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Trend to equilibrium solution for the discrete Safronov–Dubovskiĭ aggregation equation with forcing

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We consider the discrete Safronov-Dubovskiĭ aggregation equation associated with the physical condition, where particle injection and extraction take place in the dynamical system. In application, this model is used to describe the aggregation of particle-monomers in combination with sedimentation of particle-clusters. More precisely, we prove well-posedness of the considered model for a large class of aggregation kernel with source and efflux coefficients. Furthermore, over a long time period, we prove that the dynamical model attains a unique equilibrium solution with an exponential rate under a suitable condition on the forcing coefficient.

Keywords: Safronov–Dubovskiĭ aggregation equation; forcing coefficients; well-posedness; equilibrium solution; exponential convergence

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1. Introduction

Aggregation is a fundamental dynamic process to augment an animate or inanimate matter. Technically, the aggregation process describes an event where two smaller particles merge to form a large cluster. Generally, by this particulate process, although the number of daughter particles gradually decreases; however, the size of the new particle increases. In 1916, M. von Smoluchowski proposed a mathematical structure for this particulate process, well known as Smoluchowski's aggregation equation (SAE) [15]. In practical field of study, this equation is ubiquitously used to describe cloud physics [12], oceanography phenomena [2] and different models in chemistry [1]. In this context, if $a_i(t)$, $i \in \mathbb{N}\backslash\{0\}$ denotes the concentration of i-clusters at the time $t \geqslant 0$ then the SAE reads as

$$\frac{\mathrm{d}a_i(t)}{\mathrm{d}t} = \mathcal{W}_i(\mathbf{a}(t)) \quad \text{with initial data} \quad a_i(0) = a_i^{in} \geqslant 0, \tag{1.1}$$

where $\mathcal{W}_i(\mathbf{a}(t)) := \sum_{j=1}^{i-1} \mathscr{K}_{i-j,j} a_{i-j}(t) a_j(t) - \sum_{j=1}^{\infty} \mathscr{K}_{i,j} a_i(t) a_j(t)$. The first and second terms in the right-hand side of $\mathcal{W}_i(\mathbf{a}(t))$ represent the birth and death coefficients, respectively. In general, $\mathscr{K}_{i,j}$ defines the aggregation kernel and is symmetric with respect to i, j, which physically indicates the rate at which i-mers merge with j-mers to form a large $(i+j)$-cluster. It is worth to note that due to SAE (1.1) the clusters can only grow in size while no smaller particles are created or inserted in the system. Therefore, existence of a equilibrium solution cannot be achieved from the original form of SAE (1.1). In their article, Carr and Costa [**3**] proved that for a certain class of aggregation kernel the gelation phenomena may occur at any time interval. However, there is a handful number of articles on various mathematical aspects such as existence, uniqueness and large-time analysis of the solution to the SAE (1.1) and its continuous form. In later years, the SAE was extended as in [**11**, **13**] to achieve its stationary state by adding source and efflux term

$$\frac{\mathrm{d}a_i(t)}{\mathrm{d}t} = \mathcal{W}_i(\mathbf{a}(t)) + s_i - r_i a_i(t). \tag{1.2}$$

Here, s_i represents the rate at which the cluster of i-mers is injected into the system. The third term is the removal term, which indicates the cluster of size i is removed from the system at a rate $r_i a_i$. These two factors are together called forcing coefficients as they play a vital role to add or remove i-mers from the system. The existence of stationary solutions and their uniqueness is well studied in [**4**, **17**]. In the recent years, for a large class of coagulation kernels, the well-posedness and convergence to the unique equilibrium to equation (1.2) have been studied in [**14**].

During 1990 s, renowned mathematician Pavel B. Dubovskiĭ has introduced a new mechanism of discrete aggregation process, which plays a significant role to describe several physical fields of studies such as astrophysics (cloud forming), cosmology (formation of planets and galaxies), astronomy (asteroid size distribution), etc.

In the past few decades [**9**, **10**]. In literature, this discrete aggregation model is referred as the discrete Safronov–Dubovskiĭ aggregation equation (SDAE) [**6**, **8**]. According to SDAE, for the above-mentioned discrete size distribution function $a_i(t)$, we consider that $a_0(t)$ is equal to zero at any time $t \geqslant 0$. With this consideration, the governing equation, which describes the time evolution dynamics of cluster growth, is given by

$$\frac{\mathrm{d}a_i(t)}{\mathrm{d}t} = a_{i-1}(t)\sum_{j=1}^{i-1}\beta_{i-1,j}\, j a_j(t) - a_i(t)\sum_{j=1}^{i}\beta_{i,j}\, j a_j(t) - a_i(t)\sum_{j=i}^{\infty}\beta_{i,j} a_j(t). \tag{1.3}$$

We now interpret the terms appearing in the right-hand side of equation (1.3). The aggregation kernel $\beta_{i,j}$ (with $i \neq j$) denotes the collision rate of i-mers with j-mers. In general, $\beta_{i,j}$ is a non-negative and symmetric function. The first term in the right-hand side of (1.3) represents the inclusion of i-mer particles into the system. This i-mer particles formed during the collision of $(i-1)$-mers with the monomers formed by the breakup of a j-mer into j monomers. In the similar way, the second and third sum represents the removal or death of i-mers from the system due to the fusion of the monomers with i-mers and forming a larger $(i+1)$-mer. For a detailed interpretation of the second and third terms, readers can refer to [**6**].

Since the pioneer studies of Dubovskiĭ [**9**, **10**], the discrete SDAE acquired a remarkable popularity in experimental research. In the literature of Safronov–Dubovskiĭ aggregation model, there are few articles available, which deal with the mathematical aspect such as existence, uniqueness, mass conservation, large time analysis or convergence of equilibrium solution to the discrete SDAE (1.3) [**6**–**8**, **16**]. After the classical works of Dubovskiĭ, Wattis [**16**] and Davidson [**8**] reported some works on the mathematical aspect of discrete SDAE. Recently, Das and Saha [**6**] established the existence of unique mass conserving solution for a large class of unbounded coagulation rate. Therefore, to summarize, most of the current research is devoted to the existence and uniqueness of solutions of the original Safronov–Dubovskiĭ aggregation equation. It turns out that achieving steady state becomes inevitable to stabilize several cosmological phenomena or to reduce the expense of some physical experiments. Like SAE, the discrete SDAE is pure aggregation equation, thus no smaller daughter particle can be generated by the system (1.3). Similar to the SAE with source and efflux term (1.2), it can be expected that the discrete SDAE with effect of external force approaches an equilibrium state as $t \to \infty$. In this regard, we consider the discrete Safronov–Dubovskiĭ aggregation equation with source and efflux term reads

$$\frac{\mathrm{d}a_i\,(t)}{\mathrm{d}t} = \mathcal{F}_i(\mathbf{a}(t)), \quad i \geqslant 1, \tag{1.4}$$

associated with the initial data

$$a_i(0) = a_i^{in}. \tag{1.5}$$

The operator $\mathcal{F}$ is defined as

$$\begin{aligned}\mathcal{F}_i(\mathbf{a}(t)) := a_{i-1}\,(t) \sum_{j=1}^{i-1} \beta_{i-1,j}\, j a_j\,(t) - a_i\,(t) \sum_{j=1}^{i} \beta_{i,j}\, j a_j\,(t) \\ - a_i\,(t) \sum_{j=i}^{\infty} \beta_{i,j} a_j\,(t) + s_i - r_i a_i(t).\end{aligned} \tag{1.6}$$

The non-linear initial value problem (IVP) (1.4)–(1.5) represents the time evolution of the cluster growth dynamics under the effect of external force, in which the *source function* permits an external supply of particles into the system with rate s_i. On the other hand, the *efflux term* represents the rate $r_i a_i$, at which the particles are removed from the system. In application, this efflux term is often considered to define the sedimentation of particles due to gravity. We can retain the original SDAE (1.3) by setting $s_i = 0 = r_i$ in equation (1.4).

The objective of this article is devoted into two parts. First, we need to establish the well-posedness of the evolution equation (1.4) for a large class of aggregation kernels and removal rates which consolidate the model. This discussion is necessary to validate the model mathematically. Secondly, we show that for large time interval, the solutions converge to a unique equilibrium with exponential rate of convergence. This will be the first evidence in literature to discuss the stationary state solutions for the SDAE coupled with source and efflux components. Another underlying motivation for studying equation (1.4) is to work towards establishing a relation with additional differential equations, which is a common theme in the context of reaction–diffusion systems. Before the extension can be achieved, we should understand the motivation behind considering the injection and extraction terms in equation (1.4).

For particulate events, the moment functions take part an important role as some of them bear relation to significant physical entities. In this regard, for any $\mu \geqslant 0$, we define the $\mu - th$ order moment and the truncated moment of a solution to equation (1.4) as

$$\mathcal{M}_{\mu}\left(t\right)=\sum_{i=1}^{\infty} i^{\mu} a_i\left(t\right), \qquad \text{and} \qquad \mathcal{M}_{\mu}^{m}\left(t\right)=\sum_{i=1}^{m} i^{\mu} a_i\left(t\right) \quad \text{respectively.} \tag{1.7}$$

In general, zeroth-order moment denotes the total number and first-order moment represents the total mass of particles in the system. From the classical principle of conservation laws, mass can neither be created nor destroyed in any particulate system. Therefore, we can expect that the total mass $\mathcal{M}_1(t)=\sum_{i=1}^{\infty} i a_i(t)$ will remain unaltered by the original form of SDAE (1.3). In a current study, Das and Saha [**6**] studied the mass conserving behaviour for a large class of aggregation rate

$$\beta_{i,j} \leqslant \beta_0\left(1+i+j\right)^{\alpha} \quad \text{with} \quad 0 \leqslant \alpha \leqslant 1.$$

Some more evidence on mass conservation of the SDAE can be found in [**8**]. In contrast to the mass conserving behaviour, recently Das and Saha [**7**] proved the occurrence of mass-loss phenomena at any time interval for the kinetic kernels satisfying the rate

$$\beta_{\mathcal{L}}\left(i^{\alpha} j^{\beta}+i^{\beta} j^{\alpha}\right) \leqslant \beta_{i,j} \leqslant \beta_{\mathcal{U}}\left(1+i\right)^{\omega}\left(1+j\right)^{\omega} \quad \text{with} \quad 0 \leqslant \alpha \leqslant \beta \text{ and } \beta, \omega > 1.$$

Moreover, the authors also highlighted that the system (1.3) will be ill-posed for the aggregation kernel with growth rate $\beta_{i,j}=i^{\lambda}+j^{\lambda}$, with $\lambda>1$.

