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COINCIDENCES OF FIBREWISE MAPS BETWEEN SPHERE BUNDLES OVER THE CIRCLE

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Abstract When can two fibrewise maps be deformed in a fibrewise fashion until they are coincidence free? In order to get a thorough understanding of this problem (and, more generally, of minimum numbers that are closely related to it) we study the strength of natural geometric obstructions, such as ω-invariants and Nielsen numbers, as well as the related Nielsen theory.

In the setting of sphere bundles, a certain degree map $\deg_B$ turns out to play a decisive role. In many explicit cases it also yields good descriptions of the set $\mathcal{F}$ of fibrewise homotopy classes of fibrewise maps. We introduce an addition on $\mathcal{F}$, which is not always single valued but still very helpful. Furthermore, normal bordism Gysin sequences and (iterated) Freudenthal suspensions play a crucial role.

Keywords: fibrewise map and homotopy; coincidence; Nielsen number; sphere bundle; normal bordism; Gysin sequence

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1. Introduction and outline of results

Let $A_M\colon F_M \to F_M$ and $A_N\colon F_N \to F_N$ be self-diffeomorphisms of smooth, closed, connected manifolds F_M and F_N of dimensions $m-1$ and $n-1$, respectively.

Consider the fibrewise maps as in Figure 1; they commute with the obvious fibre projections p_M and p_N onto the unit circle S^1 (see [**3**, § 1.1]). Note that the dimensions of the *total spaces* M and N are m and n, respectively.

$$\begin{tikzcd}[column sep=small]
f, f_1, f_2, \dots : M := (I \times F_M)/((1,x) \sim (0, A_M(x))) \arrow[rr] \arrow[dr, "p_M"'] & & \mathrm{N} := (I \times F_N)/((1,y) \sim (0, A_N(y))) \arrow[dl, "p_N"] \\
& S^1 = I/(1 \sim 0) &
\end{tikzcd}$$

Figure 1. Diagram 1.

We are mainly interested in the following three questions.

Question 1.1. *Can the coincidence locus*

$$C(f_1, f_2) = \{x \in M \mid f_1(x) = f_2(x)\} \tag{1.1}$$

be made empty by suitable fibrewise homotopies of f_1 and f_2?

If this can be done, i.e. if the maps f_1 and f_2 can be 'deformed away from one another' in a fibrewise fashion, we say that the pair (f_1, f_2) is *loose over* S^1 (see [**3**]).

This generalizes a problem that arises very naturally in fixed-point theory (where $M = N$ and f_2 is the identity map id).

In a very general setting the looseness obstruction

$$\omega_B(f_1, f_2) \in \Omega_{m-n+1}(M; \varphi) \tag{1.2}$$

was introduced in [**3**]. This normal bordism class depends only on the fibrewise homotopy classes of f_1 and f_2 and vanishes for loose pairs. It reflects important geometric aspects of a *generic* coincidence submanifold $C(f_1, f_2)$ of M, in particular, its location in M and its stable normal bundle.

If Question 1.1 allows no affirmative answer, we want to measure somehow to what extent the pair (f_1, f_2) fails to be loose over S^1. This can be done, for example, via the *minimum number of path components*

$$\mathrm{MCC}_B(f_1, f_2) := \min\{\#\pi_0(C(f_1', f_2')) \mid f_i' \sim_B f_i,\ i = 1, 2\} \tag{1.3}$$

of coincidence subspaces in M, achieved by fibrewise deformations of f_1 and f_2 (see [**3**, (1.2)]). Clearly, $\mathrm{MCC}_B(f_1, f_2) = 0$ if and only if (f_1, f_2) is loose.

The minimum number $\mathrm{MCC}_B(f_1, f_2)$ is bounded from below by the *Nielsen number* $N_B(f_1, f_2)$ introduced in [**13**, **14**] (see also the beginning of § 4). It is a (possibly weaker) version of both types of Nielsen numbers discussed in [**3**, § 4], but agrees with them and the classical Nielsen number in the setting of classical fixed-point theory. The definition of $N_B(f_1, f_2)$ involves (in a weaker form) the refined looseness obstruction $\tilde{\omega}_B(f_1, f_2)$, which takes into account a very inconspicuous but important additional coincidence datum, namely, the constant path in N at $f_1(x) = f_2(x)$, whenever $x \in C(f_1, f_1)$ (see [**3**, (1.6)–(1.8)]).

Question 1.2. *Given fibre bundles M and N as in Figure 1, is the minimum number $\mathrm{MCC}_B(f_1, f_2)$ equal to the Nielsen number $N_B(f_1, f_2)$ for all fibrewise maps $f_1, f_2 \colon M \to N$?*

The analogous question in classical fixed-point theory was open for nearly six decades until Boju Jiang proved the answer to be negative precisely if $M = N$ is a surface with a strictly negative Euler characteristic (see [**7**] or [**1**, I(a)]).

In other settings (e.g. classical self-coincidence theory of maps from spheres to real projective spaces) answers to Questions 1.1 and 1.2 even involve the Kervaire invariant one problem or divisibility questions for Hopf invariants (cf. [**4**, **13**, **14**]).

Question 1.3. *Can we classify the fibrewise homotopy classes of fibrewise maps or, at least, get some computable estimates or bounds for their number?*

In the case when the fibres F_M and F_N are tori of arbitrary, possibly different, dimensions greater than or equal to 1, and the gluing maps A_M and A_N are Lie group automorphisms, Questions 1.1–1.3 and the whole related Nielsen coincidence theory have been recently reduced to simple, purely algebraic problems (which, nevertheless, can be highly non-trivial) (see [**11**, **12**]). In the proofs the $\tilde{\omega}_B$-invariant turned out to be absolutely crucial.

In this paper we study another case.

From now on we assume that the fibres F_M and F_N are spheres of strictly positive dimensions and that the gluing maps A_M and A_N are orthogonal.

Question 1.1 can then be answered right away: a pair (f_1, f_2) is loose over S^1 precisely if f_2 is fibrewise homotopic to $a \circ f_1$ (*here and throughout the paper a denotes the fibrewise antipodal map in a sphere bundle*). Indeed, if for each $x \in M$ the two image points $f_1(x)$ and $f_2(x)$ in the fibre $N_b \cong S^{n-1}$ of p_N over $b = p_M(x) \in S^1$ are distinct, we may use the stereographic homeomorphism from $N_b - \{f_1(x)\}$ onto the tangent space $T_{a(f_1(x))}(N_b)$, in order to deform $f_2(x)$ 'linearly' to the antipodal point $a(f_1(x))$.

This simple observation will help us to study Questions 1.2 and 1.3.

After suitable smooth isomorphisms of fibre bundles we may (and will) *assume* that A_M (see Figure 1) has the form

$$A_M(x_1, \dots, x_{m-1}, x_m) = (x_1, \dots, x_{m-1}, \pm x_m) \tag{1.4}$$

for all $x = (x_1, \dots, x_m) \in \mathbb{R}^m$, and, similarly, $A_N \colon \mathbb{R}^n \to \mathbb{R}^n$. Both gluing maps then leave the point

$$* := e_1 = (1, 0, \dots, 0) \tag{1.5}$$

in the fibre fixed. Thus, we get well-defined 'zero sections' s_{oM} (and s_{oN}) in M (and N, respectively), described by $[t] \in S^1 = I/\sim \to [t, *]$ (see Figure 1), as well as the fibrewise maps

$$f_0 = s_{oN} \circ p_M \colon M \to N \text{ (and } a \circ f_0 \colon M \to N, \text{ respectively)}, \tag{1.6}$$

which map each fibre in M fully to the zero point $*$ of the corresponding fibre in N (or to its antipodal point $a(*)$, respectively).

We are interested in the size and structure of

$$\mathcal{F} := \{f \colon M \to N \text{ fibrewise map}\}/\text{fibrewise homotopy} \tag{1.7}$$

and, at the same time, in testing the strength of the ω_B-invariant also in this context, i.e. for sphere bundles.

By construction, the normal bordism class $\omega_B(f_1, f_2)$ must necessarily vanish whenever (f_1, f_2) is a loose pair of fibrewise maps (see Question 1.1). But is this sufficient? This is also closely related to Questions 1.2 and 1.3. In §§ 3 and 4 we prove the following.

Theorem 1.4. *Given the sphere bundles M and N over S^1, the following three conditions are equivalent.*

(i) *Full obstruction condition: for all $[f_1], [f_2] \in \mathcal{F}$ we have that $\omega_B(f_1, f_2) = 0$ if and only if the pair (f_1, f_2) is loose over S^1.*

(ii) *Wecken property (see [**1**]):* $\mathrm{MCC}_B(f_1, f_2) = N_B(f_1, f_2)$ *for all $[f_1], [f_2] \in \mathcal{F}$.*

(iii) *Injectivity condition: the map*

$$\deg_B \colon \mathcal{F} \to \Omega_{m-n+1}(M; \varphi),$$

which sends $[f]$ to $\omega_B(f, a \circ f_0)$ (see (1.6)), is injective.

