1), (2), or (3). Writing $\widehat{A}_K = \widehat{A}_{K,X_1} \cdots \widehat{A}_{K,X_n}$, the basic properties of intersection numbers from [14, 3.3] imply that $I_x(\widehat{A}_K \cap F_{p/q}) = \sum_j I_x(\widehat{A}_{K,X_j} \cap F_{p/q})$. Under the assumption $(*)$, since the ring $\mathscr{R}(G)$ is independent of the presentation for $\pi_1(M)$, it is clear that the intersection multiplicity n_{Y_j} given in the proof of [12, Proposition 4.3] is equal to one for each component X_j of $X(M)$, and therefore by the same proposition the contribution of x to $\lambda_{SL(2,\mathbb{C})}(M_{p/q})$ is $\sum_j d_j i_{X_j,x}$, where $d_j = \deg(r|_{X_j})$ and $i_{X_j,x} = \frac{1}{2}I_x(E_j \cap F_{p/q})$, where E_j is the unique polynomial with no repeated factors and integer coefficients vanishing on $t^{-1}(r(X_j))$. The factor of 1/2 is due to the fact that $t \colon \Lambda \to X(\partial M)$ is generically two-to-one. Here note that the proof of [12, Proposition 4.3] holds for any one-dimensional component X_j of $X(M)$ such that $r(X_j)$ is one-dimensional even if M contains a closed incompressible surface. Note further that by [21, Corollary 2, p. 75], if Y is a component of $X(M)$ with $\dim Y > 1$, then the intersection $Y \cap r^{-1}(t(F_{p/q}))$ does not contain any zero-dimensional components and thus Y does not contribute to the Casson $SL(2, \mathbb{C})$ invariant $\lambda_{SL(2,\mathbb{C})}(M_{p/q})$.

Since $\widehat{A}_{K,X_j}$ is defined as a curve with multiplicity $d_j = \deg(r|_{X_j})$, it follows that $\widehat{A}_{K,X_j} = E_j^{d_j}$, and this implies that

$$I_x(\widehat{A}_{K,X_j} \cap F_{p/q}) = d_j I_x(E_j \cap F_{p/q}).$$

This shows that points $x \in \mathbb{C}^* \times \mathbb{C}^*$ which are not of types (1)–(3) contribute equally to $\lambda_{SL(2,\mathbb{C})}(M_{p/q})$ and $\frac{1}{2} \sum_x I_x(\widehat{A}_K \cap F_{p/q})$.

In contrast, points of types (1)–(3) may contribute differently to $\lambda_{SL(2,\mathbb{C})}(M_{p/q})$ and $\frac{1}{2} \sum_x I_x(\widehat{A}_K \cap F_{p/q})$ as follows:

Points of type (1) need not correspond to points in the character variety $X(M_{p/q})$ and will therefore sometimes contribute less to $\lambda_{SL(2,\mathbb{C})}(M_{p/q})$ than to

$$\frac{1}{2} \sum_x I_x(\widehat{A}_K \cap F_{p/q}).$$

(For more details, see [12, Section 4.1].)

By [7, Proposition 6.2], we see that points of type (2) correspond to images of reducible characters in $X(M_{p/q})$ and thus will contribute less to $\lambda_{SL(2,\mathbb{C})}(M_{p/q})$ than to $\frac{1}{2} \sum_x I_x(\widehat{A}_K \cap F_{p/q})$.

Points of type (3) correspond to images of ideal characters and as such will also contribute less to $\lambda_{SL(2,\mathbb{C})}(M_{p/q})$ than to $\frac{1}{2} \sum_x I_x(\widehat{A}_K \cap F_{p/q})$.

Since in all three cases, the points x never contribute more to $\lambda_{SL(2,\mathbb{C})}(M_{p/q})$ than to $\frac{1}{2} \sum_x I_x(\widehat{A}_K \cap F_{p/q})$, it follows that

$$(2.4) \qquad \lambda_{SL(2,\mathbb{C})}(M_{p/q}) \le \frac{1}{2} \sum_x I_x(\widehat{A}_K \cap F_{p/q}).$$

We will now show that there are at most finitely many points of types (1), (2), and (3). A type (1) point x satisfies $x = (m, \ell) = (\pm 1, \pm 1)$; thus, there are at most four of them. Because the Alexander polynomial $\Delta_K(t)$ has finitely many roots, there are finitely many points of type (2). Since the number of ideal points on any one-dimensional component X_j is finite, and since $X(M)$ has finitely many components, it follows that there are finitely many type (3) points.

We determine a bound B for the sum of the contributions of points of types (1)–(3) to $\frac{1}{2} \sum_x I_x(\widehat{A}_K \cap F_{p/q})$ that is independent of p and q. Note that this bound is not immediate, because, while the number of points of types (1)–(3) is finite, they can nevertheless lie on infinitely many surgery curves. For example, consider the point $x = (e^{6\pi i/5}, e^{2\pi i/5})$, which satisfies $m = \ell^3$ and $\ell^5 = 1$. Then x lies on $F_{p/q}$ whenever p is not a multiple of 5. If x were a point of type (3), then it could possibly lie on infinitely many surgery curves $F_{p/q}$, for p fixed and $q \to \infty$. Whether or not points of type (3) exist at all is, to the best of our knowledge, an open question; see [8, Question 5.1] and [6, Theorem 3.3]. (Note that those papers refer to type (3) points as *holes*.)