In this context, to attain the convergence to a unique equilibrium, we impose the additional force. More precisely, the current article adopts a removal rate with the coefficient r_i such that the efflux term $r_i a_i$ causes the solution of (1.4) to exhibit analogous properties as a solution to the system with strong fragmentation regime. However, the consequence to inject or remove a particle externally is that, the mass conserving behaviour cannot be expected for (1.4). However, $\mathcal{M}_1$ in general satisfies the equation

$$\frac{\mathrm{d}\mathcal{M}_1(t)}{\mathrm{d}t}=\sum_{i=1}^{\infty} i s_i-\sum_{i=1}^{\infty} i r_i a_i(t).$$

In spite of having high non-linearity due to aggregation coefficients in the considered model (1.4), our study successfully proves the existence of stationary state solutions as $t \to \infty$. In this regard, to establish well-posedness or achieve steady state, the growth rate of the source and efflux coefficients s_i and r_i play a significant role to attain the convergence to a unique equilibrium. As already mentioned, the removal coefficient r_i usually sketches the effect of sedimentation. In this circumstance, we assume that all the clusters are spherical and represented by their concentration $i \in \mathbb{N}$ along with the fractal dimension $D(\gamma)$. Under these conditions, we can obtain the scaling $r_i \sim i^{\gamma}$ [**2**, **12**].

On the other hand, concerning the source term s_i, a general assumption is that only monomers are injected, i.e. $s_i = 0$ for $i > 1$. In the present study, we allow the particle to be injected into the system at a more general rate. More precisely, we only require that, with the increment concentration of cluster, the injection rate s_i decreases sufficiently fast.

The work is organized as follows. In § 2, we state some preliminary definitions on the solution and equilibrium solution of equation (1.4), which are essentially required in the subsequent discussion. Moreover, we brief all the conditions, which we assume on the kinetic coefficients. Based on these conditions, we estimate several higher-order moments of the solution to equation (1.4) in § 3. In the subsequent § 4, we establish the existence of solution to the IVP (1.4)–(1.5). In subsection 4.1, we introduce the truncated form of the IVP (1.4) followed by the proof of local existence of solutions to the truncated problem. Later in subsection 4.2, we prove the global existence theorem with the help of some strong convergence result like Arzelà–Ascoli theorem. We prove a contraction property in § 5, which plays a key role for existence and the rate of convergence to the equilibrium solution. Moreover, with this property, we discuss the uniqueness of solution to the IVP (1.4)–(1.5) in the same section. In the second part of this article, we achieve equilibrium solution with the help of previously obtained contraction property in § 6. Furthermore, we also prove that the solution converges to the steady state with exponential rate under a suitable smallness condition on the kinetic coefficient. Finally, we end our article by drawing some conclusions of the contribution in § 7.

2. Preliminaries: definition and hypothesis

For $\mu \geqslant 0$, let ℓ^1_μ denote the weighted ℓ_1 space of real sequence $\mathbf{a} := \{a_i\}_{i=1}^{\infty}$, defined as

$$\ell^1_\mu := \left\{ \mathbf{a} = \{a_i\}_{i\in\mathbb{N}} \;\middle|\; a_i \in [0, \infty) \text{ for all } i \in \mathbb{N} \text{ and } \|\mathbf{a}\|_{\ell^1_\mu} := \sum_{i=1}^{\infty} i^{\mu} |a_i| < \infty \right\}.$$

Since the system of differential equation (1.4)–(1.5) is an infinite dimensional system, we need to define the solution.

Definition 2.1. *Let $T > 0$ be given. A solution of the Cauchy problem (1.4)–(1.5) on $[0, T)$ with the initial data $a_i(0) =: \mathbf{a}^{in} \in \ell^1_1$ is a continuous function $a_i(t) : [0, T) \to [0, \infty)$ for all $i \in \mathbb{N}$ such that*

(i) $a_i(0) = a_i^{in}$ *for all* $i \in \mathbb{N}$,

(ii) *for any* $\mu \geqslant 1$, *we have* $\mathbf{a} \in L^{\infty}([0, T), \ell^1_1) \cap C^1((0, T), \ell^1_\mu)$, *and*

(iii) *for each* $i \in \mathbb{N}$ *equation (1.4) satisfies for all* $t \in (0, T)$.

The solution $\mathbf{a}$ will be global if $T = \infty$.

Definition 2.2. *A solution $\mathbf{a}$ to the problem (1.4)–(1.5) is considered a stationary solution (global) if $\mathbf{a}$ remains independent of time, satisfying the condition:*

$$\mathcal{F}_i(\mathbf{a}) = 0, \quad i \geqslant 1.$$

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Also, **a** will be a equilibrium solution of the initial value problem (1.4)–(1.5) if it is a global stationary solution for the same problem.

In this article, we prove the results under the following assumptions on the coefficient of equation (1.4),

HYPOTHESIS 2.1.

(H_1): There exist some constants $A_* > 0$ such that $\min\{i, j\}\beta_{i,j} \leqslant A_*(i^\alpha j^\beta + i^\beta j^\alpha)$ with $\alpha \leqslant \beta$ and $\alpha, \beta \in [0, 1]$.

(H_2): The efflux coefficient has rapid growth for large cluster size, i.e. there exists a positive constant R_* such that $r_i \geqslant R_* i^\gamma$, where $\gamma > \alpha + \beta$.

(H_3): The source term has a fast decay rate, i.e. for each $\mu \geqslant 0$ there exists a positive constant S_*^μ such that $\sum_{i=1}^{\infty} i^\mu s_i \leqslant S_*^\mu$.

In lieu of the above hypotheses, the aggregation kernel enjoys the unbounded kinetic rates at infinity. With assumption (H_1), the aggregation rate covers the well-known *diffusion-controlled growth* kernel, $\beta_{i,j} = i^{-2/3} + j^{-2/3}$ (see [**5**]).

In the second hypothesis, the exponent γ plays a crucial role in our study to obtain a suitable moment estimation. Moreover, this restriction on the *efflux term* includes a significant example of sedimentation of particles aggregating due to Brownian motion, i.e. $r_i = i^{2/3}$.

Lastly, hypothesis (H_3) describes that the injected clusters gradually decrease with the size of the cluster. More precisely, the *source term* s_i is permanently zero for large i-mers.

For notational convenience, we will use the following scaling factors;

$$\widehat{A}_* := \frac{A_*}{R_*} \quad \text{and} \quad \widehat{S}_*^\mu := \frac{S_*^\mu}{R_*} \quad \text{for all } \mu \geqslant 0. \tag{2.1}$$

To estimate the moments, we note the following moment equation for a solution of equation (1.4).

LEMMA 2.3. *Let* **a** *be a solution to equation* (1.4) *and* $\varphi = \{\varphi_i\}_{i\in\mathbb{N}}$ *is a positive sequence of real numbers with at most polynomial growth, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{\infty}\varphi_i a_i = \sum_{i=1}^{\infty}\sum_{j=1}^{i}\left[j\left(\varphi_{i+1}-\varphi_i\right)-\varphi_j\right]\beta_{i,j}a_i a_j + \sum_{i=1}^{\infty}\varphi_i s_i - \sum_{i=1}^{\infty}\varphi_i r_i a_i.$$

Proof. The proof of the lemma is straightforward. □

3. Moment estimation

We obtain *a priori* estimates of several higher-order moments of solution to equation (1.4). These estimations are the primary requirement for the proofs of several theorems. In this regard, we now prove the following sequence of lemmas starting with the uniform-boundedness of the first moment, i.e. total mass. Based on this result, we derive a differential inequality for the higher-order moments, and by the application of Grönwall's inequality, we can obtain the estimation for all other moments.

LEMMA 3.1. *Assume* (H_1), (H_2) *and* (H_3) *hold; also consider that* $\mathbf{a}$ *be a solution to problem* (1.4) *with the initial data* (1.5). *Then the corresponding first moment* $\mathcal{M}_1$ *is uniformly bounded, that is, if we denote* $\mathcal{M}_1^{in} := \mathcal{M}_1(0)$, *we have*

$$\mathcal{M}_1(t) \leqslant \mathfrak{M} := \max\left\{\mathcal{M}_1^{in}, \widehat{S}_*^1\right\}.$$

In particular, there exists a time $T > 0$ *such that* $\mathcal{M}_1(t) \leqslant 2\widehat{S}_*^1$, *for all* $t \geqslant T$. *Here,* T *depends only on the constants* $\mathcal{M}_1^{in}$, $\widehat{S}_*^1$ *and* R_*.