Thus, it makes sense to take a closer look at $\deg_B$. Injectivity results will help us not only to understand under what conditions $\omega_B(f_1, f_2)$ is the *full* looseness obstruction for a pair (f_1, f_2) of fibrewise maps, but also to get valuable clues concerning Question 1.3. In this context the image of $\deg_B$ is equally interesting; it also turns out to coincide with the set of *all* possible values $\omega_B(f_1, f_2), [f_1], [f_2] \in \mathcal{F}$.

Define the integers

$$d_M := \det A_M, \qquad d_N := \det A_N, \qquad d = d_M d_N \tag{1.8}$$

(all lying in $\{\pm 1\}$), as well as the automorphism $\delta_i \colon \pi_i(S^{n-1}, *) \circlearrowleft$ (of the ith homotopy group of S^{n-1}, $i \in \mathbb{Z}$) by

$$\delta_i([u]) := d_M A_{N_*}([u]), \quad [u] \in \pi_i(S^{n-1}). \tag{1.9}$$

Example 1.5 ($n = 2$). Here N is the torus $T = S^1 \times S^1$ or the Klein bottle K, both fibred in the standard fashion over the base S^1. The Lie group structure on the fibres makes $\mathcal{F}$ into an abelian group.

Theorem 1.6. *If $n = 2$, then the composed map*

$$\mu \circ \deg_B = (r, q) \colon \mathcal{F} \to H_{m-1}(M; \tilde{\mathbb{Z}}_\varphi) = \mathbb{Z}/(d_N - 1)\mathbb{Z} \oplus \begin{cases} \mathbb{Z} & \text{if } m = 2 \text{ and } M \simeq N, \\ 0 & \text{otherwise} \end{cases}$$

is a group isomorphism. Given $[f] \in \mathcal{F}$, $r([f])$ classifies the section $f \circ s_{oM}$ of p_N up to homotopy (not necessarily through sections); $q([f])$ measures the 'mapping degree'

$$[f|] \in [F_M, F_N] \cong \pi_{m-1}(S^1)$$

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of the restriction of f to the fibre F_M. The Hurewicz homomorphism

$$\mu\colon \Omega_{m-1}(M;\varphi) \to H_{m-1}(M;\tilde{\mathbb{Z}}_\varphi)$$

maps, for example, $\omega_B(f_1, f_2)$ to the image of the fundamental class $[C(f_1, f_2)]$ (with integer coefficients, twisted like φ) under the inclusion $[f_1], [f_2] \in \mathcal{F}$.

In particular, $\deg_B$ is injective for $n = 2$, and the three conditions in Theorem 1.4 are satisfied.

The proof will be given in §§ 2, 3 and 4. Since N is also a torus bundle we can use the techniques of [**12**] together with Toda's tables of stable homotopy groups of spheres (see [**16**]). The case $m = n = 2$ was already discussed in detail in [**3**].

Remark 1.7. If $n = 2$, the apparently weaker invariant $\mu(\omega_B(f_1, f_2))$ is still a complete looseness obstruction, and $\omega_B(f_1, f_2)$ itself may contain unnecessary, redundant information (see Table 1). This very exceptional phenomenon is related to the fact that the fibre $F_N \cong S^1$ is a $K(\mathbb{Z}, 1)$-space. In general (e.g. for $F_N = S^{n-1}$, $n \geqslant 3$), $\mu(\omega_B(f_1, f_2))$ can be much weaker than the ω_B-invariant itself, which is able to capture very subtle and important coincidence aspects that get completely lost in homology (see, for example, [**9**]).

Our main tool when studying $\mathcal{F}$ (see (1.7)) and $\deg_B$ (see Theorem 1.4) in the remaining case, $n \geqslant 3$, will be the following diagram:

$$\begin{tikzcd}
& \operatorname{coker}((\delta_m - \mathrm{id})\colon \pi_m(S^{n-1})\circlearrowleft) \arrow[r, dashed, "\mathrm{act}"] \arrow[d, "E_2^\infty"] & \mathcal{F} \arrow[r, two heads, "q"] \arrow[d, "\deg_B"] & \ker((\delta_{m-1} - \mathrm{id})\colon \pi_{m-1}(S^{n-1})\circlearrowleft) \arrow[d, "E_1^\infty"] & \\
0 \arrow[r] & \operatorname{coker}((d-1)\cdot\colon \pi^S_{m-n+1}\circlearrowleft) \arrow[r, "\mathrm{incl}_*"] & \Omega_{m-n+1}(M,\psi) \arrow[r, "\pitchfork"] & \ker((d-1)\cdot\colon \pi^S_{m-n}\circlearrowleft) \arrow[r] & 0
\end{tikzcd} \tag{1.10}$$

Here the lower horizontal sequence is exact and derived from Gysin and Pontryagin–Thom isomorphisms; E_1^∞ and E_2^∞ denote stable suspension homomorphisms. The right-hand square commutes where q stands for the restriction to the fibre $F_M = S^{m-1}$ over $* = [1] = [0] \in S^1$; this yields a map $f|\colon S^{m-1} \to S^{n-1}$ such that $A_N \circ f| \sim f| \circ A_M$ (otherwise $f|$ is not compatible with the gluing maps and cannot be extended to all of M).

Given $[u] \in \pi_m(S^{n-1})$ and $[f] \in \mathcal{F}$, the restriction of f to some (suitably oriented) compact m-ball D in $M - F_M = (0, 1) \times S^{m-1}$ takes the product form

$$f|D = f^{\mathrm{h}} \times f^{\mathrm{v}}\colon D \to (0, 1) \times S^{n-1} = N - F_N.$$

After a suitable homotopy, we may assume that the second ('vertical', S^{n-1}-valued) component map f^{v} takes the constant value $* \in S^{n-1}$ over all of D. Replace it by u, interpreted as a map $(D, \partial D) \to (S^{n-1}, *)$. This procedure induces a well-defined action $([u], [f]) \to [u] * [f]$ on $\mathcal{F}$, which we indicate by the dotted arrow in (1.10). It is easy to check that

$$\deg_B([u * f]) = \deg_B([f]) + \mathrm{incl}_* \circ E_2^\infty([u]). \tag{1.11}$$

Now assume that $n \geqslant 3$. Then, the orbits of this action are precisely the inverse images $q^{-1}(\{w\})$, $w \in \ker(\delta - \mathrm{id})$. Moreover, all sections of p_N are homotopic through sections;

hence, each class $[f] \in \mathcal{F}$ has a representative f' such that $f' \circ s_{oM} = s_{oN}$. This can be used to add fibrewise maps by a kind of concatenation (as in homotopy groups, but different from the Lie group procedure employed in the case $n = 2$ (see Example 1.5)). The maps $\deg_B$ and q in (1.10) are compatible with all such additions.

These additions, even when multi-valued, have important consequences for ω_B. In §§ 3 and 4 they help us to prove Theorem 1.4 and also the following.

Proposition 1.8. *If $n \geqslant 3$, then, for all $[f_1], [f_2] \in \mathcal{F}$,*

$$\omega_B(f_1, f_2) = \deg_B([f_1]) - \deg_B([a \circ f_2])$$

(again, a denotes the fibrewise antipodal map).

Example 1.9 ($m = n = 3$, $d_M = d_N = 1$). Here, $\delta_3 = \delta_2 = \mathrm{id}$, $d = 1$, E_1^∞ is an isomorphism and $E_2^\infty \colon \pi_3(S^2) \cong \mathbb{Z} \to \pi_1^{\mathrm{S}} \cong \mathbb{Z}_2$ and $\deg_B$ are onto. However, $\mathcal{F}$ cannot possibly have a well-defined group structure such that q is a group homomorphism. In fact, for all integers $k \neq 0$ the inverse image of $k[\iota_2] \in \pi_2(S^2)$ under q has the cardinality $2|k|$, whereas $q^{-1}(\{0\})$ is infinite.

In § 2 we see that this is the only exceptional case. Indeed, we prove the following.

Theorem 1.10. *If $n \geqslant 3$ and $(m, n, d_M, d_N) \neq (3, 3, 1, 1)$, then $\mathcal{F}$ obtains a well-defined group structure such that the top horizontal line in the commuting diagram (1.10) is a short exact sequence of group homomorphisms; here, the monomorphism act is defined by*

$$\mathrm{act}([u]) := [u] * [f_0], \quad [u] \in \mathrm{coker}(\delta_m - \mathrm{id}).$$

This allows us to use the tools of standard homotopy theory.

