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Characterization of Positive Links and the s-invariant for Links

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Abstract. We characterize positive links in terms of strong quasipositivity, homogeneity, and the value of Rasmussen and Beliakova-Wehrli's s-invariant. We also study almost positive links, and in particular, determine the s-invariants of almost positive links. This result suggests that all almost positive links might be strongly quasipositive. On the other hand, it implies that almost positive links are never homogeneous links.

1 Introduction

A link is called *positive* if it has a diagram with only positive crossings, which is defined combinatorially. On the other hand, Nakamura [25] and Rudolph [37] proved that positive links are strongly quasipositive links, which are defined geometrically. It is natural to consider the following question.

Question 1.1 *What are the differences between positive links and strongly quasipositive links?*

Cromwell [9] introduced a class of links called homogeneous links. They are a generalization of positive links from the combinatorial viewpoint. Baader [3] proved that a knot is positive if and only if it is strongly quasipositive and homogeneous, answering Question 1.1 in the case of knots (see also [1]). One can obviously apply Baader's proof to the case of links and obtain the following theorem.

Theorem 1.2 ([3]) *A non-split link is positive if and only if it is strongly quasipositive and homogeneous.*

We generalize the above theorem as follows.

Theorem 1.3 *Let L be a non-split link with $\sharp L$ components. Then* (i)–(iv) *are equivalent.*

(i) *L is positive;*
(ii) *L is homogeneous and strongly quasipositive;*
(iii) *L is homogeneous, quasipositive and $g_*(L) = g(L)$;*

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(iv) *L is homogeneous and $s(L) = 2g_*(L) + \sharp L - 1 = 2g(L) + \sharp L - 1$,*

where $s(L)$ is Rasmussen and Beliakova-Wehrli's s-invariant of L, $g_*(L)$ is the four-ball genus of L, and $g(L)$ is the three-genus of L.

For the definition of Rasmussen and Beliakova-Wehrli's s-invariant, see [6, 34]. Theorem 1.3 is a generalization of [1, Theorem 1.3]. We prove Theorem 1.3 in Section 4. The key of the proof is the computation of the s-invariants of homogeneous links (see Sections 2-4). In this paper, we also study almost positive links. An *almost positive link* is a non-positive link that is represented by a diagram with exactly one negative crossing. In general, it is hard to distinguish almost positive links from positive links. We consider the following question.

Question 1.4 *What are the similarities and differences between positive links and almost positive links?*

There are some similarities between them (see [8, 9, 31, 32, 41, 44]). One of the interesting and expected similarities is Stoimenow's question:.

Question 1.5 ([42, Question 4]) *Is any almost positive link strongly quasipositive, or at least quasipositive?*

We give an evidence towards an affirmative answer to Question 1.5 as follows.

Theorem 1.6 *Let L be a non-split link with $\sharp L$ components. If L is almost positive or strongly quasipositive, then*

$$s(L) = 2g_*(L) + \sharp L - 1 = 2g(L) + \sharp L - 1.$$

Moreover, we determine the s-invariant of an almost positive link in terms of its almost positive diagram (see Theorem 5.2). We also confirm Question 1.5 for fibered almost positive knots (Theorem 6.13) and almost positive knots up to 12 crossings in Section 6.

On the other hand, there are some differences between positive links and almost positive links. In this paper, we give a significant difference between them. In fact, we prove the following corollary.

Corollary 1.7 *Any almost positive link is not homogeneous.*

Note that positive links are homogeneous, and this corollary follows from Theorems 1.3 and 1.6; see Section 5. Moreover, using Corollary 1.7, we give infinitely many knots that are pseudo-alternating and are not homogeneous (which are counterexamples of Kauffman's conjecture (Conjecture 7.2)).

Proposition 1.8 *There are infinitely many knots that are pseudo-alternating and are not homogeneous.*

This manuscript is organized as follows. In Section 2, we recall Kawamura–Lobb's inequality and homogeneous links. In Section 3, we recall strongly quasipositive links.

In Section 4, we give a characterization of positive links. In Section 5, we compute the s-invariants of almost positive links. As a corollary, we prove that any almost positive link is not homogeneous (Corollary 1.7). In Section 6, we consider the strong quasipositivities of almost positive knots with up to 12 crossings. In Section 7, we give infinitely many counterexamples of Kauffman's conjecture on pseudo-alternating links and alternative links.

Throughout this paper, we call Rasmussen and Beliakova-Wehrli's invariant an *s-invariant*. Also, we assume that all links and diagrams are oriented.

2 Kawamura–Lobb's Inequality and Homogeneous Links

In this section, we recall homogeneous links and their properties.

2.1 Kawamura–Lobb's Inequality for the s-invariant

In this subsection, we recall Kawamura–Lobb's inequality for the s-invariant.

Here we recall some definitions. For a connected diagram D, let $w(D)$ be the writhe of D, $O(D)$ the number of Seifert circles for D and $O_+(D)$ (resp. $O_-(D)$) the number of connected components of the diagram obtained from D by smoothing all negative (resp. positive) crossings of D. Kawamura [17] and Lobb [22] independently gave estimates for the s-invariant of a link, and these turned out to be the same estimate.

Theorem 2.1 ([17], [22, Theorem 1.10]) *Let D be a connected diagram of a link L. Then we obtain*

$$w(D) - O(D) + 1 + 2\big(O_+(D) - 1\big) \le s(L) \le w(D) + O(D) - 1 - 2\big(O_-(D) - 1\big).$$

2.2 Homogeneous Links

For a fixed diagram D, we consider when the upper bound and the lower bound of Kawamura–Lobb's inequality coincide. The answer is when D is homogeneous. In particular, the s-invariant of any homogeneous link is determined by its homogeneous diagram and Kawamura–Lobb's inequality. This result was given by the first author [1]. In this section, we see this result in terms of $*$-product.