Now suppose x is a type (1), (2), or (3) point. Since the surgery curves $F_{p/q}$ are all non-singular and pairwise transverse, at most finitely many of them, say $F_{p_1/q_1}, \ldots, F_{p_k/q_k}$, will intersect $\widehat{A}_K$ non-transversely at x. If such non-transverse intersections exist, then for $i = 1, \ldots, k$, let $b_{x,i} = \frac{1}{2} I_x(\widehat{A}_K \cap F_{p_i/q_i})$ and set $B_x = \max\{b_{x,1}, \ldots, b_{x,k}\}$. If every curve $F_{p/q}$ that meets $\widehat{A}_K$ at x intersects $\widehat{A}_K$ transversely at x, then set $B_x = \frac{1}{2} I_x(\widehat{A}_K \cap F_{p_0/q_0})$, where F_{p_0/q_0} is an arbitrary surgery curve containing x. Then for any surgery curve $F_{p/q}$, we have $\frac{1}{2} I_x(\widehat{A}_K \cap F_{p/q}) \le B_x$. Setting $B = \sum_x B_x$, with the sum taken over all type (1), (2), and (3) points, we see that

$$(2.5) \qquad \frac{1}{2} \sum_x I_x(\widehat{A}_K \cap F_{p/q}) - \lambda_{SL(2,\mathbb{C})}(M_{p/q}) \le B.$$

Combining equations (2.4) and (2.5) gives (2.3), and this completes the proof of the claim and the proof of the theorem. ∎

We conclude this subsection with two observations. First, note that the definition of the knot invariant $\lambda'_{SL(2,\mathbb{C})}(K)$ is an important first step in developing a Dehn surgery formula for the $SL(2,\mathbb{C})$ Casson invariant. However, as is clear from the proof above, a complete surgery formula must include correction terms measuring the difference $\lambda_{SL(2,\mathbb{C})}(M_{p/q}) - \frac{1}{2}\sum_x I_x(\widehat{A}_K \cap F_{p/q})$ at slopes p/q for which the intersection $\widehat{A}_K \cap F_{p/q}$ contains points of type (1), (2), and (3). Corrections for points of type (1) may be made analogously to the corrections for small knots in [12]. In this case, the correction terms will depend only on the parity of p. For points of type (2) and (3), new arguments will be needed.

Let G be a compact group and suppose that the Casson invariant $\lambda_G(\Sigma)$ has been defined for homology 3-spheres. Then the associated knot invariant is defined as the difference

$$\lambda'_G(K) = \lambda_G(M_{1/(q+1)}) - \lambda_G(M_{1/q}),$$

which one must show is independent of q. In that case, the limit $\lim_{q\to\infty} \frac{1}{q}\lambda_G(M_{1/q})$ exists and equals $\lambda'_G(K)$. In our case, we have seen that the limit (2.1) is independent of p, and our proof shows that

$$\lambda'_{SL(2,\mathbb{C})}(K) = \frac{1}{p}\big(\lambda_{SL(2,\mathbb{C})}(M_{p/(q+p)}) - \lambda_{SL(2,\mathbb{C})}(M_{p/q})\big)$$

provided q and p are relatively prime and $F_{p/(p+q)}$ and $F_{p/q}$ do not contain any points of types (1), (2), or (3).

2.2 The $\widehat{A}$-polynomial for Connected Sums

In the next result, we present a product formula for the $\widehat{A}$-polynomial under connected sum of two knots (*cf.* [8, Proposition 4.3], where a similar result for the A-polynomial is established).

Theorem 2.5 *If K_1 and K_2 are two oriented knots in S^3, then*

$$\widehat{A}_{K_1 \# K_2}(m, \ell) = \widehat{A}_{K_1}(m, \ell) \cdot \widehat{A}_{K_2}(m, \ell).$$

Proof Let $M_1 = S^3 \smallsetminus \tau(K_1)$ and $M_2 = S^3 \smallsetminus \tau(K_2)$ be the complements of K_1 and K_2, respectively. Further, let $K = K_1 \# K_2$ be the connected sum of the two knots and let $M = S^3 \smallsetminus \tau(K)$ be the complement. Then by Seifert–van Kampen, for appropriately chosen meridians $\mathscr{M}_1$ and $\mathscr{M}_2$ for K_1 and K_2, we see that

$$\pi_1(M) = \pi_1(M_1) *_\phi \pi_1(M_2)$$

is an amalgamated product under the homomorphism $\phi\colon \langle \mathscr{M}_1 \rangle \to \langle \mathscr{M}_2 \rangle$ given by $\phi(\mathscr{M}_1) = \mathscr{M}_2$. It follows that the representation space $R(M)$ can be viewed as a subset of the product $R(M_1) \times R(M_2)$, namely

$$R(M) = \big\{ (\rho_1, \rho_2) \in R(M_1) \times R(M_2) \mid \rho_1(\mathscr{M}_1) = \rho_2(\mathscr{M}_2) \big\}.$$

Given $\rho_i \in R(M_i)$ for $i = 1, 2$ such that $\rho_1(\mathscr{M}_1) = \rho_2(\mathscr{M}_2)$, we denote the associated point in $R(M)$ by $\rho_1 * \rho_2$. Let $\mathscr{M}$ be the meridian of $K_1 \# K_2$ corresponding to $\mathscr{M}_1$ and $\mathscr{M}_2$ under this identification.

Given a representation $\rho_1 \colon \pi_1(M_1) \to SL(2, \mathbb{C})$ such that $\rho_1(\mathscr{M}_1)$ is conjugate to a diagonal matrix, we can pull it back along the surjection $\pi_1(M) \to \pi_1(M_1)$ to get a representation $\rho \colon \pi_1(M) \to SL(2, \mathbb{C})$, and in that case $\rho = \rho_1 * \rho_2$, where $\rho_2 \colon \pi_1(M_2) \to SL(2, \mathbb{C})$ is abelian. The meridians and longitudes of K_1, K_2, and $K = K_1 \# K_2$ are related by $\mathscr{M} = \mathscr{M}_1 = \mathscr{M}_2$ and $\mathscr{L} = \mathscr{L}_1 \mathscr{L}_2$, and since ρ_2 is abelian, we see that $\rho_2(\mathscr{L}_2) = I$. It follows that $\rho(\mathscr{M}) = \rho_1(\mathscr{M}_1)$ and $\rho(\mathscr{L}) = \rho_1(\mathscr{L}_1)\rho_2(\mathscr{L}_2) = \rho_1(\mathscr{L}_1)$.