Theorem 1.11. *Assume that $n \geqslant 3$. If the two stable suspension homomorphisms E_1^∞ and E_2^∞ in (1.10) are both injective (or both surjective, respectively), then*

$$\deg_B \colon \mathcal{F} \to \Omega_{m-n+1}(M; \varphi)$$

(see Theorem 1.4) is also injective (or surjective, respectively).

Corollary 1.12. *The 'degree map' $\deg_B$ is an isomorphism in the stable dimension range $m \leqslant 2n - 4$.*

Recall that in [**3**, Theorem 1.2] the stronger assumption $m < 2n - 4$ was required for a similar conclusion.

Next consider the first non-stable dimension setting.

Theorem 1.13. *Assume that $m = 2n - 3 \geqslant 2$.*

(i) *If n is odd, then $\deg_B \colon \mathcal{F} \to \Omega_{m-2}(M; \varphi)$ is injective (or, equivalently, an isomorphism) precisely for the dimension and orientability conditions*

$$(m, n, d_M, d_N) = (3, 3, -1, \pm 1) \quad \text{or} \quad (7, 5, -1, -1) \quad \text{or} \quad (15, 9, -1, -1).$$

(Note that these are precisely the dimensions where $\pi_m(S^{n-1})$ contains an element with Hopf invariant 1.)

(ii) *If $n \leqslant 20$ is even, then $\deg_B$ is injective precisely when $n = 4$ or 8 or $(n, d) = (16, -1)$.*

This and related results will be established in § 5.

We can now describe $\mathcal{F}$ and $\deg_B$ (cf. (1.7) and Theorem 1.4) easily in the first few cases where the codimension $m - n$ is low.

Proposition 1.14. *Assume that $n \geqslant 3$.*

(i) *If $m < n - 1$, then $\mathcal{F}$ and $\deg_B$ vanish.*

(ii) *If $m = n - 1$, then*

$$\deg_B \colon \mathcal{F} \xrightarrow{\cong} \mathbb{Z}/(d-1)\mathbb{Z} = \begin{cases} \mathbb{Z} & \text{if } d_M = d_N, \\ \mathbb{Z}_2 & \text{if } d_M \neq d_N. \end{cases}$$

(iii) *If $m = n$, then the surjective map*

$$\deg_B \colon \mathcal{F} \twoheadrightarrow \Omega_1(M;\varphi) \cong \mathbb{Z}_2 \oplus \begin{cases} \mathbb{Z} & \text{if } d = +1, \text{ i.e. } M \simeq N, \\ 0 & \text{if } d = -1, \text{ i.e. } M \not\simeq N, \end{cases}$$

is a group isomorphism except when $m = n = 3$ and $d_M = 1$. In this exceptional case the infinite set $\mathcal{F}$ admits no group structure compatible with $\deg_B$ (when $d_N = +1$) or else the group homomorphism

$$\deg_B \colon \mathcal{F} \cong \mathbb{Z} \twoheadrightarrow \Omega_1(M;\varphi) \cong \mathbb{Z}_2$$

is only surjective (when $d_N = -1$).

(iv) *If $m = n + 1 \geqslant 5$, then $\mathcal{F}$ has four elements and $\deg_B$ is a group isomorphism.*

(iv′) *If $(m, n) = (4, 3)$, then the target group $\Omega_2(M;\varphi)$ of $\deg_B$ still has four elements, but the homomorphism $\deg_B$ is only surjective on $\mathcal{F} \cong \mathbb{Z}_2 \oplus \mathbb{Z}$ (when $d_M = +1$) or else only injective on $\mathcal{F} \cong \mathbb{Z}_2$ (when $d_M = -1$).*

This follows from our discussion of (1.10) and from a basic knowledge of low-dimensional homotopy groups of spheres (see [**16**]).

Remark 1.15. The remaining group extension problem in Proposition 1.14 (iv) can be solved by a simple surgery argument; $\Omega_2(M;\varphi)$ turns out to be isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (and not to $\mathbb{Z}_4$).

2. Towards the homotopy classification of fibrewise maps

In this section we study the structure of the set $\mathcal{F}$ (see (1.7)) with the help of the sequence

$$\pi_m(S^{n-1}) \overset{*}{\dashrightarrow} \mathcal{F} \xrightarrow{(q,r)} \ker((\delta_{m-1} - \mathrm{id}) \colon \pi_{m-1}(S^{n-1} \circlearrowleft) \oplus \mathscr{S} \longrightarrow 0. \tag{2.1}$$

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Here, $m, n \geqslant 2$, the dotted arrow indicates the group action described in the discussion of (1.10), $\mathscr{S}$ denotes the set of homotopy classes of sections of p_N (see Figure 1). Given a homotopy class $[f] \in \mathcal{F}$ of fibrewise maps from M to N, we restrict it to the fibre $F_M = \{1\} \times S^{m-1}$ and to the zero section s_{oM} of M(see Figure 1 and (1.6)) and define

$$q([f]) := [f|F_M] \in [S^{m-1}, S^{n-1}] = \pi_{m-1}(S^{n-1}) \tag{2.2}$$

and

$$r([f]) := [f \circ s_{oM}] \in \mathscr{S}. \tag{2.3}$$

Proposition 2.1. *The sequence (2.1) is exact in the sense that*

(i) *the map (q, r) is onto and*

(ii) *each inverse image under (q, r) is a full orbit of the $\pi_m(S^{n-1})$-action.*

Proof. (i) In Figure 1, f corresponds to a fibrewise map

$$\mathrm{id} \times f^{\mathrm{v}} \colon I \times S^{m-1} \to I \times S^{n-1}, \tag{2.4}$$

i.e. to a homotopy from $A_N \circ f| \circ A_M^{-1}$ to $f| := f|(\{1\} \times S^{m-1})$.

Such a homotopy exists precisely if $\delta_{m-1}([f|]) = [f|]$ (see (1.9)).

Sections of p_N correspond to paths in S^{n-1} (which we may deform into loops). Thus, $\mathscr{S} = \{s_{oN}\}$, except possibly when $n = 2$. But in this case we can use the group structure on the fibres of p_N to show that (q, r) is onto.

(ii) Choose a quotient map

$$c_{m-1} \colon (D^{m-1}, \partial D^{m-1}) \to (S^{m-1}, *),$$

which collapses the boundary sphere of the $(m-1)$-dimensional unit ball D^{m-1} to the base point $*$ of S^{m-1} and restricts to a diffeomorphism $D^{m-1} - \partial D^{m-1} \approx S^{m-1} - \{*\}$. Also define a characteristic map for the top cell $M - F_M \vee s_{oM}(S^1)$ in M by

$$\begin{aligned} c_M \colon I \times D^{m-1} &\to I \times S^{n-1} \\ (t, x) &\to [t, c_{m-1}(x)] \end{aligned} \tag{2.5}$$

(see Figure 1).

Given $[f_1], [f_2] \in \mathcal{F}$ with equal image under (q, r), we may assume that f_1 and f_2 agree on F_M and $s_{oM}(S^1)$. Then, the S^{n-1}-valued maps $f_1^{\mathrm{v}} \circ c_M$ and $f_2^{\mathrm{v}} \circ c_M$ (see (2.4)) coincide on the boundary ∂ of $I \times D^{m-1}$ and, hence, induce the map

$$u = f_1^{\mathrm{v}} \circ c_M \coprod\nolimits_\partial f_2^{\mathrm{v}} \circ c_M \colon S^m = I \times D^{m-1} \coprod\nolimits_\partial I \times D^{m-1} \to S^{n-1}.$$

It is not hard to see that $f_1 = (\pm u) * f_2$. □

Example 2.2 ($n = 2$). Here, N is a torus bundle as in [**12**], with a well-defined group structure (via complex multiplication) on each one-dimensional fibre. Indeed, the gluing map A_N (see (1.3)) is the identity map id or complex conjugation, respectively. Thus, N is the torus T or the Klein bottle K, respectively.

Fibrewise complex multiplication makes $\mathcal{F}$ and $\mathscr{S}$ into abelian groups. Moreover, given $k \in \mathbb{Z}$, define a section

$$s_k\colon S^1 = I/(1 \sim 0) \to N$$

by

$$s_k([t]) = [t, \mathrm{e}^{2\pi \mathrm{i} k t}], \quad t \in I.$$

According to [**12**, Proposition 4.2], we obtain an isomorphism

$$s\colon \mathbb{Z}/(d_N - 1)\mathbb{Z} \xrightarrow{\simeq} \mathscr{S},$$

which maps the residue class of k to the homotopy class of s_k. Since $\pi_m(S^1) = 0$, the resulting homomorphism

$$(s^{-1} \circ r, q)\colon \mathcal{F} \xrightarrow{\simeq} \mathbb{Z}/(d-1)\mathbb{Z} \oplus \begin{cases} \mathbb{Z} & \text{if } m = 2 \text{ and } M \simeq N, \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

is also bijective (see Proposition 2.1).