We recall the definition of $*$-product of diagram (see also [9]). The Seifert circles of a diagram is divided into two types: a Seifert circle is of *type 1* if it does not contain any other Seifert circles in one of the complementary regions of the Seifert circle in $\mathbb{R}^2$, otherwise it is of *type 2*. Let $D \subset \mathbb{R}^2$ be a knot diagram and C a type 2 Seifert circle of D. Then C separates $\mathbb{R}^2$ into two components U and V such that $U \cup V = \mathbb{R}^2$ and $U \cap V = \partial U = \partial V = C$. Let D_1 and D_2 be the diagrams obtained form $D \cap U$ and $D \cap V$ by adding suitable arcs from C, respectively. Then C *decomposes* D into a *$*$-product* of D_1 and D_2, which is denoted by $D = D_1 * D_2$. We call this decomposition a *$*$-product decomposition* of D. A diagram is *special* if D has no Seifert circles of type 2. It is not hard to see that a special positive (or negative) diagram is alternating, and a special alternating diagram is positive or negative. Clearly, any diagram is decomposed into

$$D_1 * D_2 * \cdots * D_n,$$

where D_i is a special diagram.

For a diagram, any simple closed curve in $\mathbf{R}^2$ meeting the diagram transversely at two points cuts the diagram into two parts. A diagram is *strongly prime* if one of such parts has no crossing for any simple closed curve meeting the diagram transversely at two points (see [20]). If D is not strongly prime, D is represented as a connected sum of non-trivial diagrams D_1 and D_2 on $\mathbf{R}^2$. Then we also write $D = D_1 * D_2$. Any diagram D is decomposed into $D_1 * D_2 * \cdots * D_n$, where D_i is a strongly prime diagram.

As a result, any diagram is decomposed into $D_1 * D_2 * \cdots * D_n$, where D_i is a special and strongly prime diagram. This $*$-product decomposition of D depends only on D. On the other hand, for given diagrams D_1 and D_2, a $*$-product $D_1 * D_2$ is not well defined. Throughout this section, if we write $D = D_1 * D_2$, it is one of the diagrams that have such a $*$-product decomposition.

Let $L(D)$ and $U(D)$ be the lower bound and the upper bound of Kawamura–Lobb's inequality, respectively. Namely,

$$L(D) = w(D) - O(D) + 1 + 2\big(O_+(D) - 1\big),$$
$$U(D) = w(D) + O(D) - 1 - 2\big(O_-(D) - 1\big).$$

Lemma 2.2 *Let $D_1 * D_2$ be a connected link diagram that has a $*$-product decomposition of two diagrams D_1 and D_2. Then, we have*

$$L(D_1 * D_2) = L(D_1) + L(D_2) \quad \textit{and} \quad U(D_1 * D_2) = U(D_1) + U(D_2).$$

Proof The proof follows from the following facts:

$$\omega(D_1 * D_2) = \omega(D_1) + \omega(D_2),$$
$$O(D_1 * D_2) = O(D_1) + O(D_2) - 1,$$
$$O_+(D_1 * D_2) = O_+(D_1) + O_+(D_2) - 1,$$
$$O_-(D_1 * D_2) = O_-(D_1) + O_-(D_2) - 1. \qquad \blacksquare$$

A diagram is *homogeneous* if it has a $*$-product decomposition whose factors are some special alternating diagrams. A *homogeneous link* is a link represented by a homogeneous diagram ([9], and see also [3, 4, 23]). Note that positive or negative links are homogeneous.

Let $\Delta(D)$ be the half of the difference between $U(D)$ and $L(D)$, that is,

$$\Delta(D) := \big(U(D) - L(D)\big)/2 = O(D) + 1 - O_+(D) - O_-(D).$$

The following result ensures that $\Delta(D) = 0$ for any homogeneous diagram D.

Theorem 2.3 *Let $D = D_1 * D_2 * \cdots * D_n$ be a connected homogeneous diagram of a link L, where each D_i is a special alternating diagram. Then we obtain $\Delta(D) = 0$.*

Proof We have $\Delta(D_i) = 0$ for $i = 1, \dots, n$, since any special alternating diagram is positive or negative. By Lemma 2.2 we obtain

$$L(D) = \sum L(D_i) = \sum U(D_i) = U(D). \qquad \blacksquare$$

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Corollary 2.4 *Let $D = D_1 * D_2 * \cdots * D_n$ be a connected homogeneous diagram of a link L, where each D_i is a special alternating diagram. Then we have*

$$s(L) = \sum_{i=1}^{n} s(D_i) = L(D) = U(D).$$

In particular, $s(\overline{L}) = -s(L)$.

The following theorem was proved by the first author. From Theorems 2.3 and 2.5, we see that $\Delta(D) = 0$ if and only if D is homogeneous.

Theorem 2.5 ([1]) *Let D be a connected diagram of a link L. If $\Delta(D) = 0$, then D is homogeneous.*

2.3 Kawamura's Inequality

Kawamura [16] gave another estimate for the s-invariant for any non-positive and non-negative knot. The first author [2] gave an alternative proof of the estimation by using state cycles of the Lee homology. In this section, we determine the difference between Kawamura–Lobb's inequality and Kawamura's inequality.