Note that at most finitely many characters in $X(\partial M_1)$ are characters of representations taking the meridian to a matrix of trace ± 2, and any representation $\rho_1 \colon \pi_1(M_1) \to SL(2, \mathbb{C})$ that does not take the meridian to a matrix of trace ± 2 can be conjugated so that $\rho_1(\mathscr{M}_1)$ is diagonal. Thus, using the correspondence from the previous paragraph, for any one-dimensional component X'_j of $X(M_1)$, there is a corresponding one-dimensional component X_j of $X(M)$ such that $r'(X'_j) = r(X_j)$. (Here,

$$r' \colon X(M_1) \to X(\partial M_1) \quad \text{and} \quad r \colon X(M) \to X(\partial M)$$

denote the two restriction maps.) This implies that $\widehat{A}_{K_1, X'_j}(m, \ell)$ and $\widehat{A}_{K, X_j}(m, \ell)$ contain the same factors.

We now argue that the multiplicities d'_j and d_j of the factors in $\widehat{A}_{K_1, X_j}(m, \ell)$ and $\widehat{A}_{K, X_j}(m, \ell)$ agree. To see this, recall that the multiplicities are defined in terms of the degree of the restriction maps, which in turn are defined as the cardinality of a generic fiber. Choosing $\chi \in r'(X'_j)$ a generic point so that $(r')^{-1}(\chi)$ consists entirely of irreducible characters, and noting that the pullback construction gives a one-to-one correspondence between irreducible representations $\rho_1 \colon \pi_1(M_1) \to SL(2, \mathbb{C})$ and irreducible representations $\rho = \rho_1 * \rho_2 \colon \pi_1(M) \to SL(2, \mathbb{C})$ with ρ_2 abelian, it follows that

$$
\begin{aligned}
d'_j &= \deg\big(r'|_{X'_j} \colon X'_j \to X(\partial M_1)\big) = \#\big((r')^{-1}(\chi) \cap X'_j\big) \\
&= \#\big(r^{-1}(\chi) \cap X_j\big) = \deg\big(r|_{X_j} \colon X_j \to X(\partial M_1)\big) = d_j.
\end{aligned}
$$

Since the multiplicities agree, we conclude that $\widehat{A}_{K_1, X'_j}(m, \ell) = \widehat{A}_{K, X_j}(m, \ell)$. A similar argument with the roles of K_1 and K_2 reversed shows that for any one-dimensional component X'_j of $X(M_2)$, there is a one-dimensional component X_j of $X(M)$ such that $\widehat{A}_{K_2, X'_j}(m, \ell) = \widehat{A}_{K, X_j}(m, \ell)$.

We claim that this accounts for all one-dimensional components of $X(M)$. The previous argument accounts for all characters of representations for which either ρ_1 or ρ_2 is abelian. Suppose then that X_j is a component in the character variety $X(M)$ containing the character χ_ρ of an irreducible representation $\rho = \rho_1 * \rho_2 \colon \pi_1(M) \to SL(2, \mathbb{C})$ such that neither ρ_1 nor ρ_2 is abelian. Suppose further that $r(X_j)$ is one-dimensional, since otherwise it would not contribute to $\widehat{A}_K(m, \ell)$.

Note that if both ρ_1 and ρ_2 are reducible, then by [7, Proposition 6.1], any eigenvalue μ of $\rho(\mathscr{M}) = \rho_i(\mathscr{M}_i)$ satisfies the condition that μ^2 is a root of both $\Delta_{K_1}(t)$ and $\Delta_{K_2}(t)$, the Alexander polynomials of K_1 and K_2. Further, $\rho_1(\mathscr{L}_1) = I = \rho_2(\mathscr{L}_2)$ since both representations are reducible. (This step uses the fact that $\mathscr{L}_1$ and $\mathscr{L}_2$ lie

in the second commutator subgroup of $\pi_1(M_1)$ and $\pi_2(M_2)$, respectively.) It follows that $\rho(\mathscr{L}) = \rho_1(\mathscr{L}_1)\rho_2(\mathscr{L}_2) = I$. Thus, under restriction, such representations account for at most finitely many points in $r(X_j)$. Hence, without loss of generality we may assume that ρ_1 is irreducible.

Since the meridian $\mathscr{M}$ normally generates $\pi_1(M)$, we see that $\rho(\mathscr{M}) \neq \pm I$. It follows that $\Gamma_{\rho(\mathscr{M})}$, the stabilizer subgroup of $\rho(\mathscr{M})$, is one-dimensional. We use the technique known as algebraic bending via the action of the group $\Gamma_{\rho(\mathscr{M})}$ to show that $\dim X_j > 1$. (See [15, Section 5] for a thorough explanation of this technique.)

