PSEUDO-UMBILICAL SURFACES WITH CONSTANT GAUSS CURVATURE

by BANG-YEN CHEN†

(Received 21st October 1971)

1. Introduction

Let M be a surface immersed in an m-dimensional space form $R^m(c)$ of curvature $c = 1, 0$ or -1. Let $\boldsymbol{h}$ be the second fundamental form of this immersion; it is a certain symmetric bilinear mapping $T_x \times T_x \to T_x^\perp$ for $x \in M$, where T_x is the tangent space and $T_x^\perp$ the normal space of M at x. Let $\boldsymbol{H}$ be the mean curvature vector of M in $R^m(c)$ and $\langle , \rangle$ the scalar product on $R^m(c)$. If there exists a function λ on M such that $\langle \boldsymbol{h}(X, Y), \boldsymbol{H} \rangle = \lambda \langle X, Y \rangle$ for all tangent vectors X, Y, then M is called a *pseudo-umbilical surface* of $R^m(c)$. Let D denote the covariant differentiation of $R^m(c)$ and $\boldsymbol{\eta}$ be a normal vector field. If we denote by $D^*\boldsymbol{\eta}$ the normal component of $D\boldsymbol{\eta}$, then D^* defines a connection in the normal bundle. A normal vector field $\boldsymbol{\eta}$ is said to be parallel in the normal bundle if $D^*\boldsymbol{\eta} = 0$. The length of mean curvature vector is called the *mean curvature*.

Let $\boldsymbol{e}$ be a unit normal vector at $x \in M$ in $R^m(c)$. Then the *second fundamental form* $\boldsymbol{h}(\boldsymbol{e})$ *at* $\boldsymbol{e}$ is defined by $\langle \boldsymbol{h}, \boldsymbol{e} \rangle$; it is a certain symmetric bilinear mapping $T_x \times T_x \to \mathbb{R}$. Let h_{ij}^r; $i, j = 1, 2$; $r = 3, \ldots, m$ be the coefficients of the second fundamental form $\boldsymbol{h}$ (for the details, see § 2). Then the Gauss curvature K and the normal curvature K_N are given respectively by

$$K = \sum_{r=3}^{m} (h_{11}^r h_{22}^r - h_{12}^r h_{12}^r), \tag{1}$$

$$K_N = \sum_{r,s=3}^{m} \left[\sum_{k=1}^{2} (h_{1k}^r h_{2k}^s - h_{2k}^r h_{1k}^s) \right]^2. \tag{2}$$

The mean curvature vector $\boldsymbol{H}$, the Gauss curvature K and the normal curvature K_N play the most important rôles, in differential geometry, for surfaces in space forms.

Theorem 1. *Let M be a pseudo-umbilical surface with constant Gauss curvature in a space form $R^m(c)$ of curvature c. If the mean curvature is constant and the normal curvature K_N vanishes, then M is either flat or totally umbilical in $R^m(c)$. In particular, if $c \geqq 0$, then M is either totally umbilical or contained in a Clifford torus.*

† This work was supported in part by the National Science Foundation under Grant GU-2648.

https://doi.org/10.1017/S0013091500009822 Published online by Cambridge University Press

A minimal surface of a sphere $S^{m-1} \subset E^m$ is a pseudo-umbilical surface with constant mean curvature in E^m, and the normal curvature of a surface in $S^3 \subset E^4$ is zero. Therefore by Theorem 1, we have the following strong result.

Corollary (5). *Let M be a minimal surface of a 3-sphere S^3 with constant Gauss curvature. Then M is either totally geodesic or contained in a Clifford torus in S^3.*

Remark 1. If the assumption that $K_N = 0$ is omitted, then Theorem 1 is no longer true. The Veronese surface in a euclidean space and the hyperbolic Veronese surface in a hyperbolic space are examples of pseudo-umbilical surfaces in space forms with constant Gauss curvature, constant mean curvature but with normal curvature $K_N \neq 0$ (see, for instance (2), (4)).