Let D be a diagram of a link. A Seifert circle of D is *strongly negative* (resp. *positive*) if it is not adjacent to any positive (resp. negative) crossing. Let $O_<(D)$ (resp. $O_>(D)$) be the number of the strongly negative (resp. positive) circles of D. Then we obtain the following Kawamura's inequality.

Theorem 2.6 ([16], see also [2]) *Let D be a connected diagram of a non-positive and non-negative link L. Then we obtain*

$$w(D) - O(D) + 1 + 2O_<(D) \le s(L) \le w(D) + O(D) - 1 - 2O_>(D).$$

Remark 2.7 Kawamura [16] and the first author [2] only proved the above theorem for the s-invariants of knots. However, both of their methods can be applied to the s-invariants for links.

Any strongly negative (resp. positive) circle of D is a connected component of the diagram obtained from D by smoothing all negative (resp. positive) crossings of D. Hence, if D is neither positive nor negative, we obtain

$$O_<(D) + 1 \le O_+(D) \quad \text{and} \quad O_>(D) + 1 \le O_-(D);$$

in particular, we notice that Kawamura–Lobb's inequality is sharper than Kawamura's inequality.

Let D be a connected link diagram and let S_D be the Seifert graph of D; that is, the vertices of S_D correspond to the Seifert circles of D and two vertices are connected by an edge with the label + (resp. −) if there is a positive (resp. negative) crossing of D, which is adjacent to the circles corresponding to the two vertices. Let S_D^+ (resp. S_D^-) be the graph obtained from S_D by removing all the edges with the label − (resp. +) and all the vertices corresponding to the strongly negative (resp. positive) circles of D. If D is positive (resp. negative), the graph S_D^- (resp. S_D^+) is empty. Then we have the following lemma.

Lemma 2.8 *Let D be a connected link diagram. Then we obtain*

$$O_<(D) + |S_D^+| = O_+(D) \quad \text{and} \quad O_>(D) + |S_D^-| = O_-(D),$$

where $|S_D^+|$ and $|S_D^-|$ is the number of the components of S_D^+ and S_D^-, respectively.

Proof From the definition, $O_+(D)$ is the number of the components of the graph obtained from S_D by removing all the edges with the label $-$. It is equal to the number of the strongly negative circles of D and the components of S_D^+. Hence we obtain the first equality. By the same discussion, we have the second one. ∎

Corollary 2.9 *For any diagram D, the graph S_D^+ (resp. S_D^-) is connected and not empty if and only if $O_<(D) + 1 = O_+(D)$ (resp. $O_>(D) + 1 = O_-(D)$).*

Remark 2.10 From Theorems 2.3 and 2.5, for a link diagram D, the lower bound and the upper bound of Kawamura–Lobb's inequality are equal if and only if D is homogeneous. On the other hand, from Corollary 2.9, the lower bound and the upper bound of Kawamura's inequality are equal if and only if D is homogeneous, and S_D^+ and S_D^- are connected and non-empty. Such a diagram has a $*$-product decomposition whose factors are one positive diagram and one negative diagram. In [19, Remark I.26], Lewark called such a diagram *good diagram*.

3 The s-invariants of Strongly Quasipositive Links

In this section, we give a computation of the s-invariant of strongly quasipositive links.

Recall that, for $n \in \mathbb{Z}_{>0}$, the *n-braid group B_n*, is a group that has the following presentation.

$$\left\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{rcll} \sigma_t\sigma_s &=& \sigma_s\sigma_t & (|t-s|>1) \\ \sigma_t\sigma_s\sigma_t &=& \sigma_s\sigma_t\sigma_s & (|t-s|=1) \end{array} \right\rangle.$$

Rudolph introduced the concept of a strongly quasipositive link (see [35]) as follows. For $0 < i \le j-1 < n$, we define *positive embedded band $\sigma_{i,j}$* as

$$\sigma_{i,j} := (\sigma_i, \dots, \sigma_{j-2})(\sigma_{j-1})(\sigma_i, \dots, \sigma_{j-2})^{-1} \quad \text{and} \quad \sigma_{j-1,j} := \sigma_{j-1}.$$

A link is *strongly quasipositive* if it is represented by the closure of a braid of the form

$$\beta = \prod_{k=1}^{m} \sigma_{i_k, j_k}.$$

Let L be a strongly quasipositive link represented by the closure of β. Then L bounds a surface F in S^3 naturally, called a *quasipositive surface* (see Figure 1). The Euler characteristic $\chi(F)$ of the surface is equal to $n - m$, where n is the number of strands of β, and m is the number of the positive embedded bands in β.

For a strongly quasipositive knot K, Livingston [21] and Shumakovitch [38] proved that

$$\tau(K) = s(K)/2 = g_*(K) = g(K) = g(F),$$

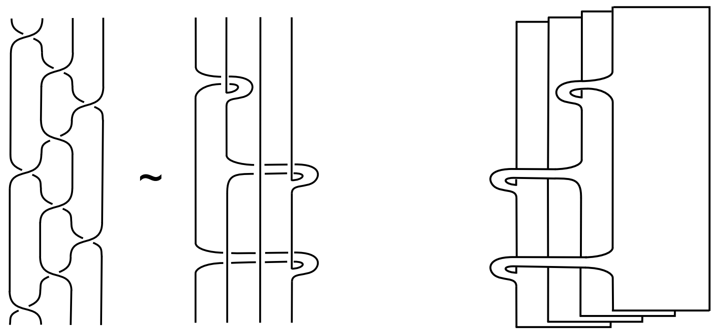

Figure 1: An example of a quasipositive surface. The closure of $\sigma_1\sigma_{2,4}\sigma_{1,4}$ bounds the right quasipositive surface.

where $\tau(K)$ is the Ozsváth–Szabó's τ-invariant of K (see [27] and [33]) and F is a quasipositive surface for K. These results are easily generalized to the s-invariant for links.