Set $\rho_A = \rho_1 * (A\rho_2 A^{-1})$ for $A \in \Gamma_{\rho(\mathscr{M})}$. Notice that ρ_A is irreducible, and that it is conjugate to ρ if and only if $A = \pm I$. Allowing A to vary over $\Gamma_{\rho(\mathscr{M})}$, the family ρ_A of irreducible representations gives rise to a one-dimensional family χ_{ρ_A} of irreducible characters in X_j such that $r(\chi_{\rho_A}) = r(\chi_\rho)$ under the restriction map $r: X(M) \to X(\partial M)$. Thus, the one-dimensional family of irreducible characters lies in the fiber $r^{-1}(\chi_\rho)$, and since $r(X_j)$ is one-dimensional by assumption, it follows that $\dim X_j > 1$. This completes the proof of the theorem. ∎

Combining Corollary 2.2 and Theorem 2.5, we see that the Casson $SL(2, \mathbb{C})$ knot invariant is additive under connected sums in S^3 provided the knots satisfy the condition $(*)$.

Corollary 2.6 *Let K_1 and K_2 be knots in S^3 such that K_1, K_2, and $K_1 \# K_2$ satisfy $(*)$. Then*

$$\lambda'_{SL(2,\mathbb{C})}(K_1 \# K_2) = \lambda'_{SL(2,\mathbb{C})}(K_1) + \lambda'_{SL(2,\mathbb{C})}(K_2).$$

2.3 Whitehead Doubles

In this subsection, we present a construction of knots K whose character variety has no one-dimensional components other than the component of reducibles. A specific example is provided by the untwisted Whitehead double of the trefoil. Note that examples of closed homology 3-spheres Σ whose character varieties $X^*(\Sigma)$ do not contain any zero-dimensional components were given in [3] using spliced sums. Similar computations for the $SU(2)$ character varieties were developed by Klassen in [17], and the idea of adapting his approach to the $SL(2, \mathbb{C})$ setting was suggested by Michael Heusener.

Given a knot J in S^3, the Whitehead double of J is the knot obtained by gluing one component of the Whitehead link L (shown in Figure 1) into the boundary of a tubular neighborhood of J. Alternatively, it is the knot whose complement is constructed by gluing the complement of J to the complement $W = S^3 \smallsetminus \tau(L)$ of L by a homeomorphism along the common 2-torus. The specific homeomorphism may introduce twists, and the resulting knot is denoted K_n and called the n-twisted double of J.

Before providing a detailed construction of K_n, we first investigate W. The fundamental group of W admits the following presentation (*cf.* [8, Lemma 9.4]):

$$(2.6) \qquad \pi_1(W) = \langle x, y \mid yxy^{-1}x^{-1}yx^{-1}y^{-1}x = xy^{-1}x^{-1}yx^{-1}y^{-1}xy \rangle.$$

If λ_x, λ_y denote the longitudes associated with the two components of L, then one can further show that

$$(2.7) \qquad \lambda_x = y^{-1}xyx^{-1}yxy^{-1}x^{-1}, \quad \lambda_y = y^{-1}x^{-1}yxy^{-1}xyx^{-1}.$$

Writing $L = \ell_1 \cup \ell_2$, notice that ℓ_1 and ℓ_2 are both unknotted, and so the complement $V = S^3 \smallsetminus \tau(\ell_2)$ is just the solid torus. We will use y, λ_y to also denote the meridian and longitude for ℓ_2 in ∂V.

We now present several lemmas that describe the character variety $X(W)$. The first lemma identifies the irreducible components of $X(W)$.

Lemma 2.7 *The character variety $X(W)$ of representations $\rho \colon \pi_1(W) \to SL(2, \mathbb{C})$ consists of two irreducible components X_0 and X_1, each of dimension two. The component X_0 contains all the reducible characters, and the component X_1 contains all the irreducible characters.*

Proof We begin by describing the component X_1 that contains all the irreducible characters. Since $\pi_1(W)$ admits a presentation with two generators and one relation, any representation $\rho \colon \pi_1(W) \to SL(2, \mathbb{C})$ that is not parabolic (defined below) can be conjugated so that

$$(2.8) \qquad \rho(x) = \begin{pmatrix} u & s \\ 0 & u^{-1} \end{pmatrix} \quad \text{and} \quad \rho(y) = \begin{pmatrix} v & 0 \\ t & v^{-1} \end{pmatrix}$$

for $u, v \in \mathbb{C}^*$ and $s, t \in \mathbb{C}$. Then ρ satisfies the relation (2.6) whenever u, v, s, t satisfy $f(s, t, u, v) = 0$, where

$$\begin{aligned}
f(s, t, u, v) = {} & u^2 v^2 (st)^3 + uv(u^2 v^2 - 2u^2 - 2v^2 + 1)(st)^2 \\
& + (u^4 + v^4 - u^2(v^2 - 1)^2 - v^2(u^2 - 1)^2)st + uv(u^2 - 1)(v^2 - 1).
\end{aligned}$$

This leads to two components of solutions, and we start by describing the component X_1 that contains all the irreducible characters. Note that if ρ is irreducible, then either $s \neq 0$ or $t \neq 0$. If $s \neq 0$, then we can conjugate ρ so that $s = 1$, and since
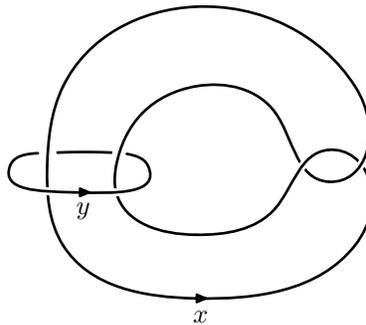
Figure 1: The Whitehead link.

$f(1, t, u, v)$ is irreducible, it follows that the set

$$U_1 = \left\{ (t, u, v) \in \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^* \mid f(1, t, u, v) = 0 \right\}$$

is an irreducible affine variety of dimension two. Note that for a point $(t, u, v) \in U_1$, the associated character χ_ρ is reducible if and only if $t = 0$, and it is abelian if and only if $t = 0$ and $v = \pm 1$.

