CHAIN CONDITIONS ON POSETS

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1. Introduction and definitions. The aim of this note is to generalize to an arbitrary partially ordered set (poset) $(P, \leqq)$ the standard lattice results on the Jordan–Dedekind Chain Condition (abbreviated hereafter to J.D.C.C.). Birkhoff **[1]** defines semimodularity for a lattice L by

(ξ) if x, y cover a and $x \neq y$, then $x \vee y$ covers x and y.

The additional assumption that L is of finite length is heavily relied upon in proving that the J.D.C.C. holds [**1**, Theorem 3, p. 68].

The semimodularity condition (ξ) has a natural generalization to an arbitrary poset P by

(σ) if x, y cover a and $x \neq y$, then there exists a $d \in P$ which covers x and y.

For a lattice the conditions (ξ) and (σ) coincide and the following is true. If P is a semimodular poset of finite length, then the J.D.C.C. holds.

In **[2]**, Rhodes has given a definition of semimodularity for a lower semilattice S. His result is that, if S satisfies a strong semimodularity condition, then S satisfies a strong chain condition.

Let $(P, \leqq)$ be a poset, and let $a, b \in P$. Then *b covers a* ($b \succ a, a \prec b$) if and only if $a < b$ and $\{x \in P: a \leqq x \leqq b\} = \{a, b\}$. Also, if $x, y \in P$, then $x \wedge y$ and $x \vee y$ mean, respectively, the greatest lower bound and least upper bound of $\{x, y\}$ if they exist. Thus $x \wedge y = a$ means that $x \wedge y$ exists and equals a. A similar statement holds for $x \vee y$.

DEFINITION 1.1. Let P be a poset. Then P is called

(i) *strongly upper semimodular* if and only if, whenever $a \wedge b$, $a \vee b$ exist and $a \succ a \wedge b$, then $a \vee b \succ b$;

(ii) *weakly upper semimodular* if and only if, whenever $a \wedge b$ exists and $a, b \succ a \wedge b$, then $a \vee b$ exists and $a \vee b \succ a, b$.

DEFINITION 1.2. Let P be a poset. Then P satisfies

(i) *the strong chain condition* if and only if, whenever $a < b$ and there is a finite maximal chain from a to b, then all maximal chains from a to b are finite and have the same length;

(*Note.* The Axiom of Choice implies that, if this condition is satisfied, then every chain from a to b is finite.)

(ii) *the weak chain condition* if and only if, whenever $a < b$ and there is a finite maximal chain from a to b, all finite maximal chains from a to b have the same length.

It will be shown that, if P is a poset which is strongly (weakly) upper semimodular, then P

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satisfies the strong (weak) chain condition. An example will show that the semimodularity conditions and chain conditions are not the same, even in a lattice. A further example will show that the weak semimodularity condition cannot be further relaxed and still imply the weak chain condition.

2. The chain conditions. Let P be a partially ordered set and let $a, x, b \in P$ with $a < x < b$. The reader is asked to use Zorn's Lemma to show that there is a chain C from a to b which contains x and which is maximal in the collection of all chains from a to b.

Throughout, $\mathbb{N}$ will denote the set of positive integers. Whenever C is a finite chain in a poset P, then $L(C)$ will denote the length of C.

THEOREM 2.1. *Let P be a strongly upper semimodular poset. Then P satisfies the strong chain condition.*

Proof. (By induction). Let $K = \{n \in \mathbb{N}$: if $a, b \in P$ with $a < b$ and C_1 is a finite maximal chain from a to b of length n and C_2 is a finite chain from a to b, then $L(C_2) \leqq n\}$. It is enough to show that $K = \mathbb{N}$.

Certainly $1 \in K$. Assume now that $n \in \mathbb{N}$, $1 < n$, and $t \in K$ whenever $1 \leqq t < n$. Let $a = a_0 \prec a_1 \prec \ldots \prec a_n = b$ be a maximal chain from a to b, and let C be a finite chain from a to b.