For the remainder of § 2 we assume that $n \geqslant 3$. All sections of p_N are then homotopic to s_{oM}, and $\mathscr{S} = 0$.

In particular, we obtain the *surjective* forgetful maps

$$\mathrm{forg} := \mathrm{forg}' \circ \mathrm{forg}''\colon \mathcal{F}'' \xrightarrow{\mathrm{forg}''} \mathcal{F}' \xrightarrow{\mathrm{forg}'} \mathcal{F}, \tag{2.7}$$

where $\mathcal{F}'$ (and $\mathcal{F}''$, respectively) denotes the set of homotopy classes of fibrewise maps $f\colon M \to N$ such that $f \circ s_{oM} \equiv s_{oN}$ ($f \circ (-s_{oM}) \equiv s_{oN}$, respectively); the (fibrewise) homotopies are required to satisfy the same conditions at every stage of the deformation.

Analogues of $\mathcal{F}'$ are interesting in their own right and have been studied in the literature (see, for example, [**6**] for section-preserving fibrewise maps over S^b, $b > 1$).

The set $\mathcal{F}''$ is useful since it has a natural group addition: given two maps f_1, f_2 that take both the section s_{oM} and its antipodal $-s_{oM}$ to s_{oN}, just stack them 'on top of one another' with respect to the x_1-coordinate (cf. (1.4) and (1.5)), i.e. just pinch the equator at $x_1 = 0$ in every fibre S^{m-1} of M (see Figure 1), to obtain $S_1^{m-1} \vee S_2^{m-1}$ and apply f_i to S_i^{m-1}, $i = 1, 2$. This well-defined addition makes $\mathcal{F}''$ into a group.

This extra structure also induces 'additions' on $\mathcal{F}$ and $\mathcal{F}'$, which are compatible, via q (see (2.2)), with the addition in $\pi_{m-1}(S^{n-1})$, e.g. define

$$[f_1] + [f_2] := \mathrm{forg}(\mathrm{forg}^{-1}(\{[f_1]\}) + \mathrm{forg}^{-1}(\{[f_2]\})) \subset \mathcal{F}. \tag{2.8}$$

However, if the forgetful map forg is not bijective, this addition may only be *multi-valued* and $\mathcal{F}$ need not be a group (see, for example, Example 2.7).

There are also well-defined group actions of $\pi_m(S^{n-1})$ on $\mathcal{F}'$ and $\mathcal{F}''$ (as introduced in the comments for (1.10)).

Proposition 2.3. *Given $[f] \in \mathcal{F}'$ and $[u_0], [u_1] \in \pi_m(S^{n-1})$, the two elements $[u_0] * [f]$ and $[u_1] * [f]$ of $\mathcal{F}'$ are equal if and only if $[u_0] = [u_1] + (\delta_m - \mathrm{id})[v]$ for some $[v] \in \pi_m(S^{n-1})$. Thus, the action of $\pi_m(S^{n-1})$ induces a well-defined group action of $\operatorname{coker}(\delta_m - \mathrm{id})$ on $\mathcal{F}'$ and yields bijections from $\operatorname{coker}(\delta_m - \mathrm{id})$ onto the inverse image $(q \circ \mathrm{forg})^{-1}(\{[f|]\}) \subset \mathcal{F}'$ for each element $[f|] \in \ker(\delta_{m-1} - \mathrm{id})$ (cf. (2.2) and (2.7)).*

Proof. Given a section-preserving fibrewise homotopy h from $u_0 * f$ to $u_1 * f$, its restriction $h|\colon F_M \times I \to F_N$ differs from the 'constant' homotopy $(x,\tau) \to f(x)$ by the local action of some $[v] \in \pi_m(S^{n-1})$; compare this with the explanations following (1.10) (and any element of $\pi_m(S^{n-1})$ may arise this way). In view of the gluing maps (see Figure 1 and (1.8)), this amounts to the action of $[v]$ (and $\delta_m([v])$, respectively) at $\{1\} \times S^{m-1} \times I$ (at $\{0\} \times S^{m-1} \times I$, respectively) in the homotopy (of 'vertical parts')

$$h^{\mathrm{v}}\colon (I \times S^{m-1} \times I, I \times \{*\} \times I) \to (S^{n-1}, \{*\}),$$

formed as in (2.4). This yields a boundary-preserving homotopy from $(u_0-(\delta_m-\mathrm{id})v)+f$ to $u_1 * f$ and Proposition 2.3 follows. □

Proposition 2.4. *If $n \geqslant 4$, then the forgetful maps* forg *and* forg$''$ *(see (2.7)) are bijective and make $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{F}''$ into (isomorphic) groups (via (2.8)); moreover,*

$$0 \longrightarrow \mathrm{coker}(\delta_m - \mathrm{id}) \xrightarrow{\mathrm{act}} \mathcal{F} \xrightarrow{q} \ker(\delta_{m-1} - \mathrm{id}) \longrightarrow 0$$

is an exact sequence of group homomorphisms. Here, we define (see (1.5))

$$\mathrm{act}([[u]]) = [u] * [f_0], \quad [u] \in \pi_m(S^{n-1}).$$

Proof. By transversality in N each homotopy from the zero section s_{oN} to itself can be deformed into the constant homotopy. This implies our first claim.

Clearly, act is a group homomorphism. Propositions 2.1 and 2.3 imply exactness. □

Next, consider the case $n = 3$. Here, forg fails, in general, to be injective and may induce only a *multi-valued* addition (see (2.8)), which is, nevertheless, compatible with q.

Theorem 2.5. *Assume that $n = 3$. Given $[f] \in \mathcal{F}$ and $u_0, u_1 \in \pi_m(S^2)$, we have that the two fibrewise homotopy classes $u_0 * [f], u_1 * [f] \in \mathcal{F}$ are equal if and only if*

$$u_0 - u_1 = \delta_m(v) - v + k[\iota_2, q[f]]$$

for some $v \in \pi_m(S^2)$ and $k \in \mathbb{Z}$, and the last term is the Whitehead product of the generator $\iota_2 \in \pi_2(S^2)$ with $q([f]) = [f|F_M] \in \pi_{m-1}(S^2)$.

Thus, the group action induces a bijection from $\pi_m(S^2)/((\delta_m-\mathrm{id})(\pi_m(S^2))+\mathbb{Z}[\iota_2,[f|]])$ onto the inverse image $q^{-1}(\{[f|]\})$.

Proof. We may assume that $f \circ s_{oM} \equiv s_{oN}$. If there is a fibrewise homotopy $h = f_\tau$, $\tau \in [0,1]$, from $u_0 * f$ to $u_1 * f$, we may also assume that it preserves the zero section for $0 \leqslant \tau \leqslant \frac{1}{2}$ and leaves the restriction to the fibre $F_M = \{1\} \times S^{m-1}$ unchanged for $\frac{1}{2} \leqslant \tau \leqslant 1$. Then, (in view of Proposition 2.3) h first deforms $u_0 * f$ to $f_{1/2} = u_{1/2} * f$, where $u_{1/2}$ has the form $u_{1/2} = u_0 + \delta v - v$ for some $v \in \pi_m(S^2)$.

Thus, it remains to study the effect of the restricted homotopy

$$h|(s_{oM}(S^1) \times [\tfrac{1}{2}, 1]), \tag{2.9}$$

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which sends all points outside of the rectangle

$$R = s_{oM}(S^1) - \{[1,*]\} \times (\tfrac{1}{2}, 1)$$

to the zero section of N. After a suitable deformation we may assume that the 'principal part' $h^{\mathrm{v}}|\colon R \to S^2$ (see (2.4)) maps finitely many disjoint open discs in R diffeomorphically onto $S^2 - \{*\}$ and their complement to the base point $*$ of S^2 (see (1.4)).