Switching the roles of s and t, we obtain a second affine variety U_2. Namely, assuming $t \neq 0$, we can conjugate ρ so that $t = 1$, and irreducibility of $f(s, 1, u, v)$ implies that

$$U_2 = \{(s, u, v) \in \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^* \mid f(s, 1, u, v) = 0\}$$

is an irreducible affine variety of dimension two. Note again that for $(s, u, v) \in U_2$, the associated character χ_ρ is reducible if and only if $s = 0$, and it is abelian if and only if $s = 0$ and $u = \pm 1$.

The varieties U_1 and U_2 provide two affine charts for the first component X_1 of $X(W)$, and by construction X_1 contains all the irreducible characters.

To understand the remaining component X_0 of $X(W)$, note that any reducible representation $\rho \colon \pi_1(W) \to SL(2, \mathbb{C})$ (including parabolic representations) can be conjugated to be upper triangular. However, if

$$\rho(x) = \begin{pmatrix} u & * \\ 0 & u^{-1} \end{pmatrix} \quad \text{and} \quad \rho(y) = \begin{pmatrix} v & * \\ 0 & v^{-1} \end{pmatrix},$$

then the character χ_ρ is equivalent to the character of the abelian representation

$$\rho'(x) = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \quad \text{and} \quad \rho'(y) = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix},$$

Thus, every reducible character $\chi_\rho \in X(W)$ is equivalent to the character of a diagonal representation, and any diagonal representation automatically satisfies (2.6). It follows that the component X_0 of reducible characters can be parameterized by $(u, v) \in \mathbb{C}^* \times \mathbb{C}^*$, which is clearly an affine variety of dimension two. Note further that the reducible characters can be identified with the characters of the representations satisfying (2.8) with $s = t = 0$. ∎

In the next lemma, we examine the reducible nonabelian representations of $\pi_1(W)$.

Lemma 2.8 *Suppose $\rho \colon \pi_1(W) \to SL(2, \mathbb{C})$ is a nonabelian reducible representation. Then there are complex numbers $u, v \neq 0, \pm 1$ such that, up to conjugation, either*

$$\rho(x) = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix} \quad and \quad \rho(y) = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}$$

or

$$\rho(x) = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \quad and \quad \rho(y) = \begin{pmatrix} \pm 1 & 0 \\ 1 & \pm 1 \end{pmatrix}.$$

For any nonabelian reducible representation ρ, its character χ_ρ is in $X_0 \cap X_1$. Thus, nonabelian reducible characters χ_ρ lie in the closure of the irreducible characters $X^*(W)$.

Proof   If $\rho(x)$ and $\rho(y)$ are set equal to either of the pairs of matrices above, one easily checks that (2.6) is satisfied. We will show that these are the only solutions possible for nonabelian reducible representations.

Suppose that ρ is a nonabelian reducible representation and conjugate it to be upper triangular. Then we have

$$\rho(x) = \begin{pmatrix} u & s \\ 0 & u^{-1} \end{pmatrix} \quad \text{and} \quad \rho(y) = \begin{pmatrix} v & t \\ 0 & v^{-1} \end{pmatrix},$$

where $u, v \in \mathbb{C}^*$ and $s, t \in \mathbb{C}$. Note that if $u = \pm 1$ and $v = \pm 1$, then ρ is parabolic and hence abelian. So either $u \neq \pm 1$ or $v \neq \pm 1$.

Assume first of all that $u \neq \pm 1$. Then we can conjugate by upper triangular matrices to arrange that $s = 0$. Since $t \neq 0$ (for otherwise ρ is abelian), we can further conjugate by diagonal matrices and arrange that $t = 1$. Then (2.6) holds if and only if $v = \pm 1$.

If, instead, $v \neq \pm 1$, then we can conjugate by upper triangular matrices to arrange that $t = 0$. Since $s \neq 0$ (for otherwise ρ is abelian), we can further conjugate by diagonal matrices and arrange that $s = 1$. Then (2.6) holds if and only if $u = \pm 1$.

Obviously, every nonabelian character χ_ρ lies on X_0, and it is equally clear by taking $s = 0$ or $t = 0$ in (2.8) that χ_ρ also lies on X_1. Further, X_1 equals the closure of $X^*(W)$, and that completes the proof of the lemma. ∎

In the following lemma, we describe the characters χ_ρ in the intersection $X_0 \cap X_1$ of the two components of $X(W)$ as consisting of reducible nonabelian characters together with the four characters of central representations.

Lemma 2.9   *Every character in the intersection $X_0 \cap X_1$ is either the character χ_ρ of a reducible nonabelian representation as in Lemma 2.8, or it is the character of a diagonal abelian representation with $\rho(x) = \pm I$ or $\rho(y) = \pm I$.*

Proof   This follows immediately from Lemma 2.8 and from the description of the abelian representations in U_1 and U_2 in the proof of Lemma 2.7 ∎

We now provide a more detailed construction of the knot K_n, the n-twisted double of the knot J in S^3. Denote the complement of J by $M = S^3 \smallsetminus \tau(J)$. Let μ_J, λ_J be the meridian and longitude for J. We can specify a homeomorphism $\phi_n \colon \partial M \to \partial V$ that is unique up to isotopy by requiring $\phi_n(\mu_J) = \lambda_y$ and $\phi_n(\lambda_J) = y + n\lambda_y$. The image of ℓ_1 in $M \cup_{\phi_n} V$ is a knot K_n in S^3 that we call the n-twisted double of J. Let $Z_n = S^3 \smallsetminus \tau(K_n)$ be the complement of the n-twisted double and note that $Z_n = M \cup_{\phi_n} W$.