Let e be a unit normal vector field of M in $R^m(c)$. If e is parallel in the normal bundle and the determinant of $h(e)$ is nowhere zero, then e is called a *non-degenerate normal vector field.* For a compact surface with Gauss curvature $K \leqq 0$, we have the following flatness theorem.

Theorem 2. *Let M be a compact surface with Gauss curvature $K \leqq 0$ in a space form $R^m(c)$. If there exists a non-degenerate normal vector field perpendicular to the mean curvature vector field, then M is flat and the normal curvature K_N vanishes.*

Remark 2. For minimal surfaces with Gauss curvature $\leqq 0$, see (1). For surfaces with mean curvature vector parallel in the normal bundle, see (3).

2. Preliminaries

Let M be a surface immersed in an m-dimensional space form $R^m(c)$ of curvature $c = 1$, 0 or -1. We choose a local field of orthonormal frames $e_1, \ldots, e_m$ in $R^m(c)$ such that, restricted to M, the vectors e_1, e_2 are tangent to M (and, consequently, $e_3, \ldots, e_m$ are normal to M). With respect to the frame field of $R^m(c)$ chosen above, let $\omega^1, \ldots, \omega^m$ be the field of dual frames. Then the structure equations of $R^m(c)$ are given by

$$d\omega^A = \sum \omega_A^B \wedge \omega^B, \quad \omega_B^A + \omega_A^B = 0, \tag{3}$$

$$d\omega_B^A = \sum \omega_C^A \wedge \omega_C^B + c\omega^A \wedge \omega^B, \quad A, B, C = 1, \ldots, m. \tag{4}$$

We restrict these forms to M. Then $\omega^r = 0$, $r, s, t = 3, \ldots, m$. Since

$$0 = d\omega^r = \omega_r^1 \wedge \omega^1 + \omega_r^2 \wedge \omega^2,$$

by Cartan's lemma we may write

$$\omega_i^r = \sum h_{ij}^r \omega j, \quad h_{ij}^r = h_{ji}^r, \quad i, j = 1, 2. \tag{5}$$

From these we obtain

$$d\omega^i = \sum \omega_i^j \wedge \omega^j, \tag{6}$$

$$d\omega_2^1 = \{c + \sum_r \det(h_{ij}^r)\}\omega^1 \wedge \omega^2 = K\omega^1 \wedge \omega^2. \tag{7}$$

$$d\omega_i^r = \sum \omega_j^r \wedge \omega_j^i + \sum \omega_s^r \wedge \omega_s^i. \tag{8}$$

The second fundamental form h and the mean curvature vector H are given respectively by

$$h = \sum h_{ij}^r \omega^i \wedge \omega^j e_r, \tag{9}$$

$$H = \tfrac{1}{2}\sum h_{ii}^r e_r. \tag{10}$$

3. Proof of Theorem 1

Let α denote the mean curvature of M. We now consider the cases $\alpha > 0$ and $\alpha = 0$ separately.

Case (i) $\alpha > 0$. In this case, we may choose our frame field in such a way that

$$H = \alpha e_3, \tag{11}$$

$$h_{12}^r = 0, \text{ for } r = 3, \ldots, m. \tag{12}$$

Since M is pseudo-umbilical, we have

$$\omega_i^3 = \alpha\omega^i, \tag{13}$$

$$\omega_1^r = h_{11}^r \omega^1, \quad \omega_2^r = -h_{11}^r \omega^2, \quad r = 4, \ldots, m. \tag{14}$$

By taking exterior differentiations of (13) and applying (6), (8) and (14), we obtain

$$\sum_{r=4}^{m} h_{ii}^r \omega_r^3 \wedge \omega^i = 0, \text{ for } i = 1, 2. \tag{15}$$