For the remainder of the proof it is convenient to use the relative homeomorphism

$$c_M\colon (I \times D^{m-1}, I \times \partial D^{m-1} \cup \{0,1\} \times D^{m-1}) \to (M, s_{oM}(S^1) \vee F_M)$$

(see (2.5)). We may assume that $f_{1/2}|T \equiv f|T \equiv s_{oM} \circ p_M|T$ on the tubular neighbourhood

$$T = c_M(I \times \{x \in D^{m-1} \mid \tfrac{1}{2} \leqslant \|x\| \leqslant 1\})$$

of $s_{oM}(S^1)$ in M. Modify $f_{1/2}$ by replacing $f_{1/2}|T$ with the map that sends $[t, c_{m-1}(x)]$ to $h|(s_{oM}(T), \|x\|)$. The resulting fibrewise map $\bar{f}_{1/2}\colon M \to N$ is homotopic to $u_1 * f$ via a homotopy that leaves $s_{oM}(S^1) \vee F_M$ unchanged.

It remains to describe $\bar{f}_{1/2}$ by the action of $\pi_m(S^2)$ on $f_{1/2}$. Any differences between these maps occur only in the top cell $M - s_{oM}(S^1) \cup F_M \cong (0,1) \times \mathring{D}^{m-1}$ of M and are entirely captured by the 'principal parts'

$$f^{\mathrm{v}}_{1/2} \circ c_M, \bar{f}^{\mathrm{v}}_{1/2} \circ c_M\colon (0,1) \times D^{m-1} \to S^2$$

(see (2.4), (2.5)), or, equivalently (via the Pontryagin–Thom construction), by the corresponding inverse images of the antipodal point $-*$ of the basepoint $*$ of S^2.

Locally, $f^{\mathrm{v}}_{1/2} \circ c_M$ can be deformed into the composite of

(i) the projection to $\mathring{D}^{m-1}$,

(ii) a pinching map *pinch* that takes a ball $D_\varepsilon \subset \mathring{D}^{m-1}$ (of radius ε, $0 < \varepsilon \ll 1$) diffeomorphically to $S^{m-1} - \{*\}$ and collapses its complement to $* \in S^{m-1}$ and

(iii) $f|\colon S^{m-1} \to S^2$.

Our modification (which yields $\bar{f}^{\mathrm{v}}_{1/2} \circ c_M$) just corresponds to adding finitely many meridian spheres $\partial D^{m-1}_{3\varepsilon}$, framed by the outwards pointing vector field in B^{m-1} and 'horizontal' vectors along the t-directions of the base S^1 of p_M.

We complete the proof by exhibiting the necessary framed bordism.

In the (t, x_1)-plane, consider the union of the t-axis L and the circle K with radius 3ε and centre $(0, 6\varepsilon)$. By embedded surgery (using the 'rectangle' Q joining small neighbourhoods of $(0,0)$ in L and of $(0, 3\varepsilon)$ in K) we get a framed bordism from $L \sqcup K$ to a curve L', which is isotopic to L (see Figure 2).

Now, embed this into the space $\mathbb{R} \times \mathbb{R}^{m-1}$ with coordinates $(t, x_1, \ldots, x_{m-1})$ and add an $(m-2)$-dimensional meridian sphere S near the point $(0, 9\varepsilon, 0, \ldots, 0)$. Throughout the resulting embedded bordism in $(0,1) \times D^{m-1} \times [\frac{1}{2}, 1]$ map each normal ε-ball of L, K, L', etc. to S^2 via $f| \circ \pitchfork$ (cf. (ii), (iii) above); similarly, map each normal disc of S

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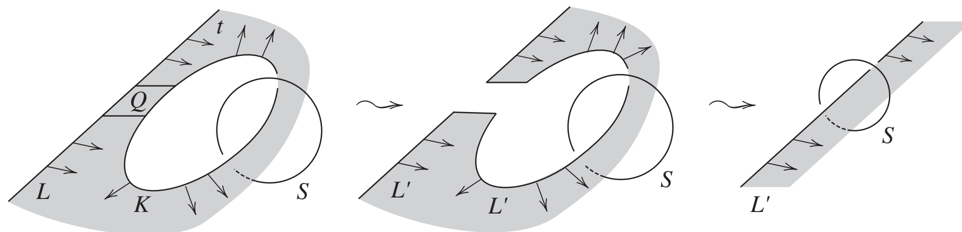

Figure 2. Framed bordism.

to S^2 via the collapsing maps $D_\varepsilon \to D_\varepsilon/\partial D_\varepsilon \cong S^2$. The contribution of $K \sqcup S$ then corresponds (via Pontryagin–Thom) to the action of the Whitehead product $\pm[\iota_2, [f|]]$.

We obtain a homotopy that stands at $\bar{f}_{1/2}$ and replaces each meridian sphere by a copy of $K \sqcup S$. If the geometric description of the modification from $f_{1/2}$ to $\bar{f}_{1/2}$ involves k meridian spheres (counted with $\pm$ signs according to their co-orientations), we conclude that $\bar{f}_{1/2} = [\pm k\iota_2, q[f]] * f_{1/2}$ and

$$\bar{u}_{1/2} = u_{1/2} \pm k[\iota_2, q[f]].$$

Now, k is the mapping degree (mod boundary) of the principal part

$$h^{\mathrm{v}}|\colon s_{oM}(S^1) \times [\tfrac{1}{2}, 1] \to S^2$$

of the restricted homotopy of Theorem 2.5. Since any integer can be realized in this way, the theorem follows (see also Proposition 2.1 (ii)). □

Explicit calculations are helped by the following.

Lemma 2.6. *The automorphism δ_i of $\pi_i(S^2)$ (see (1.8)) can be described as*

$$\delta_i = \begin{cases} d\,\mathrm{id} & \text{if } i = 2, \\ d_M\,\mathrm{id} & \text{if } i \geqslant 3. \end{cases}$$

Proof. We need consider only the case where $d_N = -1$ and the gluing map $A_N\colon S^2 \to S^2$ reverses orientations. Then, $A_{N*} = -\,\mathrm{id}$ on $\pi_2(S^2)$, as shown by the mapping degree. If $i \geqslant 3$, then $A_{N*} = \mathrm{id}$ since $\pi_i(S^2) = \mathrm{Hopf}_*(\pi_i(S^3))$ and the Hopf map $\mathrm{Hopf}\colon S^3 \to S^2$ is homotopic to $A_N \circ \mathrm{Hopf}$, as seen with the help of the Hopf invariant. □

Example 2.7 ($m = n = 3$, $d_M = d_N = 1$ (i.e. $M \cong N \cong S^1 \times S^2$)). We identify $\pi_3(S^2)$ and $\pi_2(S^2)$ with $\mathbb{Z}$ via the Hopf invariant and degree isomorphisms. Then (2.1) takes the form

$$\mathbb{Z} = \pi_3(S^2) \xrightarrow{\ \mathrm{act}\ } \mathcal{F} \xrightarrow{\ q\ } \pi_2(S^2) = \mathbb{Z}.$$

The group operation $(u, [f]) \to u * [f]$ (see the discussion of (1.10)) provides a nice geometric description for certain pairs $(u, [f]) \in \mathbb{Z} \times \mathcal{F}$. Indeed, given $j \in \mathbb{Z}$, let rot_j

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denote the fibrewise self-map of $S^1 \times S^2$ that rotates S^2 j times around the x_1-axis as we go once around the base S^1. It can be shown (cf. [**13**, **14**]) that

$$[\mathrm{rot}_j \circ f] = (jq(f)) * [f] \quad \text{and} \quad [f \circ \mathrm{rot}_j] = (j(q(f))^2) * [f]$$

for all $j \in \mathbb{Z}$ and $[f] \in \mathcal{F}$.

According to Theorem 2.5 the inverse image $q^{-1}(\{k\iota_2\})$ has cardinality $2k$ for any integer $k \neq 0$ and is infinite for $k = 0$. Thus, $\mathcal{F}$ cannot have a group structure such that q is a homomorphism.

In contrast, if $m = n = 3$ and $(d_M, d_N) \neq (1, 1)$, then all inverse images of q have the same cardinality. (For a complete description of $\mathcal{F}$ in this case see Example 2.10.)

Corollary 2.8. *If $m \geqslant 2$ and $n = 3$, then the following conditions are equivalent.*

(i) *The composed homomorphism*

$$\iota : \ker(\delta_{m-1} - \mathrm{id}) \subset \pi_{m-1}(S^2) \xrightarrow{[\iota_2, -]} \pi_m(S^2) \longrightarrow \operatorname{coker}(\delta_m - \mathrm{id})$$

is trivial.

(ii) *All inverse images $q^{-1}(\{[f|]\})$, $[f|] \in \ker(\delta_m - \mathrm{id})$, have the same cardinality.*

(iii) *The forgetful map $\mathrm{forg}' \colon \mathcal{F}' \to \mathcal{F}$ is bijective.*

(iv) *$\mathcal{F}$ allows a group structure such that $\mathrm{forg} \colon \mathcal{F}'' \to \mathcal{F}$ is a group homomorphism.*

(v) *$\mathcal{F}$ allows a group structure such that $q \colon \mathcal{F} \to \pi_{m-1}(S^{n-1})$ is a group homomorphism.*

If these conditions hold, then $\mathcal{F}$, equipped with the unique group structure determined by (iv), still fits into the exact sequence of group homomorphisms exhibited in Proposition 2.4.

