ON THE CONVERGENCE OF PRODUCT MOMENTS

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Let Θ be an abstract space and for every positive integer n let $F_{n,\theta}(x, y)$, $\theta \in \Theta$, be a family of distribution functions of random variables $(X_n, Y_n)_\theta$, $\theta \in \Theta$. For every $\theta \in \Theta$, $E_\theta g(X_n, Y_n)$ will denote the expected value of the function g of $(X_n, Y_n)_\theta$. The following proposition is proved.

PROPOSITION. Suppose the sequence of families of distribution functions $\{F_{n,\theta}; \theta \in \Theta, n = 1, 2, \ldots\}$ satisfies the following conditions:

C_1. There exists a set A dense in R_2, the 2-dimensional Euclidean Space, such that every $(x, y) \in A$ is a point of continuity of every member of a family of distribution functions $F_\theta(x, y)$, $\theta \in \Theta$ and

$$\lim_{n\to\infty} F_{n,\theta}(x, y) = F_\theta(x, y) \text{ uniformly in } \theta.$$

C_2. There exists a real number $k \geq 0$ such that

$$\lim_{n\to\infty} E_\theta |X_n|^k = E_\theta |X|^k < \infty, \quad \lim_{n\to\infty} E_\theta |Y_n|^k = E_\theta |Y|^k < \infty$$

uniformly in θ.

C_3. Given any $\epsilon \geq 0$, there exists a $c(\epsilon) > 1$ such that if $c \geq c(\epsilon)$

$$\int_{\bar{S}_c} |x|^k dF_\theta < \epsilon, \quad \int_{\bar{S}_c} |y|^k dF_\theta < \epsilon \text{ uniformly in } \theta,$$

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where $S_c = \{(x, y); -c \leq x \leq c, -c \leq y \leq c\}$.

Then for any real numbers $i, j \geq 0$ with $i + j \leq k$,

(1) $\lim_{n\to\infty} E_\theta |X_n^i Y_n^j| = E_\theta |X^i Y^j|$ uniformly in θ,

and for any non-negative integers i', j' with $i' + j' \leq k$,

(2) $-\infty < \lim_{n\to\infty} E_\theta X_n^{i'} Y_n^{j'} = E_\theta X^{i'} Y^{j'} \leq \infty$ uniformly in θ.

Proof: Consider the proof of (1) first. When $k = 0$, $i = j = 0$, so (1) is trivial. Now suppose $k > 0$. By Hölder's Inequality and C_2 if $i, j > 0$,

$$E_\theta |X^i Y^j| \leq \left[E_\theta |X^i|^{\frac{i+j}{i}} \right]^{\frac{i}{i+j}} \left[E_\theta |Y^j|^{\frac{i+j}{j}} \right]^{\frac{j}{i+j}} < \infty$$

for every $\theta \in Ⓗ$, and for sufficiently large n, say $n > n_0$,

$$E_\theta |X_n^i Y_n^j| \leq \left[E_\theta |X_n^i|^{\frac{i+j}{i}} \right]^{\frac{i}{i+j}} \left[E_\theta |Y_n^j|^{\frac{i+j}{j}} \right]^{\frac{i}{i+j}} < \infty$$

for every $\theta \in Ⓗ$. If one of i and j is zero, say $i = 0$, then by C_2 for every $\theta \in Ⓗ$, $E_\theta |Y^j| < \infty$ and hence $E_\theta |Y_n^j| < \infty$ for sufficiently large n.

Let $g(x, y) = |x^i y^j|$ and write

(3) $$|E_\theta |X_n^i Y_n^j| - E_\theta |X^i Y^j|| \leq |\int_{S_c} g dF_{n,\theta} - \int_{S_c} g dF_\theta| + \int_{\bar{S}_c} g dF_{n,\theta}$$

$$+ \int_{\bar{S}_c} g dF_\theta,$$ where $n > n_0$, $\bar{S}_c = R_2 - S_c$ and $c \geq c(\epsilon)$.

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It follows from C_1 and the Uniform Helly-Bray Theorem, which is an immediate extension to the 2-dimensional case of Theorem 29 (p. 288) given by Graves (1958), that for every $c \geq c(\epsilon)$

(4) $$\lim_{n \to \infty} \int_{S_c} |x|^k dF_{n,\theta} = \int_{S_c} |x|^k dF_\theta \text{ uniformly in } \theta.$$

So by C_2

(5) $$\lim_{n \to \infty} \int_{\bar{S}_c} |x|^k dF_{n,\theta} = \int_{\bar{S}_c} |x|^k dF_\theta \text{ uniformly in } \theta.$$

Let $R'_2 = \{(x, y); |y| \leq |x|\}$ and $R''_2 = \{(x, y); |x| < |y|\}$. Then $R'_2 \cup R''_2 = R_2$. By (5) and C_3 there exists a $n_x > n_0$ such that if $n > n_x$