On the other hand, by taking exterior differentiations of (14) and applying (6), (8) and (13), we obtain

$$dh_{ii}^r \wedge \omega^i + 2h_{ii}^r d\omega^i + \alpha\omega_3^r \wedge \omega^i = \sum_{s=4}^{m} h_{ii}^s \omega_r^s \wedge \omega^i, \tag{16}$$

for $r = 4, \ldots, m$ and $i = 1, 2$. Multiplying (16) by h_{ii}^r and summing up on r from 4 to m, we obtain

$$\sum_{r=4}^{m} h_{ii}^r dh_{ii}^r \wedge d\omega^i + 2\sum_{r=4}^{m} (h_{ii}^r)^2 d\omega^i + \alpha \sum_{r=4}^{m} h_{ii}^r \omega_3^r \wedge \omega^i$$
$$= \sum_{r,s=4}^{m} h_{ii}^r h_{ii}^s \omega_r^s \wedge \omega^i, \quad i = 1, 2. \tag{17}$$

By using $\omega_s^r + \omega_r^s = 0$ and (15), we obtain

$$\sum_{r=4}^{m} h_{ii}^r dh_{ii}^r \wedge d\omega^i + 2\sum_{r=4}^{m} (h_{ii}^r)^2 d\omega^i = 0, \quad i = 1, 2. \tag{18}$$

On the other hand, since the Gauss curvature K is constant, we have

$$\sum_{r=4}^{m} (h_{ii}^r)^2 = c + \alpha^2 + K = \text{constant}. \tag{19}$$

https://doi.org/10.1017/S0013091500009822 Published online by Cambridge University Press

Therefore, by (18) and (19) we obtain

$$(c+\alpha^2+K)d\omega^i = 0, \quad i = 1, 2. \tag{20}$$

If $c+\alpha^2+K = 0$, then $h^r_{ii} = 0$ for all $r > 3$. This implies that M is totally umbilical in $R^m(c)$. If $c+\alpha^2+K \neq 0$, then we obtain $d\omega^1 = d\omega^2 = 0$ identically on M. Therefore, by (6), we obtain $\omega^1_2 = 0$ identically. This implies that M is flat.

Case (ii) $\alpha = 0$. In this case, by the fact that $K_N = 0$, we may choose our frame field in such a way that

$$h^r_{12} = 0, \text{ for } r = 3, \ldots, m. \tag{21}$$

Hence, we have

$$\omega^r_1 = h^r_{11}\omega^1, \quad \omega^r_2 = -h^r_{11}\omega^2, \quad r = 3, \ldots, m. \tag{22}$$

Taking exterior differentiations of (22), we have

$$dh^r_{11} \wedge \omega^i + 2h^r_{11}d\omega^i = \sum_{s=3}^{m} h^s_{11}\omega^i \wedge \omega^r_s, \tag{23}$$

for $r = 3, \ldots, m$ and $i = 1, 2$. Multiplying (23) by h^r_{11} and summing up on r, we obtain

$$\sum_{r=3}^{m} (h^r_{11}dh^r_{11}) \wedge \omega^i + 2\sum_{r=3}^{m} (h^r_{11})^2 d\omega^i = 0, \quad i = 1, 2. \tag{24}$$

On the other hand, the constancy of the Gauss curvature implies that the first term of (24) vanishes. Thus we obtain

$$\sum_{r=3}^{m} (h^r_{11})^2 d\omega^i = 0, \quad i = 1, 2. \tag{25}$$

This implies that M is either totally geodesic or flat. Consequently, we see that, in both cases, M is either flat or totally umbilical in $R^m(c)$. This proves the first part of the theorem. The second part follows immediately from the first part and the last paragraph of § 1 of (**2**).