Case (i). There exist $x \in C - \{a, b\}$ and $t \in [1, n-1]$ such that $a_t \leqq x$.

If $a_t = x$, then, since $a = a_0 \prec a_1 \prec \ldots \prec a_t = x$ and $x = a_t \prec a_{t+1} \prec \ldots \prec a_n = b$ are maximal chains from a to x and from x to b of lengths t, $n-t \in K$, then

$$L(C) = L\{y \in C: y \leqq x\} + L\{y \in C: x \leqq y\} \leqq t + (n-t) = n.$$

Assume now that $a_t < x$. Since $a_t \prec a_{t+1} \prec \ldots \prec a_n = b$ is a maximal chain from a_t to b of length $n-t \varepsilon K$, there is a maximal chain from a_t to b of length $n-t$ which contains x, say $a_t = y_t \prec y_{t+1} \prec \ldots \prec y_{t+s} = x \prec y_{t+s+1} \prec \ldots \prec y_n = b$. Now, since $a = a_0 \prec a_1 \prec \ldots \prec a_t = y_t \prec y_{t+1} \prec \ldots \prec y_{t+s} = x$ and $x = y_{t+s} \prec y_{t+s+1} \prec \ldots \prec y_n = b$ are maximal of lengths $t+s$, $n-(t+s) \in K$, then

$$L(C) = L\{y \in C: y \leqq x\} + L\{y \in C: x \leqq y\} \leqq (t+s) + [n-(t+s)] = n.$$

Case (ii). For each $x \in C - \{a, b\}$ and for each $t \in [1, n-1]$, $a_t \nleqq x$. Then, for each $x \in C - \{a, b\}$, $a_1 \nleqq x$ and so $a_1 \wedge x = a_0$.

(iia). For each $x \in C - \{a, b\}, a_1 \vee x = b$. Then, since P is strongly upper semimodular, $x \prec b$ for each $x \in C - \{a, b\}$. It follows that $L(C) \leqq 2 \leqq n$.

(iib). For each $x \in C - \{a, b\}$, either $x \vee a_1$ does not exist or $x \vee a_1$ exists but is not equal to b. In either case choose $u \in P$ such that $x, a_1 < u < b$.

Since $a_1 \prec a_2 \prec \ldots \prec a_n = b$ is maximal from a_1 to b of length $n-1 \in K$, there is a maximal chain from a_1 to b of length $n-1$ which contains u, say $a_1 = y_1 \prec \ldots \prec y_r = u \prec y_{r+1} \prec \ldots \prec y_n = b$. Since $a = a_0 \prec a_1 = y_1 \prec \ldots \prec y_r = u$ is maximal from a to u of length $r \in K$, there is a maximal chain from a to u of length r which contains x, say $a = a_0 = z_0 \prec z_1 \prec \ldots \prec z_s = x \prec z_{s+1} \prec \ldots \prec z_r = u$. Now, since $a = a_0 = z_0 \prec z_1 \prec \ldots \prec z_s = x$ and $x = z_s \prec z_{s+1} \prec \ldots \prec z_r = u \prec y_{r+1} \ldots y_n = b$ are maximal of lengths $s, n-s \in K$, it follows that $L(C) \leqq s + (n-s) = n$.

In any event, $n \in K$ and so $K = \mathbb{N}$.

LEMMA 2.2. *Let P be a weakly upper semimodular poset. Let $a, b \in P$ with $a < b$ and let $a = a_0 \prec a_1 \prec \ldots \prec a_n = b$ be a finite maximal chain from a to b. Let $x \in P$ be such that $x \nleqq a_i$ for each $i \in [1, n-1]$ and $a \prec x < b$. Then $x \vee a_i \succ x \vee a_{i-1}, a_i$ for each $i \in [1, n-1]$.*

Proof. (By induction). Let $K = \{i \in [1, n-1] : x \vee a_i \succ x \vee a_{i-1}, a_i\}$.

