ON AN EXTREMAL PROBLEM IN FOURIER SERIES

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Let $f(x)$ be a bounded odd function, $-\pi < x < \pi$, $|f(x)| \leq 1$, with non-negative Fourier coefficients b_k, $k = 1, 2, \ldots$.

Otto Szász [1] proved anew the existence of a bounded set of numbers $\{\beta_n\}$, $n = 1, 2, \ldots$, such that

$$\sum_{k=1}^{n} k\, b_k \leq \beta_n n ,$$

where β_n is the smallest constant satisfying the above inequality and added that $2/\pi \leq \beta_n \leq 4/\pi$. He pointed out [1, p. 170] that $\beta_1 = 4/\pi$ and raised the question of the value of β_n for $n > 1$.

The purpose of this note is to prove that $\beta_n = 4/\pi$ for each n, $n = 1, 2, \ldots$.

Proof. For each given n, the function $f(x) = \operatorname{sgn}\{\sin nx\}$ satisfies the requirements placed on $f(x)$, where, as usual, $\operatorname{sgn} A$ is 1 for A positive, 0 for A zero, -1 for A negative.

For this function we have

$$\operatorname{sgn}\{\sin nx\} = \frac{4}{\pi} \sum_{k=o}^{\infty} \frac{\sin(2k+1)nx}{2k+1} ,$$

so that $b_n = 4/\pi$ and $b_k = 0$ for $k = 0, 1, \ldots, n-1$. Thus

$$\sum_{k=1}^{n} k\, b_k = (4/\pi)n ,$$

and the proof is complete, since Szász showed that $\beta_n \leq 4/\pi$.

REFERENCE

1. Otto Szász, Some extremum problems in the theory of Fourier series, Amer. J. of Math. 61 (1939), 165-177.

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