Theorem 3.1 ([21]) *Let L be a non-split strongly quasipositive link with $\sharp L$ components. Then*

$$s(L) = 2g_*(L) + \sharp L - 1 = 2g(L) + \sharp L - 1 = 1 - \chi(F),$$

where F is a quasipositive surface bounded by L.

Remark 3.2 In general, Theorem 3.1 does not hold for split links. In fact, if L is 2-component unlink, $s(L) = -1$ and $2g_*(L) + \sharp L - 1 = 1$.

Remark 3.3 A link is *quasipositive* if it is the closure of a braid of the form

$$\beta = \prod_{k=1}^{m} \omega_k \sigma_{i_k} \omega_k^{-1},$$

where ω_k is a word in B_n. Let K be a quasipositive knot. Then $\tau(K) = s(K)/2 = g_*(K)$. This is due to Plamenevskaya [29] and Hedden [13] for τ, and Plamenevskaya [30] and Shumakovitch [38] for s. By the same discussion, we obtain the following. Let L be a quasipositive link with $\sharp L$ components. Then we obtain $s(L) = 2g_*(L) + \sharp L - 1$.

4 Characterization of Positive Links

In this section, we prove characterizations of positive links.

Lemma 4.1 *Let D be a connected reduced homogeneous diagram of a link L with $\sharp L$ components. If $s(L) = 2g(L) + \sharp L - 1$, then D has no negative crossings.*

Proof Let D be a connected reduced homogeneous diagram of L. Then the genus of L is realized by the genus of the surface constructed by applying Seifert's algorithm to

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D (see [9]). Therefore, we obtain

$$2g(L) = 2 - \sharp L + c(D) - O(D),$$

where $c(D)$ denotes the number of crossings of D. By Theorem 2.3, we have

$$s(L) = w(D) - O(D) + 2O_+(D) - 1.$$

By assumption, $s(L) = 2g(L) + \sharp L - 1$. This implies that $O_+(D) - 1 = c_-(D)$, where $c_-(D)$ denotes the number of negative crossings of D. If there exists a non-nugatory negative crossing of D, then $O_+(D) - 1 < c_-(D)$. This contradicts the fact that $O_+(D) - 1 = c_-(D)$. Therefore, D has no negative crossing. ∎

Theorem 4.2 (Theorem 1.3) *Let L be a non-split link with $\sharp L$ components. Then* (i)–(iv) *are equivalent.*

(i) *L is positive.*
(ii) *L is homogeneous and strongly quasipositive.*
(iii) *L is homogeneous, quasipositive and $g_*(L) = g(L)$.*
(iv) *L is homogeneous and $s(L) = 2g_*(L) + \sharp L - 1 = 2g(L) + \sharp L - 1$.*

Proof (i) ⇒ (ii) A positive link is strongly quasipositive (see [25] and [37]) and homogeneous.

(ii) ⇒ (iii) If L is strongly quasipositive, then obviously L is quasipositive. Moreover, from Theorem 3.1, we have $g_*(L) = g(L)$.

(iii) ⇒ (iv) Since L is a quasipositive link, $s(L) = 2g_*(L) + \sharp L - 1$ (see Remark 3.3). By assumption, $g_*(L) = g(L)$. Therefore, $s(L) = 2g_*(L) + \sharp L - 1 = 2g(L) + \sharp L - 1$.

(iv) ⇒ (i) By Lemma 4.1, a homogeneous diagram of L with $s(L) = 2g(L) + \#L - 1$ is a positive diagram. ∎

Corollary 4.3 *Let L be an alternating link L with $\sharp L$ components. Then L is positive if and only if $s(L) = 2g(L) + \sharp L - 1$.*

Proof Cromwell [9] showed that alternating link diagrams are homogeneous. From Theorem 1.3, an alternating link L is positive if and only if L satisfies $s(L) = 2g(L) + \sharp L - 1$. ∎

The following was proved by Nakamura [26].

Corollary 4.4 ([26]) *Let L be a positive and alternating link. Then any reduced alternating diagram of L is positive.*

Proof It is known that a reduced alternating link diagram D of L are homogeneous. If L is positive, we have $s(L) = 2g(L) + \sharp L - 1$. By Lemma 4.1, the diagram D has no negative crossing; that is, D is positive. ∎

5 The s-invariants of Almost Positive Links

In this section, we compute the s-invariants of almost positive links.

A diagram is *almost positive* if it has exactly one negative crossing. Then we can see that an almost positive link is not positive and is represented by an almost positive diagram.

It is known that, for any link L, we obtain $s(L) \leq 2g_*(L) + \sharp L - 1$. On the other hand, for an almost positive link diagram D of a non-split link L, we can check $H^{0,j}_{Kh}(L) = 0$ if $j < -O(D) + w(D) = 2g(D) + \sharp L - 4$, where $H^{i,j}_{Kh}(L)$ is the Khovanov homology of L [18] and $g(D)$ is the genus of the Seifert surface obtained from D by Seifert's algorithm. Hence, we obtain

$$2g(D) + \sharp L - 3 \leq s(L) \leq 2g_*(L) + \sharp L - 1 \leq 2g(L) + \sharp L - 1 \leq 2g(D) + \sharp L - 1.$$

Stoimenow proved that the three-genera of almost positive links are computed from their almost positive diagrams as follows.