Proof. Choose $\varphi_i = i\chi_{\{i\leqslant m\}}$ in lemma 2.3 and using hypotheses (H_2) and (H_3), we have

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{M}_1^m(t)}{\mathrm{d}t} &= \sum_{i=1}^{m-1}\sum_{j=1}^{i}(i+1)\,j\beta_{i,j}a_ia_j - \sum_{i=1}^{m}\sum_{j=1}^{i} ij\beta_{i,j}a_ia_j - \sum_{i=1}^{m}\sum_{j=i}^{\infty} i\beta_{i,j}a_ia_j \\
&\quad + \sum_{i=1}^{m} is_i - \sum_{i=1}^{m} ir_ia_i \\
&\leqslant \sum_{i=1}^{m}\sum_{j=1}^{i} j\beta_{i,j}a_ia_j - \sum_{i=1}^{m}\sum_{j=i}^{m} i\beta_{i,j}a_ia_j + S_*^1 - R_*\mathcal{M}_1^m(t) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} j\beta_{i,j}a_ia_j - \sum_{i=1}^{m}\sum_{j=i}^{m}(i+j)\,\beta_{i,j}a_ia_j + \sum_{\substack{i=1\\ j=i}}^{m} i\beta_{i,i}a_i^2 + S_*^1 - R_*\mathcal{M}_1^m(t) \\
&= \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}(i+j)\,\beta_{i,j}a_ia_j - \sum_{i=1}^{m}\sum_{j=i}^{m}(i+j)\,\beta_{i,j}a_ia_j \\
&\quad + \sum_{i=1}^{m} i\beta_{i,i}a_i^2 + S_*^1 - R_*\mathcal{M}_1^m(t).
\end{aligned}$$

For first three sums, substitute $\phi_{i,j} := (i+j)\beta_{i,j}a_ic_j$ in proposition 2.1 of [6], we can obtain

$$\frac{\mathrm{d}\mathcal{M}_1^m(t)}{\mathrm{d}t} + R_*\mathcal{M}_1^m(t) \leqslant S_*^1 \quad \text{implies} \quad \mathcal{M}_1^m(t) \leqslant \left(\mathcal{M}_1^m(0) + S_*^1 t\right) - R_*\int_0^t \mathcal{M}_1^m(\xi)\mathrm{d}\xi. \tag{3.1}$$

Using Grönwall's inequality to (3.1) and using relation (2.1), we get

$$\mathcal{M}_1^m(t) \leqslant \left(\mathcal{M}_1^m(0) - \widehat{S}_*^1\right)\exp\left(-R_*t\right) + \widehat{S}_*^1. \tag{3.2}$$

Therefore, $\mathcal{M}_1^m(t) \leqslant \max\left\{\mathcal{M}_1^m(0),\, \widehat{S}_*^1\right\}$. Now using $\mathcal{M}_1^m(0) \leqslant \mathcal{M}_1^{in}$ and taking the limit $m \to \infty$, we can get the first result.

The second claim will be obvious for $\mathcal{M}_1^{in} = 0$. If $\mathcal{M}_1^{in} > 0$, the claim follows from relation (3.2) that it suffices to take $T > \max\left\{0,\, \frac{\log(\widehat{S}_*^1/\mathcal{M}_1^{in})}{R_*}\right\}$. □

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We derive a differential inequality which will be used for the higher-order (greater than one) moment estimation of the corresponding solution to equation (1.4).

Lemma 3.2. *Assume* $(H_1)--(H_3)$ *hold; also consider that* **a** *be a solution to problem* (1.4) *with the initial condition* (1.5)*. Then for each* $\mu > 1$*, we have the following differential inequality for the truncated moments corresponding to* **a***;*

$$\frac{\mathrm{d}\mathcal{M}_{\mu}^{m}(t)}{\mathrm{d}t} + \frac{R_*}{2}\mathcal{M}_{\mu+\gamma}^{m}(t) \leqslant \frac{(qR_*)^{1-p}}{2p}\left(2^{\mu+2}A_*\right)^p\left(\mathcal{M}_1^m(t)\right)^{1+p} + S_*^{\mu},$$

where $p := \frac{\mu+\gamma-1}{\gamma-\alpha-\beta}$ *and* $q := \frac{p}{p-1}$.

Proof. Taking the sum on equation (1.4) associated with the weight $\varphi_i = i^{\mu}\chi_{\{i\leqslant m\}}$, we get

$$\begin{aligned}\frac{\mathrm{d}\mathcal{M}_{\mu}^{m}(t)}{\mathrm{d}t} &= \sum_{i=1}^{m-1}\sum_{j=1}^{i}(i+1)^{\mu}\, j\beta_{i,j}a_i a_j \\ &\quad - \sum_{i=1}^{m}\left[\sum_{j=1}^{i} i^{\mu} j\beta_{i,j}a_i a_j + \sum_{j=i}^{\infty} i^{\mu}\beta_{i,j}a_i a_j - i^{\mu}s_i + i^{\mu}r_i a_i\right] \\ &\leqslant \sum_{i=1}^{m}\left[\sum_{j=i}^{m}\left(\left((j+1)^{\mu}-j^{\mu}\right)i - i^{\mu}\right)\beta_{i,j}a_i a_j + i^{\mu}s_i - i^{\mu}r_i a_i\right]\end{aligned}$$

As $\mu > 1$, we use the inequality $(i+j)^{\mu} \leqslant 2^{\mu-1}(i^{\mu}+j^{\mu})$ on the right-hand side of the above estimation

$$\begin{aligned}\frac{\mathrm{d}\mathcal{M}_{\mu}^{m}(t)}{\mathrm{d}t} &\leqslant \sum_{i=1}^{m}\left[\sum_{j=i}^{m}\left(2^{\mu}i-1\right)\left(i^{\mu}+j^{\mu}\right)\beta_{i,j}a_i a_j + i^{\mu}s_i - i^{\mu}r_i a_i\right] \\ &\leqslant \sum_{i=1}^{m}\left[\sum_{j=i}^{m} 2^{\mu} i j^{\mu}\beta_{i,j}a_i a_j + i^{\mu}s_i - i^{\mu}r_i a_i\right].\end{aligned} \tag{3.3}$$

Recalling hypothesis (H_1), that is $\beta_{i,j} \leqslant A_*(i^{\alpha}j^{\beta} + i^{\beta}j^{\alpha})$ together with (H_2) and (H_3) on inequality (3.3), we have

$$\frac{\mathrm{d}\mathcal{M}_{\mu}^{m}(t)}{\mathrm{d}t} \leqslant 2^{\mu+1}A_*\mathcal{M}_1^m(t)\mathcal{M}_{\mu+\alpha+\beta}^{m}(t) + S_*^{\mu} - R_*\mathcal{M}_{\mu+\gamma}^{m}(t). \tag{3.4}$$

Thanks to Hölder's inequality, we can write

$$\mathcal{M}_{\mu+\alpha+\beta}^{m}(t) \leqslant \left(\mathcal{M}_1^m(t)\right)^{\frac{1}{p}}\left(\mathcal{M}_{\mu+\gamma}^{m}(t)\right)^{\frac{1}{q}} \tag{3.5}$$

and thus inequality (3.4) is written as

$$\frac{\mathrm{d}\mathcal{M}_{\mu}^{m}(t)}{\mathrm{d}t} \leqslant 2^{\mu+1}A_*\left(\mathcal{M}_1^m(t)\right)^{1+\frac{1}{p}}\left(\mathcal{M}_{\mu+\gamma}^{m}(t)\right)^{\frac{1}{q}} + S_*^{\mu} - R_*\mathcal{M}_{\mu+\gamma}^{m}(t). \tag{3.6}$$

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Using Young's inequality (with ϵ) yields that

$$2^{\mu+1} A_* \left(\mathcal{M}_1^m(t)\right)^{1+\frac{1}{p}} \left(\mathcal{M}_{\mu+\gamma}^m(t)\right)^{\frac{1}{q}} \leqslant \epsilon \left(\mathcal{M}_{\mu+\gamma}^m(t)\right) + \frac{(q\epsilon)^{1-p}}{p} \left(2^{\mu+1} A_*\right)^p \left(\mathcal{M}_1^m(t)\right)^{1+p}. \tag{3.7}$$

Hence, combining both inequalities (3.6) and (3.7) for $\epsilon = \frac{R_*}{2}$, we can obtain the desired result. □

We now obtain a generalized non-linear differential inequality which will be used to estimate the higher-order moments.

LEMMA 3.3. *Let $\zeta > 0$ and $\phi \in C([0, \infty), \mathbb{R}_{\geqslant 0}) \cap C^1((t_0, \infty))$ for some $t_0 \geqslant 0$, satisfy*

$$\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} + \mathcal{A}\left[\phi(t)\right]^{1+\zeta} \leqslant \mathcal{B} \quad \textit{for all} \quad t \geqslant t_0 \tag{3.8}$$

with some constants $\mathcal{A}$ and $\mathcal{B}$, then

$$\phi(t) \leqslant \max\left\{ \left(\frac{2\mathcal{B}}{\mathcal{A}}\right)^{\frac{1}{1+\zeta}}, \left(\frac{2}{\zeta\mathcal{A}}\right)^{\frac{1}{\zeta}} (t-t_0)^{-\frac{1}{\zeta}} \right\} \quad \textit{for all} \quad t \geqslant t_0.$$

Proof. We claim that the set

$$\left\{ t > t_0 \;\middle|\; \phi(t) \leqslant \left(\frac{2\mathcal{B}}{\mathcal{A}}\right)^{\frac{1}{1+\zeta}} \right\}$$

is non-empty. Otherwise, assume the contrary that

$$\phi(t) > \left(\frac{2\mathcal{B}}{\mathcal{A}}\right)^{\frac{1}{1+\zeta}} \quad \text{for all} \quad t > t_0. \tag{3.9}$$

Combining inequalities (3.8) and (3.9), we get

$$\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} \leqslant -\mathcal{B} \quad \text{which implies} \quad \phi(t) \leqslant \phi(t_0) - (t-t_0)\,\mathcal{B} \quad \text{for all} \quad t > t_0. \tag{3.10}$$

Thus, $\phi(t) < 0$ whenever $t > t_0 + \frac{\phi(t_0)}{\mathcal{B}}$, which contradicts the non-negativity of ϕ. Next, assume that

$$T := \inf\left\{ t > t_0 \;\middle|\; \phi(t) \leqslant \left(\frac{2\mathcal{B}}{\mathcal{A}}\right)^{\frac{1}{1+\zeta}} \right\}.$$

If T does not exist, then there exist a $t' \geqslant T$ and $\epsilon > 0$ such that

$$\phi(t) > \phi(t') = \left(\frac{2\mathcal{B}}{\mathcal{A}}\right)^{\frac{1}{1+\zeta}} \quad \text{for all} \quad t \in (t', t'+\epsilon). \tag{3.11}$$

Again by similar argument as (3.10), we get $\phi(t) < \phi(t')$ for $t \in (t',\, t'+\epsilon)$, which contradicts relation (3.11).

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If $T = t_0$, then the proof is complete. Otherwise if $T > t_0$, we consider the interval (t_0, T). The definition of T gives $\mathcal{B} \leqslant \frac{\mathcal{A}}{2}[\phi(t)]^{1+\zeta}$. Using this bound on inequality (3.8), we get

$$\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} + \frac{\mathcal{A}}{2}\left[\phi(t)\right]^{1+\zeta} \leqslant 0 \quad \text{that is} \quad \phi(t) \leqslant \left(\frac{1}{(\phi(t_0))^{-\zeta} + \frac{\mathcal{A}\zeta}{2}(t-t_0)}\right)^{\frac{1}{\zeta}} \quad \text{for all} \quad t \in (t_0, T)\,.$$

Using the non-negativity of $\phi(t_0)$, we get

$$\phi(t) \leqslant \left(\frac{2}{\zeta\mathcal{A}}\right)^{\frac{1}{\zeta}} (t-t_0)^{-\frac{1}{\zeta}} \quad \text{for all} \quad t \in (t_0, T). \tag{3.12}$$

Hence, the proof is completed for all $t \geqslant t_0$. □

With the help of lemma 3.2 and lemma 3.3, we can estimate the higher-order moments of solution to (1.4).