Proof. The equivalence of parts (i)–(iii) of Corollary 2.8 follows from Proposition 2.3, Theorem 2.5 and the fact that $\operatorname{coker}(\delta_m - \mathrm{id})$ is either finite or isomorphic to $\mathbb{Z}$. Also, (iv) implies (v), since forg is onto and $q \circ \mathrm{forg}$ is a homomorphism. In turn, (ii) clearly follows from (v).

It remains to derive (iv) from (iii). Given the homotopy classes $[f_1], [f_2] \in \mathcal{F}$, we may choose representatives such that

$$f_1 \circ (-s_{oM}) \equiv f_2 \circ s_{oM} \cong s_{oN} \tag{2.10}$$

and stack f_1 on top of f_2 with respect to the x_1-coordinate (cf. (1.3) and (1.4)) as in the construction of the group structure in $\mathcal{F}''$ (see (2.8)). If Corollary 2.8 (iii) holds, the resulting homotopy class $[f_1 + f_2] \in \mathcal{F}$ is independent of our choices; indeed, any fibrewise homotopy between maps satisfying (2.10) can be replaced by one that satisfies this condition at every stage (note that f_1, when composed with a suitable reflection in M, as well as f_2, yields elements in $\mathcal{F}'$). Thus, the addition in $\mathcal{F}$, induced from $\mathcal{F}''$ via the surjective forgetful map forg (see (2.8)) is *single-valued* and makes $\mathcal{F}$ into a group. □

Example 2.9 ($m = 2$, $n = 3$). The target group of ι is trivial here and parts (i)–(v) of Corollary 2.8 are satisfied. In particular, $\mathcal{F}$ and $\mathcal{F}'$ have canonical group structures induced from $\mathcal{F}''$, and the maps forg' (see (2.7)) and act (see Proposition 2.4) yield isomorphisms

$$\mathcal{F}' \cong \mathcal{F} \cong \mathbb{Z}/(d-1)\mathbb{Z}$$

(see also Lemma 2.6).

In contrast, it is a nice exercise to show that

$$\mathcal{F}'' \cong \begin{cases} (\mathbb{Z}/(d-1)\mathbb{Z}) \oplus (\mathbb{Z}/(d-1)\mathbb{Z}) & \text{if } d_M = 1, \\ \mathbb{Z} & \text{if } d_M = -1, \end{cases}$$

and that the epimorphism $\mathrm{forg}\colon \mathcal{F}'' \to \mathcal{F}$ is bijective if and only if $d_M = d_N = -1$.

Hint. Let $S \subset M$ denote the union of the two circles $s_{oM}(S^1)$ and $(-s_{oM}(S^1))$. Then, $M - (S \cup F_M)$ consists of two 'rectangular' 2-cells, R_+ and R_-, characterized by the inequalities $\pm x_2 > 0$ and oriented via the projection to the (t, x_1)-plane (see (1.3)). Using the discs $D_\pm \subset R_\pm$ as in the discussion of (1.10), we obtain a group action of $\pi_2(S^2) \oplus \pi_2(S^2) \cong \mathbb{Z}^2$ on $\mathcal{F}''$, which we apply to $f_0 = s_{oN} \circ p_N$ to get the group epimorphism

$$\mathrm{act}''\colon \mathbb{Z} \oplus \mathbb{Z} \to \mathcal{F}''.$$

Note that $M - S$ consists of one or two annuli according to whether $d_M = -1$ or $d_M = +1$, respectively. If we slide $D_\pm$ across the fibre F_M (or the circle $(-s_{oN})(S^1)$, respectively) and take the gluing maps (with A_M, A_N our orientable convention, respectively) into account, we see that

$$\mathrm{act}(u_+, u_-) = \begin{cases} (d_N u_+, d_N u_-) & \text{if } d_M = 1, \\ (d_N u_-, d_N u_+) & \text{if } d_M = -1 \end{cases}$$

and

$$\mathrm{forg}(\mathrm{act}(u_+, u_-)) = -\,\mathrm{forg}(\mathrm{act}(u_-, u_+))$$

for all $(u_+, u_-) \in \mathbb{Z} \oplus \mathbb{Z}$.

Example 2.10 ($m = n = 3$). Except in the case $d_M = d_N = 1$, discussed in Example 2.7, Lemma 2.6 implies that $\iota \cong 0$. Then, $\mathcal{F}$ has a canonical group structure (see Corollary 2.8) and we get that

$$\mathcal{F} = \begin{cases} \mathbb{Z} & \text{if } (d_M, d_N) = (1, -1), \\ \mathbb{Z}_2 & \text{if } (d_M, d_N) = (-1, 1), \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } (d_M, d_N) = (-1, -1). \end{cases}$$

Example 2.11 ($m > n = 3$). In this case $\iota \equiv 0$ since the $[\iota_2, \eta] = 0$ for $\eta \in \pi_{m-1}(S^2)$ (see [**5**, Theorem 2.4]). Therefore, by Corollary 2.8 we have the short exact sequence of groups

$$
\begin{tikzcd}
0 \arrow[r] & \pi_m(S^2)/(d_m-1)\pi_m(S^2) \arrow[r] & \mathcal{F} \arrow[r] & \ker((d_m-1)\cdot\colon \pi_{m-1}(S^2)\circlearrowleft) \arrow[r] \arrow[ll, dashed, bend left, "\iota"] & 0.
\end{tikzcd}
$$

3. The invariants $\boldsymbol{\omega_B}$ and $\mathbf{deg}_{\boldsymbol{B}}$

In this section we briefly recall those geometric phenomena that are captured by the looseness obstruction ω_B and the 'degree' map $\deg_B$. For a better understanding of the target group of these invariants we study the normal bordism Gysin sequence and relate it to the sequences in (2.1), Proposition 2.4 and Example 2.11. In particular, we establish (1.10).

Given the homotopy classes $[f_1],[f_2] \in \mathcal{F}$, we may choose representatives such that the resulting map

$$(f_1, f_2)\colon M \to N \times_B N$$

into the indicated fibrewise product is smooth and transverse to the *fibrewise* diagonal $\triangle \subset N \times_B N$. The coincidence locus

$$C = (f_1, f_2)^{-1}(\triangle) = \{x \in M \mid f_1(x) = f_2(x)\} \tag{3.1}$$

is then an $(m-n+1)$-manifold in M, equipped with a description of its stable normal bundle in terms of the vector bundle

$$\varphi = p_M^*(\otimes^{(1-d)/2}\lambda). \tag{3.2}$$

(This reflects the tangent bundles of M and N and their twisting due to the gluing maps, cf. (1.4) and (1.8); λ denotes the non-trivial line bundle over the basis S^1. Thus, φ is trivial if $d = 1$ and $\varphi = p_M^*(\lambda)$ if $d = -1$.) These data represent the normal bordism class

$$\omega_B(f_1, f_2) \in \Omega_{m-n+1}(M; \varphi),$$

which depends only on $[f_1],[f_2] \in \mathcal{F}$ (for more details see [**3**]). For $m, n \geqslant 2$ the group $\Omega_{m-n+1}(M; \varphi)$ fits into a long exact Gysin sequence

$$\cdots \to \pi_i^{\mathrm{S}} \oplus \pi_{i-m+1}^{\mathrm{S}} \xrightarrow{(d-1)\oplus(d_N-1)} \pi_i^{\mathrm{S}} \oplus \pi_{i-m+1}^{\mathrm{S}} \xrightarrow{\mathrm{incl}_*} \Omega_i(M;\varphi) \xrightarrow{\pitchfork} \pi_{i-1}^{\mathrm{S}} \oplus \pi_{i-m}^{\mathrm{S}} \to \cdots \tag{3.3}$$

(see [**12**, Theorem 5.4]). It is obtained from the normal bordism sequence of the pair $(M, M - F_M)$ of spaces, using identifications via the Thom–Gysin isomorphisms

$$\Omega_{*+1}(M, M - F_M; \varphi) \underset{\cong}{\xrightarrow{\pitchfork_{\mathrm{rel}}}} \Omega_*^{\mathrm{fr}}(F_M = S^{m-1}) \xrightarrow{\cong} \Omega_*^{\mathrm{fr}} \oplus \Omega_{*-m+1}^{\mathrm{fr}}, \tag{3.4}$$

and the Thom–Pontryagin isomorphism between the framed bordism groups Ω_*^{fr} of a point and the stable homotopy groups π_*^{S} of spheres.