Theorem 5.1 ([42, Corollary 5 and the proof of Theorems 5 and 6]) *Let D be an almost positive diagram of a non-split link L with a negative crossing p.*

(i) *If there is no (positive) crossing joining the same two Seifert circles of D as the circles that are connected by the negative crossing p, we have $g(L) = g(D)$ (see the left of Figure 2).*

(ii) *If there is a (positive) crossing joining the same two Seifert circles of D as the circles that are connected by the negative crossing p, we have $g(L) = g(D) - 1$ (see the right of Figure 2).*

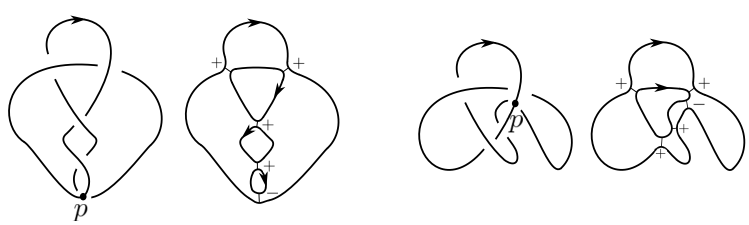

Figure 2: In the left picture, there is no crossing joining the same two Seifert circles as the two circles that are connected by the negative crossing p. In the right picture, there is a crossing joining the same two Seifert circles as the two circles that are connected by the negative crossing p.

By the same discussion as [43], we can compute the s-invariants of almost positive links as follows.

Theorem 5.2 *Let D be an almost positive diagram of a link L with negative crossing p.*

(i) *If there is no crossing joining the same two Seifert circles of D as the two circles that are connected by the negative crossing p, we obtain*

$$s(L) + 1 - \sharp L = 2g_*(L) = 2g(L) = 2g(D).$$

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(ii) *Otherwise, we obtain*

$$s(L)+1-\sharp L=2g_*(L)=2g(L)=2g(D)-2.$$

Proof Let D_+ be the positive diagram obtained from D by the crossing change at p and let L_+ be the link represented by D_+. By well known properties of the s-invariant, we obtain

$$s(L_+)-2\le s(L)\le s(L_+), \tag{5.1}$$

$$|s(L)|\le 2g_*(L)+\sharp L-1\le 2g(L)+\sharp L-1, \tag{5.2}$$

$$s(L_+)+1-\sharp L=2g_*(L_+)=2g(L_+)=2g(D_+)(=2g(D)). \tag{5.3}$$

(i) Suppose that there is no (positive) crossing joining the same two Seifert circles as the circles that are connected by the negative crossing p. By (5.1), we can see that $s(L)=s(L_+)$ or $s(L_+)-2$. By Lemma 5.3 and (5.3), we have $s(L)\ne 2g(D)+\sharp L-3=s(L_+)-2$. Hence, we obtain $s(L)=s(L_+)=2g(D)+\sharp L-1$. By (5.2), we have

$$2g(D)+\sharp L-1=s(L)\le 2g_*(L)+\sharp L-1\le 2g(L)+\sharp L-1\le 2g(D)+\sharp L-1.$$

(ii) Suppose that there is a (positive) crossing joining the same two Seifert circles as the circles which are connected by the negative crossing p: By Theorem 5.1, (5.2), and (5.3), we obtain

$$\begin{aligned}2g(D)+\sharp L-3=s(L_+)-2\le s(L)&\le 2g_*(L)+\sharp L-1\\ &\le 2g(L)+\sharp L-1=2g(D)+\sharp L-3.\end{aligned}$$ ∎

Proof of Corollary 1.7 By Theorem 1.3, a homogeneous link L satisfying $s(L)=2g_*(L)+\sharp L-1=2g(L)+\sharp L-1$ is a positive link. By Theorem 5.2, all almost positive links satisfy $s(L)=2g_*(L)+\sharp L-1=2g(L)+\sharp L-1$. Hence, any almost positive link is not homogeneous. ∎

Proof of Theorem 1.6 Theorem 1.6 follows from Theorems 3.1 and 5.2. ∎

Lemma 5.3 ([43, Lemma 3.4]) *Let D be an almost positive link diagram of a non-split link L with a negative crossing p. If there is no (positive) crossing of D joining the same two Seifert circles as the circles which are connected by the negative crossing p, we have $H_{Kh}^{0,2g(D)+\sharp L-4}(L)=0$, where $H_{Kh}^{i,j}(L)$ is the Khovanov homology of L and $\sharp L$ is the number of the components of L.*

6 Strong Quasipositivities of Almost Positive Knots with up to 12 Crossings

In order to present evidence towards an affirmative answer to Stoimenow's question (Question 1.5), in this section, we check the strong quasipositivities of almost positive knots with up to 12 crossings. In Subsection 6.1, we find all knots that are or may be almost positive with up to 12 crossings. In Subsection 6.2, we check the strong quasipositivities of these knots.

6.1 The Positivities and Almost Positivities of Knots up to 12 Crossings

In this subsection, we consider the positivities and almost positivities of knots with up to 12 crossings. Here, we call a knot positive if the knot or the mirror image of the knot has a positive diagram. By using Proposition 6.4, Theorems 1.3, 6.1–6.3, 6.5–6.6, and Lemma 6.7, we can determine the positivities and almost positivities of knots with up to 12 crossings except for $12_{n148}, 12_{n276}, 12_{n329}, 12_{n366}, 12_{n402}, 12_{n528}$, and 12_{n660}, which have almost positive diagrams (here we used KnotInfo [7] due to Cha and Livingston, and the Mathematica Package KnotTheory [5]). See Table 1.