Lemma 3.4. *Assume $(H_1) - -(H_3)$ hold; also consider that* $\mathbf{a}$ *be a solution to problem (1.4) with the corresponding first moment $\mathcal{M}_1$ and initial condition (1.5). Then for any $\mu > 1$,*

$$\mathcal{M}_\mu(t) \leqslant \max\left\{\left(\frac{2^{1+p}\left(2^{\mu+1}\widehat{A}_*\right)^p q^{1-p}}{p}\mathfrak{M}^{1+p+\lambda_\mu} + 2\widehat{S}_*^{\mu}\mathfrak{M}^{\lambda_\mu}\right)^{\frac{1}{1+\lambda_\mu}}, \mathfrak{M}\left(\frac{4}{R_*\lambda_\mu t}\right)^{1/\lambda_\mu}\right\},$$

where $\lambda_\mu := \frac{\gamma}{\mu-1}$ and p, q defined in lemma 3.2. In particular, there exists a constant $\mathcal{C}_\mu$ such that $\mathcal{M}_\mu(t) \leqslant \mathcal{C}_\mu(1 + t^{-1/\lambda_\mu})$. Note that $\mathcal{C}_\mu$ depends only on the constants namely, α, β, γ, $\widehat{S}_*^1$, $\widehat{S}_*^\mu$, $\widehat{A}_*$ and R_*.

Proof. Thanks to the Hölder's inequality, which gives the following result for the higher-order truncated moments

$$\left(\mathcal{M}_\mu^m(t)\right)^{\frac{\mu+\gamma-1}{\mu-1}} \leqslant \left(\mathcal{M}_{\mu+\gamma}^m(t)\right)\left(\mathcal{M}_1^m(t)\right)^{\frac{\gamma}{\mu-1}}. \tag{3.13}$$

The uniform boundedness of the first truncated moment $\mathcal{M}_1^m$ yields

$$\mathcal{M}_{\mu+\gamma}^m(t) \geqslant \left(\mathcal{M}_\mu^m(t)\right)^{\frac{\mu+\gamma-1}{\mu-1}} \mathfrak{M}^{-\frac{\gamma}{\mu-1}}.$$

Using the bound $\mathcal{M}_1^m(t) \leqslant \max\left\{\mathcal{M}_1^m(0),\ \widehat{S}_*^1\right\}$ and the above inequality on lemma 3.2, we have the following required differential inequality

$$\frac{\mathrm{d}\mathcal{M}_\mu^m(t)}{\mathrm{d}t} + \frac{R_*}{2}\mathfrak{M}^{-\frac{\gamma}{\mu-1}}\left(\mathcal{M}_\mu^m(t)\right)^{1+\lambda_\mu} \leqslant \frac{(qR_*)^{1-p}\left(2^{\mu+2}A_*\right)^p}{2p}\mathfrak{M}^{1+p} + S_*^\mu. \tag{3.14}$$

Now, consider

$$\mathcal{A} := \frac{R_*}{2}\mathfrak{M}^{-\frac{\gamma}{\mu-1}} \quad \text{and} \quad \mathcal{B} := \frac{(qR_*)^{1-p}\left(2^{\mu+2}A_*\right)^p}{2p}\mathfrak{M}^{1+p} + S_*^\mu$$

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and apply lemma 3.3 for $t_0 = 0$, we can get the claim. Finally, we can get the estimation $\mathcal{M}_\mu(t) \leqslant \mathcal{C}_\mu(1 + t^{-1/\lambda_\mu})$ by considering $\mathcal{C}_\mu := \max\left\{(\frac{2\mathcal{B}}{\mathcal{A}})^{\frac{1}{1+\lambda_\mu}}, (\frac{2}{\zeta\mathcal{A}})^{\frac{1}{\lambda_\mu}}\right\}$. □

The following lemma estimates the higher-order moments in which the estimates are free from the initial data.

LEMMA 3.5. *Assume* $(H_1) - -(H_3)$ *hold; also consider that* $\mathbf{a}$ *be a solution to problem* (1.4) *with the corresponding first moment* $\mathcal{M}_1$. *If* $\lambda_\mu = \frac{\gamma}{\mu-1}$ *and* p, q *defined in lemma* 3.2, *then for any* $\mu > 1$ *there exists a time* $T > 0$ *(depends on* $\mathbf{a}$*) such that*

$$\mathcal{M}_\mu(t) \leqslant \left\{ \frac{2^{2+\lambda_\mu}}{p} \left(2^{\mu+3}\widehat{A}_*\right)^p q^{1-p} (\widehat{S}_*^1)^{1+p+\lambda_\mu} + 2^{2+\lambda_\mu} \widehat{S}_*^\mu \left(\widehat{S}_*^1\right)^{\lambda_\mu} \right\}^{\frac{1}{1+\lambda_\mu}} \tag{3.15}$$

and

$$\mathcal{M}_\mu(t) \leqslant 4 \left\{ \frac{\left(2^{\mu+3}\widehat{A}_*\right)}{p} q^{1-p} \left(\widehat{S}_*^1\right)^{1+p} + \widehat{S}_*^\mu \right\} \quad \textit{for all} \quad t \geqslant T. \tag{3.16}$$

Proof. From lemma 3.1, we get $T_1 > 0$ such that $\mathcal{M}_1(t) \leqslant 2S_*^1$ for all $t \geqslant T_1$. Using this inequality on relation (3.13) and proceed similarly as lemma 3.3 to get

$$\begin{aligned} &\frac{\mathrm{d}\mathcal{M}_\mu^m(t)}{\mathrm{d}t} + \frac{R_* \left(\widehat{S}_*^1\right)^{-\frac{\gamma}{\mu-1}}}{2^{1+\frac{\gamma}{\mu-1}}} \left(\mathcal{M}_\mu^m(t)\right)^{1+\lambda_\mu} \\ &\quad \leqslant \frac{(qR_*)^{1-p} \left(2^{\mu+3}A_*\right)^p}{p} \left(\widehat{S}_*^1\right)^{1+p} + S_*^\mu \quad \text{for all} \quad t \geqslant T_1. \end{aligned}$$

Setting $t_0 = T_1$, $\mathcal{A} := \frac{R_*(\widehat{S}_*^1)^{-\frac{\gamma}{\mu-1}}}{2^{1+\frac{\gamma}{\mu-1}}}$, $\mathcal{B} := \frac{(qR_*)^{1-p}(2^{\mu+3}A_*)^p}{p}(\widehat{S}_*^1)^{1+p} + S_*^\mu$ and applying lemma 3.3, we have

$$\mathcal{M}_\mu^m(t) \leqslant \max\left\{ \left(\frac{2\mathcal{B}}{\mathcal{A}}\right)^{\frac{1}{1+\lambda_\mu}}, \left(\frac{2}{\zeta\mathcal{A}}\right)^{\frac{1}{\lambda_\mu}} (t - T_1)^{-\frac{1}{\lambda_\mu}} \right\} \quad \text{for all} \quad t \geqslant T_1.$$

Choose $T_2 \geqslant T_1$, such that

$$\left(\frac{2}{\zeta\mathcal{A}}\right)^{\frac{1}{\lambda_\mu}} (t - T_1)^{-\frac{1}{\lambda_\mu}} \leqslant \left(\frac{2\mathcal{B}}{\mathcal{A}}\right)^{\frac{1}{1+\lambda_\mu}} \tag{3.17}$$

for all $t \geqslant T_2$. Now substituting the values of $\mathcal{A}$, $\mathcal{B}$ and taking limit $m \to \infty$, we obtain estimate (3.15) by recalling relation (2.1).

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We now proceed to get inequality (3.16). Since $\gamma \geqslant 0$, we have the following results hold true for any higher-order moments

$$\mathcal{M}_{\mu}^{m}(t) \leqslant \mathcal{M}_{\mu+\gamma}^{m}(t) \quad \text{and} \quad \mathcal{M}_{\mu}(t) \leqslant \mathcal{M}_{\mu+\gamma}(t) \tag{3.18}$$

for all $t \geqslant 0$. Combining these results together with $\mathcal{M}_1(T) \leqslant 2S_*^1$ for all $t \geqslant T_2$, lemma 3.2 generates

$$\frac{\mathrm{d}\mathcal{M}_{\mu}^{m}(t)}{\mathrm{d}t} + \frac{R_*}{2}\mathcal{M}_{\mu}^{m}(t) \leqslant \mathcal{B}.$$

Direct application of Grönwall's inequality and estimation (3.15) together give

$$\mathcal{M}_{\mu}^{m}(t) \leqslant \left(\frac{2\mathcal{B}}{\mathcal{A}}\right)^{\frac{1}{1+\lambda_\mu}} \exp\left[-\frac{R_*}{2}(t-T_2)\right] + \left(1-\exp\left[-\frac{R_*}{2}(t-T_2)\right]\right)\frac{2\mathcal{B}}{R_*}. \tag{3.19}$$

Suppose that $T \geqslant T_2$ such that

$$\left(\frac{2\mathcal{B}}{\mathcal{A}}\right)^{\frac{1}{1+\lambda_\mu}} \exp\left[-\frac{R_*}{2}(t-T_2)\right] \leqslant \frac{2\mathcal{B}}{R_*}.$$

Setting the above inequality on (3.19) and passing the limit $m \to \infty$, we obtain estimate (3.16) for $t \geqslant T$ with the help of relation (2.1). □

Remark 3.6. The above lemma 3.5 gives two different estimates of the moment function $\mathcal{M}_{\mu}(t)$. One of the estimates (3.15) depends on γ, and the second estimate (3.16) is independent of γ.