The homomorphism incl_* is induced by the (shifted) inclusion

$$F_M = \{1\} \times S^{m-1} \approx \{\tfrac{1}{2}\} \times S^{m-1} \subset M - F_M \subset M$$

(see Figure 1).

Furthermore, given an element of $\Omega_*(M;\varphi)$, we may represent it by a singular manifold $c\colon C \to M$ (equipped with a vector bundle isomorphism $\bar{c}$ relating the stable normal bundle of C to $c^*(\varphi)$, and assume that C is smooth and transverse to the fibre $F_M \subset M$. Then, the 'intersection manifold' $c|\colon c^{-1}(F_M) \to F_M$, together with the stable framing $\bar{c}$ induced from $c|$, represents $\pitchfork([C, c, \bar{c}])$. In particular, for all $[f_1], [f_2] \in \mathcal{F}$, $\omega_B(f_1, f_2)$ gets mapped to the ω-invariant (see [**10**]) of the restricted maps $f_1|, f_2|\colon F_M \to F_N$, i.e.

$$\pitchfork(\omega_B(f_1, f_2)) = \omega(f_1|, f_2|). \tag{3.5}$$

The inverse of $\pitchfork_{\mathrm{rel}}$ (see (3.4)) takes a framed singular manifold $g\colon Q \to F_M = \{1\} \times S^{m-1}$ to the composite

$$[\tfrac{1}{2}, 1 + \tfrac{1}{2}] \times Q \xrightarrow{\ \mathrm{id} \times g\ } [\tfrac{1}{2}, 1 + \tfrac{1}{2}] \times S^{m-1} \longrightarrow M$$

(see Figure 1). In view of the gluing maps (see (1.4) and (1.8)) the resulting relative bordism class gets mapped, by the boundary homomorphism, to the difference

$$(-1)^d [Q, A_M \circ g] - [Q, g] \in \Omega^{\mathrm{fr}}_*(F_M)$$

of the two contributions from the top end (at the level $1 + \frac{1}{2}$) and the bottom end (at $\frac{1}{2}$), respectively.

Example 3.1 ($n = 2$). Here (3.3) gives rise to the top line in the commuting diagram of short exact sequences

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & (\pi^{\mathrm{S}}_{m-1}/(d-1)\cdot\pi^{\mathrm{S}}_{m-1}) \oplus (\mathbb{Z}/(d_N-1)\mathbb{Z}) \arrow[r] \arrow[d, "\mu_{F_M}"] & \Omega_{m-1}(M;\psi) \arrow[r] \arrow[d, "\mu_M"] \arrow[l, dashed, bend left, "\pitchfork'"] & \ker((d-1)\cdot\colon \pi^{\mathrm{S}}_{m-2} \circlearrowleft) \arrow[r] \arrow[d, "\mu_{F_M}"] & 0 \\
0 \arrow[r] & H_{m-1}(F_M;\mathbb{Z})/(d_N-1)H_{m-1}(F_M;\mathbb{Z}) \arrow[r] & H_{m-1}(M;\tilde{\mathbb{Z}}_d) \arrow[r, "\pitchfork_H"'] \arrow[l, dashed, bend left, "\pitchfork'_H"] & \ker((d-1)\cdot\colon H_{m-2}(F_M;\mathbb{Z}) \circlearrowleft) \arrow[r] & 0
\end{tikzcd} \tag{3.6}$$

(see [**12**, Proposition 5.4]). Similarly, the bottom line is derived from the homology Gysin sequence, which is based on the pair $(M, M - F_M)$ and on the local system $\tilde{\mathbb{Z}}_d$ of integer coefficients (twisted like the orientation bundle $p_M^*(\otimes^d \lambda)$ of φ (see (3.2))). The vertical arrows are (induced by) the Hurewicz homomorphism of M and F_M, respectively.

Transverse intersection with the zero section $s_{oM}(S^1) \approx S^1$ in M yields the partial splitting

$$\pitchfork'\colon \Omega_{m-1}(M;\varphi) \to \Omega_0(S^1; \otimes^{d_N}\lambda) \cong \mathbb{Z}/(d_N - 1)\mathbb{Z}$$

in the top line of (3.6) (for calculation of low-dimensional normal bordism groups see, for example, [**8**, Theorem 9.3]). This commutes with the Hurewicz homomorphisms and

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Table 1. *The cardinality of* $\ker \mu_M$*, which measures the relative size of* $\deg_B(\mathcal{F})$ *in* $\Omega_{m-1}(M;\varphi)$*, e.g. if* $d=-1$ *and* $m \geqslant 3$*, then* $|\ker \mu_M| = |\Omega_{m-1}(M;\varphi)|/|\mathcal{F}|$ *is the quotient of the indicated cardinalities.*

m	2	3	4	5	6	7	8	9	10
$d=1$	2	4	48	24	1	2	480	960	32
$d=-1$	2	4	4	2	1	2	4	8	32

the splitting

$$\begin{aligned}\pitchfork'_H\colon H_{m-1}(M;\tilde{\mathbb{Z}}_d) \to H_0(S^1;\tilde{\mathbb{Z}}_{d_N}) &\cong \mathbb{Z}/(d_N-1)\mathbb{Z}\\ &\cong H_{m-1}(F_M;\mathbb{Z})/(d_N-1)H_{m-1}(F_M;\mathbb{Z})\end{aligned}$$

of the bottom line in (3.6). Note that $\pitchfork'_H$ can also be expressed via cap products with the Thom class

$$U_\xi \in H^{m-1}(\xi,\xi-\text{zero section};\tilde{\mathbb{Z}}_{d_M}) \cong H^{m-1}(M,M-S^1;\tilde{\mathbb{Z}}_{d_M})$$

of the tangent bundle $\xi \cong (\otimes^{d_M}\lambda)\oplus\mathbb{R}^{m-2}$ of M along the fibres of p_M (see [**15**, §5, Exercise J6]).

We obtain the isomorphism

$$(\pitchfork'_H,\pitchfork_H)\colon H_{m-1}(M;\tilde{\mathbb{Z}}_d) \xrightarrow{\cong} \mathbb{Z}/(d_N-1)\mathbb{Z} \oplus \begin{cases}\mathbb{Z} & \text{if } m=2 \text{ and } d=1,\\ 0 & \text{otherwise.}\end{cases}$$

Furthermore, the composite map (see (1.7) and (3.6))

$$\mathcal{F} \xrightarrow{\deg_B} \Omega_{m-1}(M;\varphi) \xrightarrow{\mu_M} H_{m-1}(M;\tilde{\mathbb{Z}}_d), \tag{3.7}$$

when followed by $(\pitchfork'_H,\pitchfork_H)$, equals the classifying isomorphism $(s^{-1}\circ r,q)$ in (2.6).

We conclude that $\mu_M\circ\deg_B$ itself is an isomorphism. Hence, $\deg_B$ is injective but in general not onto. The kernel of μ_M (which is finite whenever $m \geqslant 2 = n$) measures how many aspects of the $\deg_B$-values in $\Omega_{m-1}(M;\varphi)$ are redundant. For example, if $m=2$, then $\ker\mu_M \cong \pi_1^{\mathrm{S}} \cong \mathbb{Z}_2$ and $\deg_B$ assumes only 'every other' possible value in $\Omega_{m-1}(M;\varphi)$. For further low dimensions of M the cardinalities

$$|\ker\mu_M| = |\pi_{m-1}^{\mathrm{S}}/(d-1)\pi_{m-1}^{\mathrm{S}}||\ker((d-1)\colon \pi_{m-2}^{\mathrm{S}}\circlearrowleft)|$$

(see (3.6)) may depend on the sign of $d=\pm1$ and are listed in Table 1 (based on Toda's tables in [**16**, Chapter XIV]).

Finally, observe that not only is the injectivity condition (ii) satisfied whenever $m \geqslant n = 2$, so is Theorem 1.4 (i). Indeed, using the group structure in $\mathcal{F}$ we see that

$$\omega_B(f_1,f_2) = \deg_B(f_1-f_2+a\circ f_0) \tag{3.8}$$

for all $[f_1],[f_2]\in\mathcal{F}$ (see [**12**, Theorem 3.1 (i)]). In particular, if $\omega_B(f_1,f_2)=0$, then $[f_2]=[f_1+a\circ f_0]=[a\circ f_1]$ and the pair (f_1,f_2) is loose.

For the remainder of this paper we assume that $n \geqslant 3$. All sections of p_N (see Figure 1) are then fibrewise homotopic and $[f_0] = [a \circ f_0]$ in $\mathcal{F}$ (see (1.5)).