Theorem 6.1 ([32, Corollary 1.7], [41, Corollary 6.1]) *Nontrivial almost positive links have negative signature.*

Theorem 6.2 ([9, Corollaries 2.1 and 2.2], [45]) *If L is an almost positive link or a positive link, then all coefficients of its Conway polynomial are non-negative.*

Theorem 6.3 ([42, Theorem 6]) *If L is an almost positive link, then*

$$\operatorname{maxdeg}_z \nabla_L(z) = \operatorname{maxdeg}_z P_L(v,z) = 1 - \chi(L),$$

where ∇_L is the Conway polynomial and $P_L(v,z)$ is the HOMFLYPT polynomial.

Proposition 6.4 ([40, Prop. 6.2]) *Let K be an almost positive knot with $g(K) \geq 3$. Then its signature $\sigma(K)$ is smaller than or equal to -4.*

Theorem 6.5 ([9, Corollary 5.1]) *If L is a homogeneous link and the coefficient of the maximal degree term of its Conway polynomial is ± 1, then the number of the crossings of a homogeneous diagram of L is at most $2 \cdot \operatorname{maxdeg}_z \nabla_L(z)$, where $\operatorname{maxdeg}_z \nabla_L(z)$ is the maximal degree of the Conway polynomial of L. In particular, the minimal crossing number of L is at most $2 \cdot \operatorname{maxdeg}_z \nabla_L(z)$.*

Theorem 6.6 ([14, Theorem 1.4]) *Positive knots up to genus two are quasialternating.*

For the definition of quasialternating links, see [28].

Lemma 6.7 *The knot 12_{n638} is a positive knot.*

Proof See Figure 3. ■

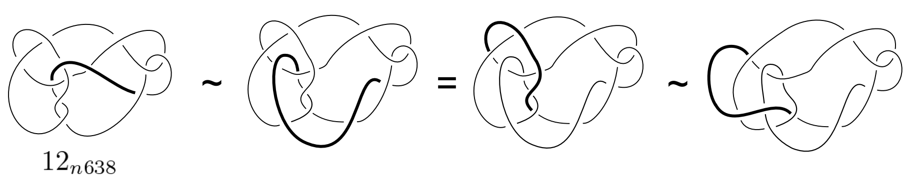

Figure 3: The knot 12_{n638} has a positive diagram.

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Remark 6.8 In the above process, we find some almost positive knots, 10_{145}, 12_{n149}, 12_{n332}, 12_{n404}, 12_{n432}, and 12_{n642}. They have almost positive diagrams, and 10_{145}, 12_{n404} and 12_{n642} are not homogeneous by Theorem 6.5. The knots 12_{n149}, 12_{n332}, and 12_{n432} are not positive by Theorem 6.6.

The knots 12_{n148}, 12_{n276}, 12_{n329}, 12_{n366}, 12_{n402}, 12_{n528}, and 12_{n660} are either positive or almost positive, since they have almost positive diagrams. In general, it is hard to check whether given almost positive link diagram represents a positive link or not.

Question 6.9 *Are the knots* 12_{n148}, 12_{n276}, 12_{n329}, 12_{n366}, 12_{n402}, 12_{n528} *and* 12_{n660} *non-positive? (If so, they are almost positive knots.)*

Remark 6.10 In [41, Example 6.1] and [42, Corollary 10], Stoimenow introduced infinitely many almost positive knots.

	≤ 11 crossings	12 crossings
total	801	2176
non-positive (negative) knots	693	$2031 \leq, \leq 2038$
positive (negative) knots	108	$138 \leq, \leq 145$
almost positive (negative) knots	1	$5 \leq, \leq 12$

Table 1: The positivities of knots with up to 12 crossings. To determine the almost positivities of some knots, we use Theorem 6.3 and Proposition 6.4. The only almost positive knot with up to 11 crossings is 10_{145}. The knots, 12_{n149}, 12_{n332}, 12_{n404}, 12_{n432}, and 12_{n642} are almost positive. Are 12_{n148}, 12_{n276}, 12_{n329}, 12_{n366}, 12_{n402}, 12_{n528}, and 12_{n660} almost positive?

6.2 Strong Quasipositivities of Almost Positive Knots with up to 12 Crossings

We check the strong quasipositivities of almost positive knots with up to 12 crossings. In this section, we call a knot strongly quasipositive if the knot or the mirror image of the knot is strongly quasipositive. From Table 1, the 6 knots, 10_{145}, 12_{n149}, 12_{n332}, 12_{n404}, 12_{n432}, and 12_{n642} are almost positive. In addition, the 7 knots, 12_{n148}, 12_{n276}, 12_{n329}, 12_{n366}, 12_{n402}, 12_{n528}, and 12_{n660} may be almost positive, and other knots with up to 12 crossings are not almost positive.

From Lemmas 6.12 and 6.14 and Table 1, we obtain the following proposition. The proposition is evidence towards an affirmative answer to Question 1.5.