4. Existence of a solution

To prove the existence theorem, we adopt the similar approach which is frequently used for the discrete Safronov–Dubovskiĭ aggregation equation [6]. We truncate all the coefficients β, s, r and the initial data $\mathbf{a}^{in}$ of equation (1.4) as follows

$$\beta_{i,j}^{m} := \begin{cases} \beta_{i,j}, & \text{when} \quad 1 \leqslant i,j \leqslant m, \\ 0, & \text{otherwise,} \end{cases} \quad , \quad s_i^m := \begin{cases} s_i, & \text{when} \quad 1 \leqslant i \leqslant m, \\ 0, & \text{otherwise,} \end{cases}$$
$$r_i^m := \begin{cases} r_i, & \text{when} \quad 1 \leqslant i \leqslant m, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad a_i^{m,in} := \begin{cases} a_i^{in}, & \text{when} \quad 1 \leqslant i \leqslant m, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

Therefore, the above truncations generate the following system of $m(\geqslant 2)$ ordinary differential equations of model (1.4);

$$\begin{aligned} \frac{\mathrm{d}a_i^m}{\mathrm{d}t} = \quad & a_{i-1}^m \textstyle\sum_{j=1}^{i-1} j\beta_{i-1,j}^m a_j^m - a_i^m \sum_{j=1}^{i} j\beta_{i,j}^m a_j^m - a_i^m \sum_{j=i}^{m} \beta_{i,j}^m a_j^m + s_i^m - r_i^m a_i^m, \\ & \text{when} \quad i \leqslant m, \\ a_i^m(t) = \quad & 0, \qquad \text{when} \quad i > m. \end{aligned} \tag{4.2}$$

Note that $\mathbf{a}^m = (a_i^m)_{i\in\mathbb{N}}$ is also a solution of equation (1.4) where all the coefficients are defined as in (4.1). Also from (4.1), it is clear that $\beta_{i,j}^m$, s_i^m and r_i^m satisfy

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the assumptions (H_1), (H_2) and (H_3), respectively. So, all the moment estimations derived in § 3 remain valid for the truncated solution $\mathbf{a}^m$.

4.1. Existence of solution for finite dimensional system

The following proposition proves the existence of unique global solution to the truncated system (4.2).

PROPOSITION 4.1. *Assume that* $\mathbf{a}^{m,in} = \{a_i^{m,in}\}_{i\in\mathbb{N}} \in \ell_1^1$ *and non-negative. Then system* (4.2) *has unique solution with* $\mathbf{a}^m = (a_i^m)_{i\in\mathbb{N}} \in C^1([0,\, \infty),\, \ell_1^1)$ *for each* $i \in \mathbb{N}$*.*

Proof. Existence and uniqueness of solution of finite dimensional system (4.2) follows from the classical argument from theory of ordinary differential equations. In this regard, we define a polynomial function $f_i : \mathbb{R}^m \to \mathbb{R}^m$ as

$$f_i\left(x_1,\ldots,x_m\right) = x_{i-1}\sum_{j=1}^{i-1} j\beta_{i-1,j}x_j - x_i\sum_{j=1}^{i} j\beta_{i,j}x_j - x_i\sum_{j=i}^{m}\beta_{i,j}x_j + s_i^m - r_i^m x_i$$

such that the truncated equation (4.2) is written as

$$\frac{\mathrm{d}a_i^m(t)}{\mathrm{d}t} = f_i\left(a_1^m,\ldots,a_m^m\right), \quad \text{for all} \quad i=1,\ldots,m.$$

Therefore, each f_i is a polynomial function of components a_i, moreover it is locally Lipschitz continuous. Therefore, the existence and uniqueness of local solution $\mathbf{a}^m$ to the Cauchy problem (2.1) follows from the standard Picard–Lindelöf existence theorem. Thus, there exists a maximal time interval $a_i^m \in C^1([0,\, T_*))$ for all $i = 1,\, \ldots,\, m$.

Again for the non-negativity of the solution, we consider that for arbitrary $\epsilon > 0$, there exists a solution a_i^ϵ for the system, that is

$$\frac{\mathrm{d}a_i^\epsilon(t)}{\mathrm{d}t} = f_i\left(a_1^\epsilon,\ldots,a_m^\epsilon\right) + \epsilon. \quad \text{for all} \quad i=1,\ldots,m.$$

Also consider, for some $t_0 > 0$ and $1 \leqslant i \leqslant m$, we have $a_i^\epsilon(t_0) > 0$ and $a_r^\epsilon(t_0) = 0$ when $r \neq i$. Then, $\frac{\mathrm{d}a_r^\epsilon(t)}{\mathrm{d}t} = a_{r-1}^\epsilon(t)\sum_{j=1}^{r-1} j\beta_{r-1,j}a_j^\epsilon(t) + s_i + \epsilon > 0$. Taking $\epsilon \longrightarrow 0$, we get the non-negativity result.

Finally, for the global existence of solution (i.e. $T_* = \infty$), we use lemma 3.1 as follows

$$0 \leqslant a_i(t) \leqslant i^{-1}\sum_{i=1}^{m} ia_i \leqslant \mathfrak{M}. \tag{4.3}$$

So, the above result (4.3) shows that the solution cannot blow up on $[0,\, T_*)$, which implies the solution $\mathbf{a}^m \in C^1([0,\, \infty),\, \ell_1^1)$. □

4.2. Global existence theorem

In this section, we state and prove the main existence theorem of the global solution to the IVP (1.4)=(1.5). Before this, we will prove two important results

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which will be used to prove the compactness and passing the limit $n \to \infty$ on the local solution obtained from proposition 4.1.

LEMMA 4.2. *Assume conditions $(H_1)-(H_3)$ hold. Also consider that $\mathbf{a}^m$ be a solution to problem (4.2) with the corresponding initial first moment $\mathcal{M}_1^{in} := \mathcal{M}_1(0)$. Then for each fixed $i \in \mathbb{N}$, there exists a positive constant $\Re$ (depends on i but not on m) such that*

$$\left\| \frac{\mathrm{d}a_i^m(t)}{\mathrm{d}t} \right\|_{L^\infty(0,\infty)} \leqslant \Re \quad \text{for all} \quad m \geqslant i.$$

Proof. Using non-negativity of the coefficient β, s and the solution $\mathbf{a}^m$, together with condition (H_1) on equation (4.2), we get

$$\begin{aligned} \left| \frac{\mathrm{d}a_i^m(t)}{\mathrm{d}t} \right| \leqslant A_* \sum_{j=1}^{i-1} \left((i-1)^\alpha j^\beta + (i-1)^\beta j^\alpha \right) a_{i-1}^m a_j^m + \\ A_* \sum_{j=1}^{i} \left(i^\alpha j^\beta + i^\beta + j^\alpha \right) a_i^m a_j^m \\ + A_* \sum_{j=i}^{m} \left(i^\alpha j^\beta + i^\beta + j^\alpha \right) a_i^m a_j^m + s_i + r_i a_i^m. \end{aligned} \tag{4.4}$$

Applying the estimation $i^\mu a_i^m \leqslant \|a^m\|_{\ell^1_\mu}$ on the above inequality, we have

$$\left| \frac{\mathrm{d}a_i^m(t)}{\mathrm{d}t} \right| \leqslant 6A_* \|a^m\|_{\ell^1_\alpha} \|a^m\|_{\ell^1_\beta} + s_i \leqslant 6A_* \left(\mathcal{M}_1^m(t) \right)^2 + S_*^1 + \frac{r_i}{i} \|a^m\|_{\ell^1_1}.$$

Applying lemma 3.1, we conclude that

$$\left\| \frac{\mathrm{d}a_i^m(t)}{\mathrm{d}t} \right\|_{L^\infty(0,\infty)} \leqslant 6A_* \mathfrak{M}^2 + S_*^1 + \frac{r_i}{i} \mathfrak{M}. \qquad \square$$

PROPOSITION 4.3. *Let all the conditions $(H_1)-(H_3)$ hold. Also consider for the non-negative sequences $(\beta_{i,j}^n)$, $(r^n)_{n\in\mathbb{N}}$, $(s^n)_{n\in\mathbb{N}}$ and $(\mathbf{a}^n)_{n\in\mathbb{N}} \in L^\infty([0,\infty), \ell^1_1)$ for each $n \in \mathbb{N}$, there exists some sequence $\mathbf{a} = (a_i)_{i\in\mathbb{N}}$, $r = (r_i)_{i\in\mathbb{N}}$ and $s = (s_i)_{i\in\mathbb{N}}$ such that*

$$\beta_{i,j}^n \to \beta_{i,j}, \quad r_i^n \to r_i, \quad s_i^n \to s_i \quad \text{as} \quad n \to \infty \quad \text{for each } i, j \in \mathbb{N}.$$

as well as $a_i^n(t) \to a_i(t)$ as $n \to \infty$ uniformly on compact subset of $[0,\infty)$. Moreover, there exists a constant Ω_1 for $(\mathbf{a}^n)_{n\in\mathbb{N}}$ satisfying

$$\mathcal{M}_1^n(t) := \sum_{i=1}^{\infty} i a_i^n(t) \leqslant \Omega_1 \quad \text{for all} \quad t \geqslant 0$$

and for all $\mu > 1$ there exists some constant Ω_μ and $\lambda \in [0,1)$ for $(\mathbf{a}^n)_{n\in\mathbb{N}}$ such that

$$\mathcal{M}_\mu^n(t) := \sum_{i=1}^{\infty} i a_i^n(t) \leqslant \Omega_\mu \left(1 + t^{-\lambda} \right) \quad \text{for all} \quad t \geqslant 0.$$

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Then

(i) *For each fixed* $i \in \mathbb{N}$

$$\lim_{n\to\infty}\left(\sum_{j=1}^{i-1} j\beta^n_{i-1,j}a^n_{i-1}a^n_j - \sum_{j=1}^{i} j\beta^n_{i,j}a^n_i a^n_j - \sum_{j=i}^{\infty} \beta^n_{i,j}a^n_i a^n_j + s^n_i - r^n_i a^n_i\right)$$
$$= \sum_{j=1}^{i-1} j\beta_{i-1,j}a_{i-1}a_j - \sum_{j=1}^{i} j\beta_{i,j}a_i a_j - \sum_{j=i}^{\infty} \beta_{i,j}a_i a_j + s_i - r_i a_i \qquad (4.5)$$

converges uniformly on each compact subset of $(0, \infty)$.