(6) $$\int_{\bar{S}_c \cap R'_2} g dF_{n,\theta} \leq \int_{\bar{S}_c} |x|^k dF_{n,\theta} \leq \int_{\bar{S}_c} |x|^k dF_\theta + \epsilon < 2\epsilon$$

uniformly in θ. Similarly there exists a $n_y > n_0$ such that if $n > n_y$

(7) $$\int_{\bar{S}_c \cap R''_2} g dF_{n,\theta} < 2\epsilon \quad \text{uniformly in } \theta.$$

By C_2

(8) $$\int_{\bar{S}_c} g dF_\theta \leq \int_{\bar{S}_c \cap R'_2} |x|^k dF_\theta + \int_{\bar{S}_c \cap R''_2} |y|^k dF_\theta < 2\epsilon$$

uniformly in θ. Again by the Uniform Helly-Bray Theorem there exists n_1 such that if $n > n_1$

(9) $$\left| \int_{S_c} g dF_{n,\theta} - \int_{S_c} g dF_\theta \right| < \epsilon \quad \text{uniformly in } \theta.$$

Then it follows from (3), (6), (7), (8) and (9) that given any $\epsilon > 0$, there exists $n_2 = \max(n_x, n_y, n_1)$ such that if $n > n_2$

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$$\left| E_\theta |X_n^i Y_n^j| - E_\theta |X^i Y^j| \right| < \epsilon + 4\epsilon + 2\epsilon = 7\epsilon \quad \text{uniformly in } \theta.$$

The proof of (2) is similar.

REMARK 1. The significance of the Proposition is that usually it is easier to calculate the moments of the marginal distributions since the product moments involve multiple integrals.

REMARK 2. A simplification in the above proof, may be obtained by replacing C_2 and C_3 by the following new condition:

C_4. There exists a bounded subset T of R_2 such that $\{(x, y) | 0 < F_\theta(x, y) < 1\}$ is contained in T for all $\theta \in \text{Ⓗ}$ and also, for sufficiently large n, $\{(x, y) | 0 < F_{n,\theta}(x, y) < 1\}$ is contained in T for all $\theta \in \text{Ⓗ}$.

For, if C_1 and C_4 are satisfied, by the Uniform Helly-Bray Theorem (1) and (2) hold for any $i, j \geqslant 0$.

REMARK 3. A special case of the Proposition is that no θ is involved in $F_{n,\theta}$ and F_θ.

REMARK 4. The convergence of the absolute moments in C_2 cannot be weakened to simply the convergence of moments. This can be seen from the following example. (For simplicity, the case in which no θ is involved in $F_{n,\theta}$, F_θ, and X_n and Y_n, X and Y are independent, is considered). Define the density functions of the marginal distributions of X_n and Y_n by

$$f_{1n}(x) = \begin{cases} \frac{1}{2n} & \text{if } x = -n, \\ (1 - \frac{1}{n}) & \text{if } -1/2 \leqslant x \leqslant 1/2, \\ \frac{1}{2n} & \text{if } x = n, \\ 0 & \text{elsewhere} \end{cases}$$

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$$f_{2n}(y) = \begin{cases} \frac{1}{2n} & \text{if } y = -n \\ (1 - \frac{1}{n}) & \text{if } -1/2 \leq y \leq 1/2, \\ \frac{1}{2n} & \text{if } y = n, \\ 0 & \text{elsewhere.} \end{cases}$$

Let the density function of (X_n, Y_n) be

$$f_n(x, y) = f_{1n}(x) f_{2n}(y)$$

and the density function of (X, Y) be

$$f(x, y) = \begin{cases} 1 \text{ if } -1/2 \leq x \leq 1/2, \ -1/2 \leq y \leq 1/2 \\ 0 \text{ elsewhere.} \end{cases}$$

Then it is easily seen that for every $(x, y) \in R_2$

$$\lim_{n\to\infty} f_n(x, y) = f(x, y).$$

and hence

$$\lim_{n\to\infty} F_n(x, y) = F(x, y).$$

Now $$EX_n^3 = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^3 f_{1n}(x)dx + \frac{(-n)^3}{2n} + \frac{n^3}{2n} = 0 = EX^3,$$

$$EY_n^3 = 0 = EY^3.$$

But $$\lim_{n\to\infty} EX_n^2 = \lim_{n\to\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 f_{1n}(x)dx + \frac{(-n)^2}{2n} + \frac{(n)^2}{2n} = \infty \neq EX^2 = \frac{1}{12}.$$

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So (1) and (2) do not hold. Also it can be seen that C_4 is not satisfied.

REMARK 5. The Proposition can be immediately extended to the case of a sequence of families of s-dimensional random variables $\{(X_{1n}, X_{2n}, \ldots, X_{sn})_\theta ; \theta \in \textcircled{H}, n = 1, 2, \ldots\}$.

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