Proposition 6.11 *All almost positive knots with up to* 12 *crossings are strongly quasipositive.*

Lemma 6.12 *The* 9 *knots,* 10_{145}, 12_{n148}, 12_{n276}, 12_{n329}, 12_{n366}, 12_{n402}, 12_{n528}, 12_{n642} *and* 12_{n660} *are strongly quasipositive.*

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Proof It is known that these knots are fibered (KnotInfo [7]). These knots are positive or almost positive, because they have almost positive diagrams. Note that positive links are strongly quasipositive (see [25, 37]). By Theorem 6.13, these knots are strongly quasipositive. ■

Theorem 6.13 *All fibered almost positive knots are strongly quasipositive.*

Proof Let K be a fibered almost positive knot and let D be an almost positive diagram. Obviously, the diagram D has a $*$-product decomposition whose factors are some positive diagrams $D_1, \dots, D_{n-1}$ and one special almost positive diagram D_n. Let S and S_i be the Seifert surfaces obtained from D and D_i, respectively. Note that $S_1, \dots, S_{n-1}$ are quasipositive surfaces (see [25, 37]). We consider the following two cases.

(i) Suppose that there is no crossing joining the same two Seifert circles of D as the two circles that are connected by the negative crossing. In this case, by Theorem 5.1, the surface S has minimal genus. In particular, the surface is the fiber surface. By Gabai's results [10, 11], the Seifert surface S_i is also the fiber surface. Then, by Goda–Hirasawa–Yamamoto's result [12, Corollary 1.8], the fiber surface S_n is a plumbing of positive Hopf bands. Since the positive Hopf band is a quasipositive surface and plumbings preserve the quasipositivities of surfaces [36], the surface S_n is quasipositive. Hence, the surface S is quasipositive, since it is a Murasugi sum of the quasipositive surfaces $S_1, \dots, S_n$ (see [36]). In particular, the knot K is strongly quasipositive.

(ii) In other cases, by the same discussion as Theorem 5.2(ii), we have

$$\tau(K) = g_*(K) = g(K) = g(D) - 1,$$

where $\tau(K)$ is Ozsváth–Szabó's τ-invariant of K. Hedden [13, Theorem 1.2] proved that for a fibered knot K', the knot is strongly quasipositive if and only if $\tau(K') = g_*(K') = g(K')$. Hence, K is strongly quasipositive. ■

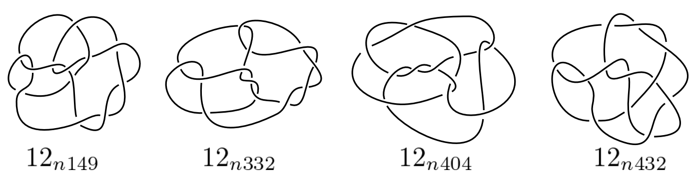

Figure 4: 12_{n149}, 12_{n332}, 12_{n404} and 12_{n432}.

Lemma 6.14 *The knots* 12_{n149}, 12_{n332}, 12_{n404}, *and* 12_{n432} *(see Figure 4) are strongly quasipositive.*

Proof Firstly, we check the strong quasipositivity of 12_{n149}. As the pictures in Figure 5 show, the canonical Seifert surface of a positive knot diagram is obtained from a Seifert surface of 12_{n149} by two deplumbings. Note that the canonical Seifert surface of

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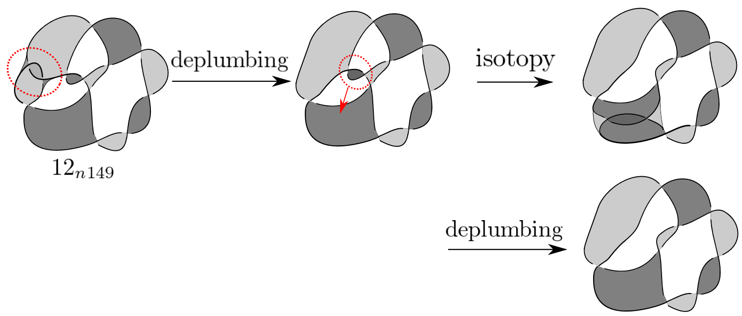

Figure 5: The top left picture is the canonical Seifert surface of an almost positive diagram of 12_{n149}. These pictures show that the Seifert surface is quasipositive.

a positive knot diagram is quasipositive (see [25,37]). Since plumbings and deplumbings preserve the quasipositivities of surfaces (see [36]), this Seifert surface of 12_{n149} is quasipositive. Hence, 12_{n149} is strongly quasipositive. By the same discussion, we can prove that 12_{n332}, 12_{n404}, and 12_{n432} are strongly quasipositive (see Figures 6, 7, and 8). ∎

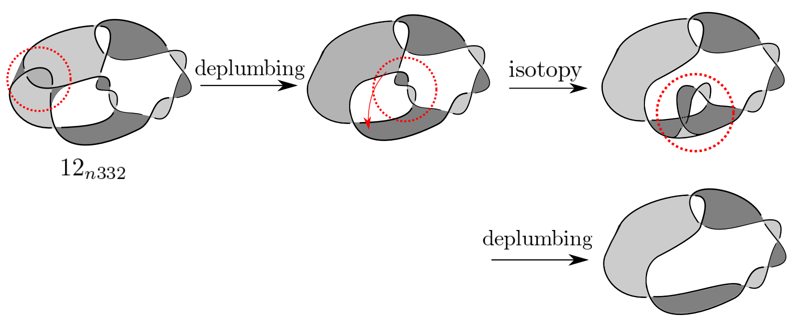

Figure 6: A proof of the strong quasipositivity of 12_{n332}.

7 Infinitely Many Counterexamples of Kauffman's Conjecture on Pseudo-alternating Links and Alternative Links

In this section, we give infinitely many counterexamples of Kauffman's conjecture on pseudo-alternating links and alternative links.