Using Seifert–van Kampen, we see that

$$\pi_1(Z_n) = \pi_1(M) *_{(\phi_n)_*} \pi_1(W).$$

Given a representation $\rho \colon \pi_1(Z_n) \to SL(2, \mathbb{C})$, by restricting, we obtain representations $\rho_1 \colon \pi_1(M) \to SL(2, \mathbb{C})$ and $\rho_2 \colon \pi_1(W) \to SL(2, \mathbb{C})$. Note that the representations ρ_1, ρ_2 satisfy

$$(2.9) \qquad \rho_1(\mu_J) = \rho_2(\lambda_y) \quad \text{and} \quad \rho_1(\lambda_J) = \rho_2\left(y(\lambda_y)^n\right),$$

and that any pair $(\rho_1, \rho_2) \in R(M) \times R(W)$ of representations satisfying (2.9) uniquely determines a representation $\rho \colon \pi_1(Z_n) \to SL(2, \mathbb{C})$. In this case, we write $\rho = \rho_1 * \rho_2$.

Lemma 2.10 *If $\rho\colon \pi_1(Z_n) \to SL(2,\mathbb{C})$ is nonabelian, then its restriction $\rho_2 = \rho|_{\pi_1(W)}$ to W is nonabelian. If ρ_2 is reducible and nonabelian, then*

$$\chi_{\rho_2}(y) = \pm 2 \quad and \quad \chi_{\rho_2}(\lambda_y) = 2.$$

Proof Suppose ρ_2 is abelian. By (2.7) we see that λ_y is a product of commutators, and it follows that $\rho_2(\lambda_y) = I$. Thus (2.9) implies that $\rho_1(\mu_J) = I$, and since μ_J normally generates $\pi_1(M)$, it follows that ρ_1 is trivial. But this implies that $\rho = \rho_1 * \rho_2$ is abelian, a contradiction.

Now suppose ρ_2 is reducible and nonabelian. Conjugating, we may assume that ρ_2 is upper triangular with

$$\rho_2(x) = \begin{pmatrix} u & 1 \\ 0 & u^{-1} \end{pmatrix} \quad \text{and} \quad \rho_2(y) = \begin{pmatrix} v & t \\ 0 & v^{-1} \end{pmatrix}.$$

Using (2.7), one easily sees that $\rho(\lambda_y)$ must be upper triangular with trace 2. Since $\rho_2(y)$ commutes with $\rho_2(\lambda_y)$, either $\rho_2(\lambda_y) = I$ or $v = \pm 1$.

If $\rho_2(\lambda_y) = I$, then (2.9) shows that $\rho_1(\mu_J) = I$, which implies that ρ_1 is trivial. Further, $\rho_2(y) = \rho_1(\lambda_J)\rho_2(\lambda_y)^{-n} = I$, which shows that ρ_2 is abelian, a contradiction. Thus, $v = \pm 1$, and the lemma follows. ∎

Recall that $X^*(G)$ denotes the subset of characters χ_ρ of irreducible representations $\rho\colon G \to SL(2,\mathbb{C})$. For the character variety of the n-twisted double K_n, we define

$$X^{*,ab}(Z_n) = \left\{ \chi_\rho \in X^*(Z_n) \mid \rho|_{\pi_1(M)} \text{ is abelian} \right\}$$

$$X^{*,na}(Z_n) = \left\{ \chi_\rho \in X^*(Z_n) \mid \rho|_{\pi_1(M)} \text{ is nonabelian} \right\}.$$

Let U_n be the n-twisted double of the unknot, and observe that U_n is a twist knot.

Lemma 2.11 $X^{*,ab}(Z_n) \cong X^*(U_n)$.

Proof A representation $\rho_1\colon \pi_1(M) \to SL(2,\mathbb{C})$ is abelian if and only if it factors through the abelianization map $\pi_1(M) \to H_1(M) \cong \mathbb{Z}$. Thus, irreducible representations ρ of $\pi_1(Z_n) = \pi_1(M) *_{(\phi_n)_*} \pi_1(W)$ that are abelian on $\pi_1(M)$ are in one-to-one correspondence with representations of $\mathbb{Z} *_{(\phi_n)_*} \pi_1(W) = \pi_1(S^3 \smallsetminus \tau(U_n))$. ∎
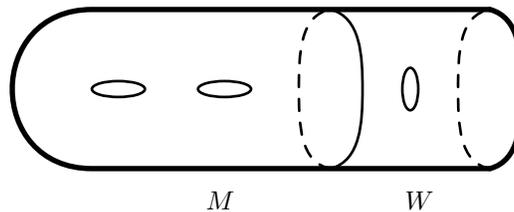
Figure 2: The complement $Z_n = S^3 \smallsetminus \tau(K_n)$ of the n-twisted Whitehead double.

The next result gives a slightly stronger statement for untwisted doubled knots K_0. Henceforth for such knots we omit the 0-subscripts, writing K rather than K_0 for the untwisted double and Z rather than Z_0 for its complement.

Proposition 2.12 *Let J be a knot in S^3 and let $M = S^3 \setminus \tau(J)$ be its complement. Let K be the untwisted double of J and let $Z = S^3 \setminus \tau(K)$ be its complement.*

(i)   *If $\rho: \pi_1(Z) \to SL(2, \mathbb{C})$ is irreducible, then $\rho_1 = \rho|_{\pi_1(M)}$ is irreducible.*

(ii)   *If $\rho: \pi_1(Z) \to SL(2, \mathbb{C})$ is reducible, then ρ_1 is trivial and ρ is abelian.*

Proof   Given a representation $\rho: \pi_1(Z) \to SL(2, \mathbb{C})$ with $\rho_1 = \rho|_{\pi_1(M)}$ reducible, we show that ρ_1 is trivial and ρ is abelian. This will establish both claims of the proposition.