(ii) *For each fixed* $i \in \mathbb{N}$ *and for any* $t > 0$

$$\lim_{n\to\infty}\left[\int_0^t \left(\sum_{j=1}^{i-1} j\beta^n_{i-1,j}a^n_{i-1}(\xi)a^n_j(\xi) - \sum_{j=1}^{i} j\beta^n_{i,j}a^n_i(\xi)a^n_j(\xi) - \sum_{j=i}^{\infty} \beta^n_{i,j}a^n_i(\xi)a^n_j(\xi) + s^n_i - r^n_i a^n_i(\xi)\right)\right] \mathrm{d}\xi$$
$$= \int_0^t \left(\sum_{j=1}^{i-1} j\beta_{i-1,j}a_{i-1}(\xi)a_j(\xi) - \sum_{j=1}^{i} j\beta_{i,j}a_i(\xi)a_j(\xi) - \sum_{j=i}^{\infty} \beta_{i,j}a_i(\xi)a_j(\xi) + s_i - r_i a_i(\xi)\right) \mathrm{d}\xi. \qquad (4.6)$$

Proof. With the help of Fatou's lemma and the assumption of the proposition, we can say that the limit sequence $(a_i)_{i\in\mathbb{N}}$ satisfies the estimates

$$\mathcal{M}_1(t) := \sum_{i=1}^{\infty} i a_i(t) \leqslant \Omega_1 \quad \text{and} \quad \mathcal{M}_\mu(t) := \sum_{i=1}^{\infty} i a_i(t) \leqslant \Omega_\mu\left(1 + t^{-\lambda}\right). \qquad (4.7)$$

We deduce result 4.5 by proving the convergence for each term separately, under the assumption $s^n_i \longrightarrow s_i$ as $n \longrightarrow \infty$ already assumed in the proposition. The locally uniform convergence of $a^n_i(t)$ to $a_i(t)$ on $[0, \infty)$ and finite number of terms in the following terms, we can get

$$\sum_{j=1}^{i-1} j\beta^n_{i-1,j}a^n_{i-1}a^n_j \xrightarrow{n\to\infty} \sum_{j=1}^{i-1} j\beta_{i-1,j}a_{i-1}a_j,$$
$$\sum_{j=1}^{i} j\beta^n_{i,j}a^n_i a^n_j \xrightarrow{n\to\infty} \sum_{j=1}^{i} j\beta_{i,j}a_i a_j \quad \text{and} \qquad (4.8)$$
$$r^n_i a^n_i \xrightarrow{n\to\infty} r_i a_i$$

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are also locally uniformly convergent on $[0, \infty)$. It remains to estimate the difference $\sum_{j=i}^{\infty} \beta_{i,j}^n a_i^n a_j^n - \sum_{j=i}^{\infty} \beta_{i,j} a_i a_j$. Choose some $Z \in \mathbb{N}$ (which we fix later) and rewrite the difference in the following way

$$\left|\sum_{j=i}^{\infty} \beta_{i,j}^n a_i^n a_j^n - \sum_{j=i}^{\infty} \beta_{i,j} a_i a_j\right| \leqslant |a_i^n - a_i| \sum_{j=i}^{\infty} \beta_{i,j}^n a_j^n + a_i \sum_{j=i}^{Z-1} \left(\left|\beta_{i,j}^n - \beta_{i,j}\right| a_j^n + \beta_{i,j} \left|a_j^n - a_j\right| \right) + a_i \sum_{j=Z}^{\infty} \left(\beta_{i,j}^n a_j^n + \beta_{i,j}.a_j \right). \tag{4.9}$$

Using condition (H_1) in the first term on the right-hand side of (4.9), we get

$$|a_i^n - a_i| \sum_{j=i}^{\infty} \beta_{i,j}^n a_j^n \leqslant A_* |a_i^n - a_i| \sum_{j=i}^{\infty} \left(i^\alpha j^\beta + i^\beta j^\alpha \right) a_j^n \leqslant 2A_* i^\beta |a_i^n - a_i| \sum_{j=i}^{\infty} j a_j^n \leqslant 2A_* i^\beta \Omega_1 |a_i^n - a_i|. \tag{4.10}$$

By using the assumption of proposition 4.1, we get the right-hand side of (4.10) that converges to zero as $n \longrightarrow \infty$ locally uniformly on $[0, \infty)$. To estimate the second term on the right-hand side of (4.9), we follow the same argument as (4.8). For the completeness, each term of the sum converges to zero locally uniformly on $[0, \infty)$ and since the sum contains a fixed number of terms,

$$\lim_{n\to\infty} \left(a_i \sum_{j=i}^{Z-1} \left(\left|\beta_{i,j}^n - \beta_{i,j}\right| a_j^n + \beta_{i,j} \left|a_j^n - a_j\right| \right) \right) = 0 \tag{4.11}$$

converges locally uniformly on $[0, \infty)$. Finally, to estimate the third term on the right-hand side of (4.9), we use the relation $\beta_{i,j} \leqslant A_*(i^\alpha j^\beta + i^\beta j^\alpha) \leqslant 2A_* i^\beta j^\beta$ (since $\alpha \leqslant \beta < \mu$) and $\lambda \in [0, 1)$ we get

$$a_i(\xi) \sum_{j=Z}^{\infty} \left(\beta_{i,j}^n a_j^n(\xi) + \beta_{i,j} a_j(\xi) \right) \leqslant 2A_* Z^{\beta-\mu} i^\beta a_i(\xi) \sum_{j=Z}^{\infty} j^\mu \left(a_j^n(\xi) + a_j(\xi) \right) \leqslant 2A_* \Omega_1 Z^{\beta-\mu} \left(\mathcal{M}_\mu^n(\xi) + \mathcal{M}_\mu(\xi) \right) \leqslant 2A_* \Omega_1 \Omega_\mu \left(1 + \xi^{-\lambda} \right) Z^{\beta-\mu} \longrightarrow 0 \quad \text{as} \quad Z \longrightarrow 0 \tag{4.12}$$

converges locally uniformly on $[0, \infty)$. Since the right-hand side of (4.11) is independent of n, the right-hand side of (4.9) is arbitrarily small by taking $n \longrightarrow \infty$ and then $Z \longrightarrow \infty$. This estimation together with (4.8), we get the convergence result (4.5).

Moreover, since all the estimations (4.8), (4.10) and (4.11) are uniform with respect to ξ and since $\int_0^t (1+\xi^{-\lambda})\mathrm{d}\xi = t + t^{1-\lambda}/(1-\lambda)$, the convergence result (4.6) follows directly. $\square$

We are now at the stage where we can prove the global existence theorem for the IVP (1.4).

THEOREM 4.4 Global existence theorem. *Consider that* $\mathbf{a}^{in} \in \ell_1^1$. *Under the assumptions* $(H_1)-(H_3)$, *there exists at least one global solution* $\mathbf{a}$ *to* (1.4)–(1.5) *with initial condition* $\mathbf{a}^{in}$.

Proof. Let $(a^m)_{m\in\mathbb{N}}$ be the solution of m-dimensional system (4.2) obtained from proposition 4.1 and lemma 3.1 states that, $\mathbf{a}^m$ have a uniformly bounded first moment, i.e. $\mathcal{M}_1^m(t) \leqslant \mathfrak{M}$. Therefore, lemma 4.2 guarantees that $(a^m)_{m\in\mathbb{N}}$ is uniformly bounded on $C^{0,1}(0,T)$ for all fixed $T\in(0,\infty)$. Thus, by Arzelà–Ascoli theorem we ensure that there exists a subsequence of $(a_i^m)_{m\in\mathbb{N}}$ (for notational convenience, we will not relabel) and a continuous sequence $\mathbf{a} = (a_i)_{i\in\mathbb{N}}$ such that

$$a_i^m \longrightarrow a_i \quad \text{as} \quad m \longrightarrow \infty \tag{4.13}$$

converges locally uniformly on $[0,\infty)$. Moreover, with the help of lemmas 3.1, 3.4 and Fatou's lemma, we have for some $\mu > 1$ and $\lambda \in [0,1)$ that

$$\mathcal{M}_1(t) \leqslant \mathfrak{M} \quad \text{and} \quad \mathcal{M}_\mu(t) \leqslant \Omega_\mu\left(1+t^{-\lambda}\right). \tag{4.14}$$

Since $(a_i^m)_{m\in\mathbb{N}}$ is a solution of (4.2), we have

$$\begin{aligned} a_i^m(t) - a_i^{m,in} = \int_0^t \Bigg(& a_{i-1}^m(\xi)\sum_{j=1}^{i-1} j\beta_{i-1,j}^m a_j^m(\xi) - a_i^m(\xi)\sum_{j=1}^{i} j\beta_{i,j}^m a_j^m(\xi) \\ & - a_i^m(\xi)\sum_{j=i}^{\infty}\beta_{i,j}^m a_j^m(\xi) + s_i^m - r_i^m a_i^m(\xi)\Bigg)\mathrm{d}\xi. \end{aligned} \tag{4.15}$$

Applying proposition 4.1 by taking the limit $m \longrightarrow \infty$ on equation (4.15), we get

$$\begin{aligned} a_i(t) - a_i^{in} = \int_0^t \Bigg(& a_{i-1}(\xi)\sum_{j=1}^{i-1} j\beta_{i-1,j} a_j(\xi) - a_i(\xi)\sum_{j=1}^{i} j\beta_{i,j} a_j(\xi) \\ & - a_i(\xi)\sum_{j=i}^{\infty}\beta_{i,j} a_j(\xi) + s_i - r_i a_i(\xi)\Bigg)\mathrm{d}\xi. \end{aligned} \tag{4.16}$$

Now, consider (4.14) with the convergence (4.13), we get $\mathbf{a} \in L^\infty([0,\infty),\ell_1^1) \cap C^1((0,\infty),\ell_\mu^1)$. Moreover, continuity of the initial data and equation (4.16) implies that $\mathbf{a} \in C^1([0,\infty))$. So, by the help of Leibnitz rule differentiating (4.16) with respect to t establishes that $\mathbf{a}(t)$ solves (1.4). Hence, the proof of theorem 4.4 is completed. $\square$

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5. A contraction property

This property plays a key roll to prove the existence and the rate of convergence to the equilibrium solution. Moreover, relying on this contraction property, we will also prove the uniqueness of the solution to equation (1.4).