At first, we recall the definition of pseudo-alternating links [24]. A *primitive flat surface* is the canonical Seifert surface obtained from a special alternating diagram by Seifert's algorithm. A *generalized flat surface* is an orientable surface obtained from

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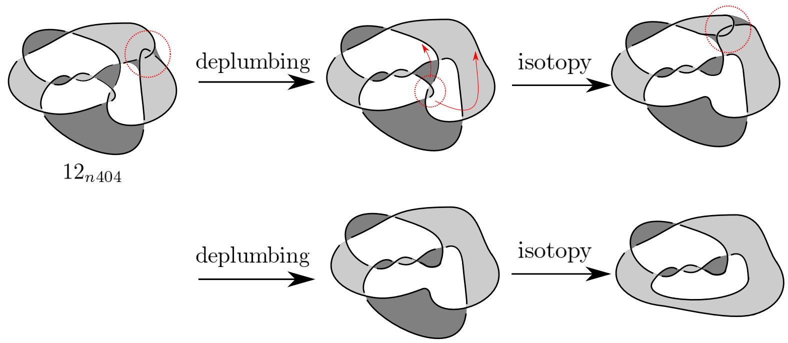

Figure 7: A proof of the strong quasipositivity of 12_{n404}.

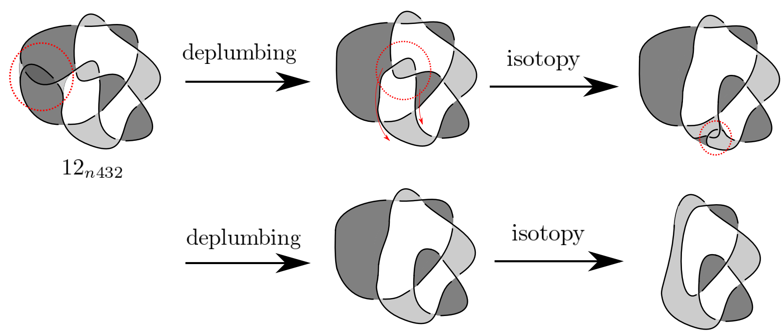

Figure 8: A proof of the strong quasipositivity of 12_{n432}.

some primitive flat surfaces by Murasugi sum along their Seifert disks (for example, see the bottom figure in Figure 10). Then a link is *pseudo-alternating* if it bounds a generalized flat surface.

Next, we recall the definition of alternative links [15]. For a link diagram D, the *spaces of D* are the connected components of the complement of the Seifert circles of D in S^2. We draw an edge joining two Seifert circles at the place where a crossing of D connects the circles. Moreover, we assign the sign “+" (resp. “−") to an edge if the crossing corresponding to the edge is positive (resp. negative). Then a diagram D is *alternative* if for each space X of D, all the edges in X have the same sign.

From the definitions, we have the following corollary.

Corollary 7.1 *All alternative links are homogeneous. All homogeneous links are pseudo-alternating.*

Kauffman conjectured that all pseudo-alternating links are alternative.

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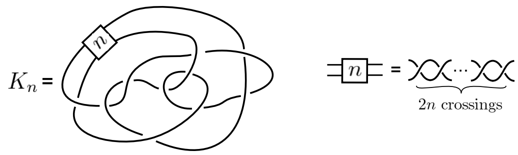

Figure 9: The knot K_n introduced by Stoimenow [41, Example 6.1], where $n \geq 0$ is the number of the full twists. Stoimenow proved that K_n is almost positive.

Conjecture 7.2 ([15]) All pseudo-alternating links are alternative.

However, this conjecture is false. In fact, Silvero [39] introduced two counterexamples: 10_{145} and $L9n18$.

Here, we prove that the infinitely many almost positive knots introduced by Stoimenow (which contains 10_{145}) are counterexamples for this conjecture.

Proposition 7.3 *Let K_n be the knot depicted in Figure 9. Then K_n is non-alternative and is pseudo-alternating.*

Proof Stoimenow [41, Example 6.1] proved that K_n is almost positive. By Corollary 1.7, the knot K_n is not homogeneous, in particular, not alternative. On the other hand, by Figure 10, the knot K_n bounds a generalized flat surface. ∎

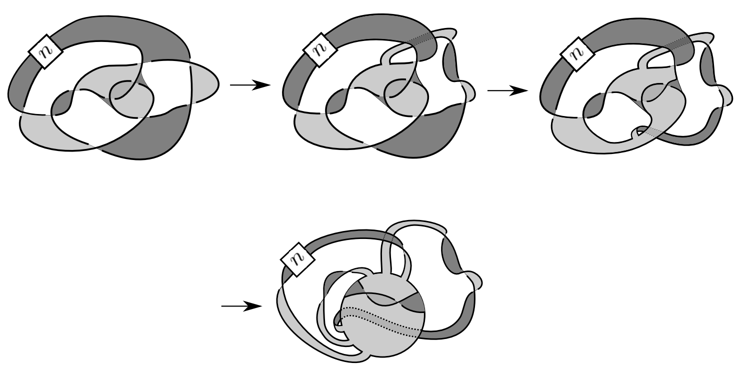

Figure 10: The top left picture is a Seifert surface of K_n. By isotopy, the surface changes into the bottom surface, which is a generalized flat surface.

Proof of Proposition 1.8 This follows from Proposition 7.3. ∎

Finally, we give two questions.

Question 7.4 *Are all almost positive links pseudo-alternating?*

Question 7.5 *Are all homogeneous links alternative?*

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