4. Proof of Theorem 2

Let M be a compact surface with Gauss curvature $K \leqq 0$ in a space form $R^m(c)$. If there exists a non-degenerate normal vector field $\boldsymbol{e}$ over M, which is perpendicular to the mean curvature vector $\boldsymbol{H}$, then we may choose our frame field in such a way that $\boldsymbol{e}_3 = \boldsymbol{e}$ and $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ are in the principal direction of $\boldsymbol{e}$. Since $\boldsymbol{e}$ is perpendicular to the mean curvature vector field $\boldsymbol{H}$, we have

$$\omega^3_1 = g\omega^1, \quad \omega^3_2 = -g\omega^2, \quad g > 0. \tag{26}$$

The parallelism of $\boldsymbol{e}$ in the normal bundle implies

$$\omega^3_r = 0, \quad \text{for } r = 4, \ldots, m. \tag{27}$$

By taking exterior differentiations of (26) and applying (27) we obtain

$$2gd\omega^i + dg \wedge \omega^i = 0, \quad i = 1, 2. \tag{28}$$

https://doi.org/10.1017/S0013091500009822 Published online by Cambridge University Press

From (28) we can consider local coordinates (u, v) in an open neighbourhood U of a point $p \in M$ such that

$$ds^2 = Edu^2 + Gdv^2, \quad \omega^1 = \sqrt{E}du, \quad \omega^2 = \sqrt{G}dv, \tag{29}$$

where ds^2 is the first fundamental form and E and G are local positive functions on U. From (28), equation (29) becomes

$$d(gE) \wedge du = 0, \quad d(gG) \wedge dv = 0, \tag{30}$$

which shows that (gE) is a function of u and (gG) is a function of v. By making the following coordinate transformation:

$$u' = \int (gE)^{\frac{1}{2}} du, \quad v' = \int (gG)^{\frac{1}{2}} dv, \tag{31}$$

we see that there exists a neighbourhood V of each point $p \in M$ such that there exist isothermal coordinates (u, v) in V such that

$$\begin{cases} ds^2 = f\{du^2 + dv^2\}, \quad \omega^1 = \sqrt{f}du, \quad \omega^2 = \sqrt{f}dv, \\ gf = 1 \end{cases} \tag{32}$$

where $f = f(u, v)$ is a positive function defined on V. It is well-known that the Gauss curvature K is given by

$$K = -\frac{1}{2f} \Delta \log(f), \tag{33}$$

with respect to the isothermal coordinates (u, v). Hence the condition $K \leqq 0$ with $gf = 1$ implies $\Delta \log(g) = -\Delta \log(f) \leqq 0$. By Hopf's lemma, we see that $\log(g)$ is a constant on M. Hence the Gauss curvature

$$K = -\frac{1}{2f} \Delta \log(f) = \frac{g}{2} \Delta \log(g) = 0.$$

This implies that M is flat. By taking exterior differentiation of (29) we obtain

$$\omega_1^3 \wedge \omega_1^r + \omega_2^3 \wedge \omega_2^r = 0, \text{ for } r > 3. \tag{34}$$

Substituting (26) into (34) we obtain $gh_{12}^r = 0$, for $r > 3$. This implies $K_N = 0$. This completes the proof of the theorem.

REFERENCES

(1) B.-Y. Chen, Minimal surfaces of S^m with Gauss curvature $\leqq 0$, *Proc. Amer. Math. Soc.* **31** (1972), 235-238.

(2) B.-Y. Chen, Pseudo-umbilical submanifolds of a Riemannian manifold of constant curvature, III, *J. Differential Geometry* (to appear).

(3) B.-Y. Chen and G. D. Ludden, Rigidity theorems for surfaces in euclidean space, *Bull. Amer. Math. Soc.* **78** (1972), 72-73.

(4) S. S. Chern, M. do Carmo and S. Kobayashi, Minimal submanifolds of a sphere with second fundamental form of constant length, *Functional Analysis and Related Fields* (Springer-Verlag, 1970), 60-75.

(5) H. B. Lawson, Local rigidity theorems for minimal hypersurfaces, *Ann. of Math.* (2) **89** (1969), 187-197.

Michigan State University
East Lansing, Michigan 48823