LEMMA 5.1. *Suppose the assumptions* $(H_1)-(H_3)$ *hold, also* $\mathbf{c}=(c_i)_{i\in\mathbb{N}}$ *and* $\mathbf{d}=(d_i)_{i\in\mathbb{N}}$ *be two solutions to equation* (1.4). *Then for* $\mu\geqslant 1$ *there exists a constant* k_μ*, such that*

$$\frac{\mathrm{d}}{\mathrm{dt}}\sum_{i=1}^{\infty} i^{\mu}\left|c_i-d_i\right| \leqslant\left(2 A_*\left(2^\mu+2+k_\mu\right) \sum_{j=1}^{\infty} j^{\mu+\beta}\left(c_j+d_j\right)-R_*\right) \sum_{i=1}^{\infty} i^\mu\left|c_i-d_i\right| .$$

Proof. Consider lemma 2.3 for the solutions $\mathbf{c}=(c_i)_{i\in\mathbb{N}}$, $\mathbf{d}=(d_i)_{i\in\mathbb{N}}$ and taking the difference by setting $\varphi_i=i^\mu\operatorname{sgn}(c_i-d_i)$, we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\sum_{i=1}^{\infty} i^{\mu}\left|c_i-d_i\right| &= \sum_{i=1}^{\infty}\sum_{j=i}^{\infty}\left[i\left(\varphi_{j+1}-\varphi_j\right)-\varphi_i\right]\beta_{i,j}\left(c_ic_j-d_id_j\right)-\sum_{i=1}^{\infty}\varphi_i\left(c_i-d_i\right)r_i\\
&=\sum_{i=1}^{\infty}\sum_{j=i}^{\infty}\left[i\left(\varphi_{j+1}-\varphi_j\right)-\varphi_i\right]\beta_{i,j}\left[c_j\left(c_i-d_i\right)+d_i\left(c_i-d_i\right)\right]\\
&\quad-\sum_{i=1}^{\infty}\varphi_i\left(c_i-d_i\right)r_i\\
&=\sum_{i=1}^{\infty}\sum_{j=i}^{\infty}\left[i\left(\varphi_{j+1}-\varphi_j\right)-\varphi_i\right]\beta_{i,j}c_j\left(c_i-d_i\right)\\
&\quad+\sum_{i=1}^{\infty}\sum_{j=i}^{\infty}\left[i\left(\varphi_{j+1}-\varphi_j\right)-\varphi_i\right]\beta_{i,j}d_i\left(c_i-d_i\right)\\
&\quad-\sum_{i=1}^{\infty}\varphi_i\left(c_i-d_i\right)r_i. \qquad (5.1)
\end{aligned}$$

Putting the explicit form of φ_i, we can get the following estimations

$$\begin{aligned}
&\left[i\left(\varphi_{j+1}-\varphi_j\right)-\varphi_i\right]\left(c_i-d_i\right)\\
&\quad=\left(c_i-d_i\right)\left[i\left((j+1)^\mu\operatorname{sgn}\left(c_{j+1}-d_{j+1}\right)-j^\mu\operatorname{sgn}\left(c_j-d_j\right)\right)-i^\mu\operatorname{sgn}\left(c_i-d_i\right)\right]\\
&\quad=\left(c_i-d_i\right)\left[i(j+1)^\mu\operatorname{sgn}\left(c_{j+1}-d_{j+1}\right)-ij^\mu\operatorname{sgn}\left(c_j-d_j\right)\right]-i^\mu\left|c_i-d_i\right|\\
&\quad\leqslant\left|c_i-d_i\right|\left[i(1+j)^\mu+ij^\mu-i^\mu\right]\\
&\quad\leqslant\left(2^\mu+1\right)ij^\mu\left|c_i-d_i\right| \qquad (5.2)
\end{aligned}$$

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$$\begin{aligned}
&[i(\varphi_{j+1}-\varphi_j)-\varphi_i](c_j-d_j)\\
&\quad=(c_j-d_j)\left[i(j+1)^{\mu}\operatorname{sgn}(c_{j+1}-d_{j+1})-i^{\mu}\operatorname{sgn}(c_i-d_i)\right]-ij^{\mu}|c_j-d_j|\\
&\quad\leqslant|c_j-d_j|\left[i(1+j)^{\mu}+i^{\mu}-ij^{\mu}\right]\\
&\quad\leqslant|c_j-d_j|\left[k_{\mu}ij^{\mu-1}+i^{\mu}\right]
\end{aligned}\tag{5.3}$$

for some constant k_μ. Use estimations (5.2) and (5.3) on (5.1), to get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{\infty}i^{\mu}|c_i-d_i| &\leqslant (2^{\mu}+1)\sum_{i=1}^{\infty}\sum_{j=i}^{\infty}ij^{\mu}\beta_{i,j}c_j|c_i-d_i|\\
&\quad+\sum_{i=1}^{\infty}\sum_{j=i}^{\infty}\left[k_{\mu}ij^{\mu-1}+i^{\mu}\right]\beta_{i,j}d_i|c_j-d_j|-\sum_{i=1}^{\infty}i^{\mu}|c_i-d_i|r_i\\
&\leqslant 2A_*\Bigg[(2^{\mu}+1)\sum_{i=1}^{\infty}\sum_{j=i}^{\infty}i^{\beta}j^{\mu+\beta}c_j|c_i-d_i|\\
&\quad+(k_{\mu}+1)\sum_{i=1}^{\infty}\sum_{j=i}^{\infty}i^{\beta}j^{\mu+\beta-1}d_i|c_j-d_j|\Bigg]\\
&\quad-\sum_{i=1}^{\infty}i^{\mu}|c_i-d_i|r_i\\
&\leqslant 2A_*(k_{\mu}+2^{\mu}+2)\sum_{i=1}^{\infty}\sum_{j=i}^{\infty}\left(i^{\beta}j^{\mu+\beta}c_j+j^{\beta}i^{\mu+\beta-1}d_j\right)|c_i-d_i|\\
&\quad-R_*\sum_{i=1}^{\infty}i^{\mu}|c_i-d_i|\\
&\leqslant\left(2A_*(k_{\mu}+2^{\mu}+2)\sum_{j=1}^{\infty}j^{\mu+\beta}(c_j+d_j)-R_*\right)\sum_{i=1}^{\infty}i^{\mu}|c_i-d_i|.
\end{aligned}$$

□

Lemma 5.2. *Suppose assumptions $(H_1)-(H_3)$ hold and we redefine p, q and λ as*

$$p=\frac{\mu+\gamma-1}{\gamma-\alpha-\beta},\quad q=\frac{p}{p-1}\quad \textit{and}\quad \lambda_{\mu+\beta}=\frac{\gamma}{\mu+\beta-1}\quad \textit{respectively.}$$

Then for any pair of solutions $\mathbf{c}=(c_i)_{i\in\mathbb{N}}$ and $\mathbf{d}=(d_i)_{i\in\mathbb{N}}$, there exists a time T_* such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{\infty}i^{\mu}|c_i-d_i|\leqslant-\eta\sum_{i=1}^{\infty}i^{\mu}|c_i-d_i|\qquad \textit{for almost every}\quad t\geqslant T_*,$$

where $\eta := \max\{\eta_1, \eta_2\}$ *with* η_1 *and* η_2 *defined as*

$$\begin{aligned}\eta_1 &:= R_* \\ &- 4A_*\left(k_\mu + 2^\mu + 2\right)\left\{\frac{q^{1-p}2^{2+\lambda_{\mu+\beta}}}{p}\left(2^{\mu+\beta+3}\widehat{A}_*\right)^p (\widehat{S}_*^1)^{1+p+\lambda_{\mu+\beta}}\right. \\ &\quad \left. + 2^{2+\lambda_{\mu+\beta}}\widehat{S}_*^{\mu+\beta}\left(\widehat{S}_*^1\right)^{\lambda_{\mu+\beta}}\right\}^{\frac{1}{1+\lambda_{\mu+\beta}}} \\ &> 0\end{aligned}$$

and

$$\eta_2 := R_* - 16A_*\left(k_\mu + 2^\mu + 2\right)\left\{\frac{\left(2^{\mu+\beta+3}\widehat{A}_*\right)}{p} q^{1-p}\left(\widehat{S}_*^1\right)^{1+p} + \widehat{S}_*^{\mu+\beta}\right\} > 0.$$

Proof. We need to show that there exists a time T_* such that

$$2A_*\left(k_\mu + 2^\mu + 2\right)\sum_{j=1}^{\infty} j^{\mu+\beta}\left(c_j + d_j\right) - R_* \leqslant -\eta \qquad \text{for almost every} \quad t \geqslant T_*. \tag{5.4}$$

According to lemma 3.5, for a large time T_* the left-hand side of the inequality (5.4)

$$\begin{aligned}&2A_*\left(k_\mu + 2^\mu + 2\right)\sum_{j=1}^{\infty} j^{\mu+\beta}\left(c_j + d_j\right) - R_* \\ &\quad \leqslant 4A_*\left(k_\mu + 2^\mu + 2\right)\min\left\{4\left(\frac{\left(2^{\mu+\beta+3}\widehat{A}_*\right)}{p} q^{1-p}\left(\widehat{S}_*^1\right)^{1+p} + \widehat{S}_*^{\mu+\beta}\right),\right. \\ &\quad \left(\frac{1}{p}2^{2+\lambda_{\mu+\beta}}\left(2^{\mu+\beta+3}\widehat{A}_*\right)^p q^{1-p}(\widehat{S}_*^1)^{1+p+\lambda_{\mu+\beta}}\right. \\ &\quad \left.\left. + 2^{2+\lambda_{\mu+\beta}}\widehat{S}_*^{\mu+\beta}\left(\widehat{S}_*^1\right)^{\lambda_{\mu+\beta}}\right)^{\frac{1}{1+\lambda_{\mu+\beta}}}\right\} - R_*\end{aligned}$$

for all $t \geqslant T_*$. Now, using the definition of η_1 and η_2 and take $\eta := \max\{\eta_1, \eta_2\}$, we can conclude the proof. □

The uniqueness of the solution to equation (1.4) is obtained through lemma 5.1.