Suppose that ρ_1 is reducible. Since λ_J lies in the second commutator subgroup of $\pi_1(M)$, it follows that $\rho_1(\lambda_J) = I$, and (2.9) shows that $\rho_2(y) = I$. Equation (2.7) implies that $\rho_2(\lambda_y) = I$, and applying (2.9) again shows that $\rho_1(\mu_J) = I$. Since μ_J normally generates $\pi_1(M)$, this implies ρ_1 is trivial. Moreover, since $\rho_2(y) = I$ and $\pi_1(W)$ is generated by x and y, the image of $\rho = \rho_1 * \rho_2$ is generated by $\rho_2(x)$, and it follows that ρ is abelian.   ∎

The Alexander polynomial of the n-twisted Whitehead double K_n is given by $\Delta_{K_n}(t) = nt^2 + (1 - 2n)t + n$. By [7, Proposition 6.1], any reducible nonabelian representation $\rho: \pi_1(Z_n) \to SL(2, \mathbb{C})$ has meridional eigenvalue equal to a square root of $\Delta_{K_n}(t)$. Since the untwisted double K has trivial Alexander polynomial, this shows that $\pi_1(Z)$ does not admit any nonabelian reducible $SL(2, \mathbb{C})$ representations. In particular, every nonabelian representation $\rho: \pi_1(Z) \to SL(2, \mathbb{C})$ is automatically irreducible, and the component $Y_0 \subset X(Z)$ of reducible characters is disjoint from the other components $Y_j \subset X(Z)$.

The next result shows that for most untwisted doubles, the variety of irreducible characters has no one-dimensional components.

Proposition 2.13 *Let J be a knot in S^3 such that the A-polynomial is not divisible by $\ell - 1$, $\ell + 1$, or $\ell^2 + m$, and let $M = S^3 \setminus \tau(K)$ and $r: X(M) \to X(\partial M)$. Suppose that every component X_j of $X(M)$ has a one-dimensional image $r(X_j)$.*

If K denotes the untwisted double of J and $Z = S^3 \setminus \tau(K)$ its complement, then every component Y_j of $X(Z)$ of irreducible characters has $\dim Y_j > 1$.

Note that most knots satisfy the above hypotheses; for a specific example, take J to be the trefoil.

Proof   We write $Z = M \cup_T W$, where T is the 2-torus along which M and W are identified, and we use r_1 and r_2 to denote the restriction maps from $R(M)$ and $R(W)$ to $R(T)$, giving the diagram:
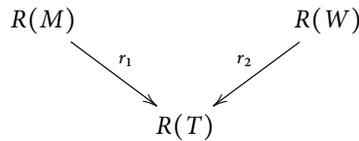
$$R(M) \qquad\qquad\qquad R(W)$$
$$\searrow {\scriptstyle r_1} \qquad {\scriptstyle r_2} \swarrow$$
$$R(T)$$

We use this diagram to identify $R(Z)$ with pairs $(\rho_1, \rho_2) \in R(M) \times R(W)$ that satisfy $r_1(\rho_1) = r_2(\rho_2)$, and in that case we write $\rho = \rho_1 * \rho_2$. Recall from the presentation (2.6) of $\pi_1(W)$ and from the gluing equations (2.9) for the untwisted double that

$$r_1(\rho_1) = r_2(\rho_2) \quad \text{if and only if} \quad \rho_1(\mu_J) = \rho_2(\lambda_y) \text{ and } \rho_1(\lambda_J) = \rho_2(y).$$

Recall from Lemma 2.7 that $X(W) = X_0(W) \cup X_1(W)$. Further, by Lemma 2.8, every nonabelian reducible representation $\rho_2 : \pi_1(W) \to SL(2, \mathbb{C})$ is a limit of irreducible representations.

In their proof of [8, Lemma 9.4], Cooper and Long show that the image of $X_1^*(W)$ under restriction $X(W) \to X(T)$ is the Zariski-open subset given by the complement of the *forbidden* curves $m - 1, m + 1, \ell + m^2$. Here, $T \subset \partial W$ is the boundary of the component ℓ_2 of the Whitehead link L labelled y in Figure 1, and m is an eigenvalue of $\rho(y)$ and ℓ an eigenvalue of $\rho(\lambda_y)$. In particular any curve of representations of $\pi_1(T)$ with characters in the image of $X_1^*(W)$ extends continuously to a curve of representations of $\pi_1(W)$. We use this to show that every component $X_j \subset X(M)$ containing irreducible characters and with image $r(X_j) \subset X(T)$ not coincident with a forbidden curve gives rise to a component $Y_j \subset X^*(Z)$ of dimension at least 2, and that every component of $X^*(Z)$ arises this way.

Let X_j be a component of $X(M)$ containing an irreducible character, and let R_j be the corresponding component in the space $R(M)$ of $SL(2, \mathbb{C})$ representations, so that under $t : R(M) \to X(M)$, we have $t(R_j) = X_j$. By hypothesis, the image $r(X_j)$ is one-dimensional and does not coincide with any of the forbidden curves $\ell - 1, \ell + 1$, and $m + \ell^2$. (Note that the meridian and longitude of the knot J are opposite to those of $T \subset \partial W$, and thus we must exchange ℓ and m when viewing the forbidden curves in $X(\partial M)$.) Consequently, the intersection of $r(X_j)$ with the three forbidden curves consists of at most finitely many characters. Set

$$R_j' = \{\rho_1 \in R_j \mid \rho_1 \text{ is irreducible and } r(\chi_{\rho_1}) \text{ does not lie on a forbidden curve}\}.$$

Since X_j contains only finitely many reducible characters, under the composition $R_j \xrightarrow{t} X_j \xrightarrow{r} X(T)$, this excludes at most finitely many points from $r(X_j)$. Now the image $r(X_j)$ is one-dimensional by hypothesis, and so it follows that R_j' is non-empty and Zariski-open in R_j. Further, [8, Lemma 9.4] shows that every $\rho_1 \in R_j'$ extends to a representation $\rho : \pi_1(Z) \to SL(2, \mathbb{C})$ whose restriction $\rho_2 = \rho|_{\pi_1(W)}$ is irreducible.