Since $a_1, x \succ a$, it follows that $x \wedge a_1 = a$. Since P is weakly upper semimodular, then $x \vee a_1$ exists and $x \vee a_1 \succ x, a_1$. But $x = x \vee a_0$ and so $x \vee a_1 \succ x \vee a_0, a_1$. Thus $1 \in K$.

Assume now that $i \in [1, n-2]$ and $i \in K$. Then $x \vee a_i \succ a_i$ and $a_{i+1} \succ a_i$. Notice that $x \vee a_i \neq a_{i+1}$ since otherwise $x \leqq a_{i+1}$, contradicting the hypothesis. Since $a_{i+1} \succ a_i$, it follows that $(x \vee a_i) \wedge a_{i+1} = a_i$. Since P is weakly upper semimodular, then $(x \vee a_i) \vee a_{i+1} \succ x \vee a_i, a_{i+1}$; that is, $x \vee a_{i+1} \succ x \vee a_i, a_{i+1}$. Thus $i+1 \in K$.

THEOREM 2.3. *Let P be a weakly upper semimodular poset. Then P satisfies the weak chain condition.*

Proof. (By induction). Let $K = \{n \in \mathbb{N}$: if $a, b \in P$ with $a < b$ and there is a finite maximal chain from a to b of length n, then all finite maximal chains from a to b have length $n\}$.

Certainly $1 \in K$. Assume now that $n \in K, a < b$, and that $a = a_0 \prec a_1 \prec \ldots \prec a_{n+1} = b$, $a = b_0 \prec b_1 \prec \ldots \prec b_{m+1} = b$ are finite maximal chains from a to b of lengths $n+1$ and $m+1$, respectively. Consider b_1 and choose j minimal with respect to $b_1 \leqq a_j$. If $j = 1$, then $b_1 = a_1$ since $a_1 \succ a$. It follows immediately that $n = m$ and hence $n+1 = m+1$. Assume $j \neq 1$. Then $b_1 < a_j$, since $b_1 \succ a_0$. By Lemma 2.2, $b_1 = b_1 \vee a_0 \prec b_1 \vee a_1 \prec \ldots \prec b_1 \vee a_{j-1} = a_j \prec a_{j+1} \prec \ldots \prec a_{n+1} = b$ is maximal of length n. Thus $n = m$ and again $n+1 = m+1$. In any event, $n+1 \in K$ and hence $K = \mathbb{N}$.

EXAMPLE 2.4. Let $L = \{(x, 0) : x \in \mathbb{R}, 0 \leqq x \leqq 1\} \cup \{(1, y) : y \in \mathbb{R}, 0 \leqq y \leqq 1\} \cup \{(0, 1)\}$, where $\mathbb{R}$ is the set of real numbers. Let L be ordered by the usual cartesian ordering. Then L is a lattice which is weakly upper semimodular, but L does not satisfy the strong chain condition.

Definition 1.1.2 might be considered to be a bit disappointing in the light of Definition 1.1.1. One might hope that 1.1.2 would read that whenever $a \wedge b$, $a \vee b$ exist and $a, b \succ a \wedge b$, then $a \vee b \succ a, b$. The next example is to illustrate that even for a lower semilattice S, the weakened definition need not imply the weak chain condition.

EXAMPLE 2.5. Let $S = \{(0,0), (0,\frac{1}{2}), (\frac{1}{2},1), (\frac{1}{2},0), (1,0), (1,\frac{1}{2}), (1,1)\} \cup \{(x,x) : x \in \mathbb{R}$ and $\frac{1}{2} < x < 1\}$. Let S be ordered by the usual cartesian ordering. Then S is a lower semilattice such that, whenever $a \wedge b$, $a \vee b$ exist and $a, b \succ a \wedge b$, then $a \vee b \succ a, b$. However, there are maximal chains from $(0,0)$ to $(1,1)$ of lengths 3 and 4.

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