PROPOSITION 5.3 Uniqueness of global solution. *Let all the conditions of theorem 4.4 hold. Then there exists atmost one global solution to equation (1.4).*

Proof. Consider $\mathbf{c} = (c_i)$ and $\mathbf{d} = (d_i)$ be two solutions to equation (1.4). Set $\mu = 1$ in lemma 5.1, to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{\infty} i\left|c_i - d_i\right| \leqslant \left(10A_*\sum_{j=1}^{\infty} j^{\beta+1}\left(c_j + d_j\right) - R_*\right)\sum_{i=1}^{\infty} i\left|c_i - d_i\right|.$$

Recalling lemma 3.4, for $R_* > 0$ the above assumption can take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{\infty} i\left|c_i - d_i\right| \leqslant \left(20A_*\mathcal{C}_{1+\beta}\left(1 + t^{-\beta/\gamma}\right)\right)\sum_{i=1}^{\infty} i\left|c_i - d_i\right|.$$

The fact $\beta < \gamma$ gives $t \to (1 + t^{-\beta/\gamma})$ is integrable at zero. Thus, due to the Grönwall's inequality, we can obtain $\mathbf{c} = \mathbf{d}$. □

6. Existence and convergence of equilibrium solution

In this section, we will establish the existence of a unique equilibrium solution. Moreover, with the help of lemma 5.2, we will show that any solution to equation (1.4) converges to this equilibrium with an exponential rate.

THEOREM 6.1. *Let conditions $(H_1) - (H_3)$ hold. Moreover, for each $\mu \geqslant 1$ either*

$$2A_*\left(k_\mu + 2^\mu + 2\right)\left\{\frac{1}{p}2^{2+\lambda_{\mu+\beta}}\left(2^{\mu+\beta+3}\widehat{A}_*\right)^p q^{1-p}(\widehat{S}_*^1)^{1+p+\lambda_{\mu+\beta}} + 2^{2+\lambda_{\mu+\beta}}\widehat{S}_*^{\mu+\beta}\left(\widehat{S}_*^1\right)^{\lambda_{\mu+\beta}}\right\}^{\frac{1}{1+\lambda_{\mu+\beta}}} - R_* < 0$$

or

$$8A_*\left(k_\mu + 2^\mu + 2\right)\left\{\frac{\left(2^{\mu+\beta+3}\widehat{A}_*\right)}{p}q^{1-p}\left(\widehat{S}_*^1\right)^{1+p} + \widehat{S}_*^{\mu+\beta}\right\} - R_* < 0,$$

where $p = \frac{\mu+\gamma-1}{\gamma-\alpha-\beta}$, $q = \frac{p}{p-1}$ *and* $\lambda_{\mu+\beta} = \frac{\gamma}{\mu+\beta-1}$. *Then*

(i) *there exists a unique stationary solution $Q = (Q_i)_{i\in\mathbb{N}} \in \ell_\mu^1$ for all $\mu \geqslant 1$,*

(ii) *for each solution **a**, there exists some constant $K, \eta > 0$ (independent of **a**) and a time $T_c > 0$ such that*

$$\|\mathbf{a}(t) - Q\|_{\ell_\mu^1} \leqslant K\exp[-\eta t] \quad \textit{for all} \quad t \geqslant T_c.$$

We will prove the following lemma, which is a direct consequence of lemma 5.2.

LEMMA 6.2. *Let all the conditions of theorem 6.1 hold. Then for any solution $\mathbf{a} = (a_i)_{i\in\mathbb{N}}$ to equation (1.4), we have*

$$\lim_{t\to\infty}\left|\frac{\mathrm{d}}{\mathrm{d}t}a_i(t)\right| = 0 \quad \textit{for all} \quad i \in \mathbb{N}.$$

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Proof. For any $h > 0$, we introduce a shifted (along time) sequence $\mathbf{a}^h = (a_i^h)_{i\in\mathbb{N}}$, defined as $a_i^h(t) = a_i(t+h)$. Therefore, $\mathbf{a}^h$ is again a solution to equation (1.4). Applying lemma 5.2 with these two solutions $\mathbf{a}$ and $\mathbf{a}^h$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{\infty} i^{\mu}\left|a_i(t+h)-a_i(t)\right| \leqslant -\eta\sum_{i=1}^{\infty} i^{\mu}\left|a_i(t+h)-a_i(t)\right| \qquad \text{for all} \quad t \geqslant T_*. \tag{6.1}$$

Integrating

$$\sum_{i=1}^{\infty} i^{\mu}\left|a_i(t+h)-a_i(t)\right| \leqslant \sum_{i=1}^{\infty} i^{\mu}\left|a_i(T_*{+}h) - a_i(T_*)\right| \exp\left[-\eta\left(t-T_*\right)\right] \qquad \text{for all} \quad t \geqslant T_*.$$

Since $\mathbf{a} \in C^1((0,\,\infty),\,\ell^1_\mu)$, so taking the limit $h \to 0$ yields

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}a_i(t)\right\|_{\ell^1_\mu} \leqslant \left\|\frac{\mathrm{d}}{\mathrm{d}t}a_i(T_*)\right\|_{\ell^1_\mu} \exp\left[-\eta\left(t-T_*\right)\right] \qquad \text{for all} \quad t \geqslant T_*.$$

Further taking limit $t \to \infty$ on the above estimation, we obtain

$$\lim_{t\to\infty}\left\|\frac{\mathrm{d}}{\mathrm{d}t}a_i(t)\right\|_{\ell^1_\mu} = 0.$$

Which completes the proof. □

Now we have reached the point where we can prove the existence and uniqueness of the equilibrium solution. Consequently, we will prove exponential convergence to the equilibrium solution.

Proof of theorem 6.1. We will prove the theorem in three steps.

Existence of equilibrium solution: Let $\mathbf{a} = (a_i)_{i\in\mathbb{N}}$ be a solution to equation (1.4) associated with some initial data $\mathbf{a}^{in}$. According to lemmas 3.1 and 3.5, for each $\mu > 1$ we have

$$\mathcal{M}_1(t) \leqslant 2\widehat{S}^1_* \quad \text{and} \quad \mathcal{M}_\mu(t) \leqslant \Omega_\mu\left(1+t^{-\lambda_\mu}\right) \quad \text{for sufficiently large } t. \tag{6.2}$$

This yields the existence of a sequence (t_n) which satisfies $t_n \to \infty$ as $n \to \infty$ and non-negative sequence (Q_i) such that

$$\mathbf{a}(t_n) \longrightarrow Q \quad \text{in } \ell^1_\mu \quad \text{for all } \mu \geqslant 1 \quad \text{as } n \longrightarrow \infty.$$

In particular, with the help of Fatou's lemma and estimation (6.2), the sequence Q also satisfies

$$\sum_{i=1}^{\infty} iQ_i \leqslant 2\widehat{S}^1_* \quad \text{and} \quad \sum_{i=1}^{\infty} i^{\mu}Q_i \leqslant \Omega_\mu \quad \text{for all} \quad \mu > 1.$$

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It is now a straightforward application of proposition 4.1 to the stationary sequence $\mathbf{a}(t_n)$ and taking into account lemma 5.2, we get

$$\mathcal{F}_i(Q) = \lim_{n\to\infty} \mathcal{F}_i(\mathbf{a}(t_n)) = \lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}t} a_i(t_n) = 0.$$

This proves that Q is an equilibrium solution to equation (1.4).

Uniqueness of equilibrium solution: On the contrary, we assume that there exist two equilibrium solutions Q^1 and Q^2 to equation (1.4). Since Q^1 and Q^2 both are solutions of equation (1.4), applying lemma 6.2 we can obtain

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{\infty} i^{\mu} \left|Q_i^1 - Q_i^2\right| \leqslant -\eta \sum_{i=1}^{\infty} i^{\mu} \left|Q_i^1 - Q_i^2\right| \qquad \text{for all } \mu \geqslant 1,$$

from which the uniqueness follows.

Convergence to equilibrium: Applying lemma 5.2 for any solution $\mathbf{a}$ and equilibrium solution Q_i, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{\infty} i^{\mu} |a_i - Q_i| \leqslant -\eta \sum_{i=1}^{\infty} i^{\mu} |a_i - Q_i| \qquad \text{for all } t \geqslant T_*.$$

By integrating the above differential inequality, we can obtain

$$\sum_{i=1}^{\infty} i^{\mu} |a_i(t) - Q_i| \leqslant \sum_{i=1}^{\infty} i^{\mu} |a_i(T_*) - Q_i| \exp[-\eta(t - T_*)] \quad \text{for all} \quad t \geqslant T_*. \tag{6.3}$$

According to lemma 3.5, there exists a constant $\tilde{K}$ which is independent of $\mathbf{a}$ and a time $\tilde{T}$ such that

$$\sum_{i=1}^{\infty} i^{\mu} |a_i(T_*) - Q_i| \leqslant \sum_{i=1}^{\infty} i^{\mu} (a_i + Q_i) \leqslant 2\tilde{K} \quad \text{for all} \quad t \geqslant \tilde{T}. \tag{6.4}$$

Finally, take $T_c = \max\{T_*, \tilde{T}\}$, $K := 2\tilde{K}\exp(T_*)$ and using esimation (6.4) on (6.3) we get

$$\sum_{i=1}^{\infty} i^{\mu} |a_i(t) - Q_i| \leqslant K \exp[-\eta t] \quad \text{for all} \quad t \geqslant T_c,$$

which concludes the proof. □

7. Concluding remarks

In this article, we consider an extension of the Safronov–Dubovskiĭ aggregation equation, where particles input and output can take place. A complete theoretical investigation for well-posedness and convergence to the steady-state solution to the IVP (1.4)–(1.5) has been done. Here all the associated kinetic coefficients are unbounded and cover a large class of physical kinetic rates. Initially, we truncated system (1.4) in a finite-dimensional system and with the help of standard Picard–Lindelöf theorem, we have proved the local existence theorem. After that, a strong convergence result, which is known as Arzelà–Ascoli theorem, ensures the convergence of the sequence of solution whose limit function is proved to be a solution of the IVP (1.4)–(1.5). In the subsequent sections, by proving some contraction property of the solution, for a long time limit, we established that the solution converges to a unique equilibrium solution with the exponential convergence rate.

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Competing interest
None.

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