Let $Y_j \subset X(Z)$ be the component of irreducible characters χ_ρ with $\rho_1 \in R_j$, and let $f : Y_j \to X(T)$ be the map given by $\chi_\rho \mapsto \chi_{\rho_0}$, where $\rho_0 = \rho|_{\pi_1(T)}$ is the restriction of ρ to the splitting torus T. By the previous construction, we see that $f(Y_j)$ contains $(r \circ t)(R_j')$, which is a curve with at most finitely many points removed. In fact, the construction shows that a Zariski-open subset of Y_j consists of characters χ_ρ with $\rho = \rho_1 * \rho_2$, where $\rho_1 : \pi_1(M) \to SL(2, \mathbb{C})$ and $\rho_2 : \pi_1(W) \to SL(2, \mathbb{C})$ are both irreducible representations. By Lemma 2.10 and Proposition 2.12, every irreducible character $\chi_\rho \in X(Z)$ is the character of a representation $\rho = \rho_1 * \rho_2$ with ρ_1 irreducible and ρ_2 nonabelian. Let R_j be the component of $R(M)$ containing ρ_1. If ρ_2 is irreducible, then $\chi_\rho \in Y_j$ for the component $Y_j \subset X(Z)$ constructed above. If instead ρ_2 is reducible and nonabelian, then it lies in the Zariski closure of R_j', and it follows that ρ lies in the Zariski closure of Y_j.

We now use algebraic bending [15] to show that $\dim(Y_j) \geq 2$. Given $\rho = \rho_1 * \rho_2$ with ρ_1 and ρ_2 irreducible, let Γ_0 be the stabilizer subgroup of the restriction $\rho_0 = \rho|_{\pi_1(T)}$. Since $\pi_1(T)$ is abelian and ρ_1 is irreducible, it follows that Γ_0 is one-dimensional. (Indeed, Γ_0 is isomorphic to either $\mathbb{C}^*$ or $\mathbb{C}$ depending on whether ρ_0 is diagonal or parabolic.) For $A \in \Gamma_0$, define $\rho_A = \rho_1 * (A\rho_2 A^{-1})$. Clearly then $\rho_A \colon \pi_1(Z) \to SL(2, \mathbb{C})$ is irreducible and $f(\chi_{\rho_A}) = f(\chi_\rho)$. On the other hand, for $A \neq \pm I$, one can show that ρ_A is not conjugate to ρ. This gives a one-dimensional family of irreducible characters in the fiber $f^{-1}(\chi_{\rho_0})$, and since $f(Y_j)$ is one-dimensional, this implies that $\dim Y_j > 1$. ∎

Corollary 2.14 *If J is a knot in S^3 satisfying the hypotheses of Proposition* 2.13*, then its untwisted double K has $\widehat{A}_K(m, \ell) = 1$ and $\lambda'_{SL(2,\mathbb{C})}(K) = 0$.*

Proof Since $\widehat{A}_K(m, \ell)$ is defined using only components of $X(Z)$ of dimension one, we see immediately that $\widehat{A}_K(m, \ell) = 1$. The assertion $\lambda'_{SL(2,\mathbb{C})}(K) = 0$ follows from Corollary 2.2 if K satisfies condition $(*)$. If, on the other hand, $(*)$ does not hold, then no component Y_j of $X(Z)$ with dimension greater than one contributes to $\lambda_{SL(2,\mathbb{C})}(Z_{p/q})$ for any surgery p/q, as noted in the proof of Theorem 2.1. Thus, clearly, $\lambda'_{SL(2,\mathbb{C})}(K) = 0$ in this case too. ∎

Taking J to be the trefoil, this implies that its untwisted double has trivial $\widehat{A}$-polynomial and vanishing $SL(2, \mathbb{C})$ Casson knot invariant. In particular, neither the $\widehat{A}$-polynomial nor the $SL(2, \mathbb{C})$ Casson knot invariant detect the unknot.

In conclusion, it would be interesting to find a way to incorporate higher-dimensional components of the $SL(2, \mathbb{C})$ character variety $X(\Sigma)$ into the definition of the $SL(2, \mathbb{C})$ Casson invariant for 3-manifolds Σ. The resulting invariant would coincide with $\lambda_{SL(2,\mathbb{C})}(\Sigma)$ in the case where the character variety $X(\Sigma)$ is zero-dimensional, and an intriguing problem would then be to establish a surgery formula for the new invariant and to define an associated invariant of knots. It is reasonable to believe that the knot invariant would be related to an appropriately defined generalization of the A-polynomial in much the same way that $\lambda'_{SL(2,\mathbb{C})}$ is related to the $\widehat{A}$-polynomial. In particular, since the m-degree of the A-polynomial is known to detect the unknot [1], one would expect that an $SL(2, \mathbb{C})$ knot invariant that incorporates higher-dimensional components of $X(M)$ would be a powerful tool in low-dimensional topology.

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Mathematics & Statistics, McMaster University, Hamilton, ON
e-mail: boden@mcmaster.ca

Mathematics & Statistics, The College of New Jersey, Ewing, NJ, USA
e-mail: ccurtis@tcnj.edu