Therefore, given $[f], [f_i], [\bar{f}_i] \in \mathcal{F}$, $i = 1, 2$, we conclude that

$$\deg_B([f]) = \omega_B(f, f_0) \tag{3.9}$$

(see (1.7)). Moreover,

$$\omega_B(f_1 + \bar{f}_1, f_2 + \bar{f}_2) = \omega_B(f_1, f_2) + \omega_B(\bar{f}_1, \bar{f}_2); \tag{3.10}$$

this is independent of how we represent $[f_i] = \operatorname{forg}([f_i''])$ and $[\bar{f}_i] = \operatorname{forg}([\bar{f}_i''])$ by elements $[f_i''], [\bar{f}_i''] \in \mathcal{F}''$, $i = 1, 2$, in order to perform the addition in $\mathcal{F}$ (see (2.8)). Indeed, use a small deformation of the zero section s_{oN} in N to a nearby disjoint section s and deform $\bar{f}_i''$ accordingly until $\bar{f}_i'' \circ s_{oM} = \bar{f}_i'' \circ (a \circ s_{oM}) = s$, $i = 1, 2$. The coincidences of the resulting (slightly modified) sum with the map $f_1 + f_2$ lie disjointly in the two open half-spheres of M described by $\varepsilon x_1 > 0$ (and diffeomorphic to the locus $\varepsilon x_1 > -1$), $\varepsilon = \pm 1$ (cf. (1.3), (1.4)). Each of these half-spheres contributes one of the summands in (3.10).

In particular, we see that

$$\begin{aligned} \deg_B([f_1] + [f_2]) &= \deg_B([f_1]) + \deg_B([f_2]) \\ &= \omega_B(f_1 + f_2, f_0 + f_0); \end{aligned} \tag{3.11}$$

this does not depend on the choices involved in forming the sums in $\mathcal{F}$ (see (2.8)).

Furthermore, since a pair (f_1, f_2) is loose if and only if $[f_2] = [a \circ f_1]$ (see the discussion preceding (1.3)), it follows that

$$\omega_B(a \circ f, f_0) + \omega_B(f_0, f) = \omega_B(a \circ f, f) = 0.$$

Therefore,

$$\omega_B(f_0, f) = -\deg_B([a \circ f]), \quad [f] \in \mathcal{F}. \tag{3.12}$$

Moreover,

$$\omega_B(f_1, f_2) = \deg_B([f_1]) - \deg_B([a \circ f_2]) \tag{3.13}$$

for all $[f_1], [f_2] \in \mathcal{F}$ whenever $n \geqslant 3$; just note that both sides of the last equation agree with

$$\omega_B(f_1 + f_0, f_0 + f_2) = \omega_B(f_1, f_0) + \omega_B(f_0, f_2).$$

Equation (3.13) now yields a proof of Theorem 1.4 for the case $n \neq 2$. Indeed, if $\deg_B$ is injective and $\omega_B(f_1, f_2) = 0$, then $[f_1] = [a \circ f_2]$ and the pair (f_1, f_2) is loose. Conversely, if the vanishing of $\omega_B(\cdot, \cdot)$ always implies looseness and if the $\deg_B([f_1]) = \deg_B([f_2])$, then $\omega_B(f_1, a \circ f_2) = 0$ and $[f_1] = [f_2]$.

Many of our results are summarized in the discussion of (1.10). The horizontal sequences in this diagram have been established in (2.1) and (3.3) ($\mathscr{S}$, r and π^{S}_{2-n} being trivial for $n \geqslant 3$), Propositions 2.1, 2.4 and Theorem 2.5.

The right-hand square in (1.10) commutes (at least up to a $\pm$ sign), since

$$\pitchfork (\deg_B([f])) = \omega(f|F_M, a \circ f_0|F_M)$$

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is the (stabilized) framed bordism class of the inverse image of the point $a(e_1) \in F_N = S^{n-1}$; this corresponds to $\pm E_1{}^{\infty}(q[f])$ via the Pontryagin–Thom isomorphism.

Given $[f] \in \mathcal{F}$ and $[u] \in \pi_m(S^{n-1})$, a generic coincidence manifold of the pair $(u * f, a \circ f_0)$ consists of the framed manifold $u^{-1}(\{a(*)\})$, mapped into a small ball $\mathring{B} \subset M$, and the coincidence manifold of $(f, a \circ f_0)$. This implies (1.11).

4. Nielsen theory

In this section we complete the proof of Theorem 1.4 by showing that its conditions (i) and (ii) are equivalent.

First, we recall the definition of the Nielsen number $N_B(f_1, f_2)$ according to the (new) convention adopted in [**13**, **14**]. Given a generic pair (f_1, f_2) of fibrewise maps from M to N, the Nielsen equivalence relation among coincidence points (see [**3**, Definition 4.1]) yields a decomposition of the coincidence manifold $C(f_1, f_2)$ (see (1.1)) and results in a decomposition

$$\omega_B(f_1, f_2) = \sum_A \omega_B(f_1, f_2)_A \in \Omega_{m-n+1}(M, \tilde{\varphi})$$

of the ω_B-invariant as a sum where the summands are labelled by the elements A of a certain 'geometric Reidemeister set' $\pi_0(E_B(f_1, f_2))$ (see [**3**, § 3]). In this paper we follow [**13**, **14**] and *define* $N_B(f_1, f_2)$ *to be the number of non-trivial summands* in *this* sum decomposition (and not, as in [**3**], in the analogous decomposition for $\tilde{\omega}_B(f_1, f_2)$).

Consider the case $n = 2$ (see Examples 2.2 and 3.1). Given any homotopy classes $[f_1], [f_2] \in \mathcal{F}$, we claim that $\mathrm{MCC}_B(f_1, f_2) = N_B(f_1, f_2)$. If $m = n = 2$, this follows from [**12**, Theorem 1.1], since M and N are torus bundles; the precise values of $\mathrm{MCC}_B = N_B$ have been calculated in [**3**] and [**12**]. If $m > n = 2$, we can still add and subtract f_1, f_2 in a fibrewise fashion. Then, the minimum numbers of the pairs (f_1, f_2) and $(f_1 - f_2 + a \circ f_0, f_2 - f_2 + a \circ f_0)$ $(\sim_B (s_k \circ p_M, a \circ f_0)$ for some $k \in \mathbb{Z}$, where $k \in \{0, 1\}$ if $d_N = -1$ (see Example 2.2, (2.6))) agree, and so do the Nielsen numbers. But the coincidence manifold $C(s_k \circ p_M, a \circ f_0)$ is empty (if $k = 0$) or else consists of the (connected) fibres $\{(2j-1)/2k\} \times S^{m-1}$, $j = 1, \dots, k$, which are pairwise Nielsen inequivalent (see [**3**, Definition 4.1] and [**12**]). Thus,

$$\mathrm{MCC}_B(s_k \circ p_M, a \circ f_0) = N_B(s_k \circ p_M, a \circ f_0) = |k|.$$

(For the case of fixed points in S^1-fibrations over an arbitrary base, see [**2**, Theorems 3.6 and 3.7].)

In the remaining case, $n \geqslant 3$, the fibre F_N is simply connected. Then, by definition, $N_B(f_1, f_2)$ equals 0 or 1 according to whether $\omega_B(f_1, f_2)$ vanishes or not (see [**3**, (4.2)]). Therefore, the following result completes the proof of Theorem 1.4.

Proposition 4.1. *If* $n \geqslant 3$, *then* $\mathrm{MCC}_B(f_1, f_2) \leqslant 1$ *for all* $[f_1], [f_2] \in \mathcal{F}$.

Proof. Choose classes $[f_1''], [f_2''] \in \mathcal{F}''$ such that

$$\mathrm{forg}([f_1'']) = [a \circ f_1], \quad \mathrm{forg}([f_2'']) = [f_2] \in \mathcal{F},$$

and consider the pair

$$(a \circ (f_2'' + (-f_2'') + f_1''), f_2'' + f_0 + f_0),$$

which represents $([f_1], [f_2])$ (see (1.5) and the discussion following (2.7)). Clearly, the first summands $a \circ f_2''$, f_2'' do not contribute to the coincidence set. We construct a fibrewise homotopy of the second summand $f := a \circ ((-f_2'') + f_1'')$, which keeps f unchanged at $s_{oM}(S^1)$ (where it is attached to $a \circ f_2''$).

In the end, the coincidence set

$$C(f, f_0) = C(a \circ (f_2'' + (-f_2'') + f_1''), f_2'' + f_0 + f_0)$$

will be made path connected.

After a small deformation, f is generic and the 'zero point' $* \in F_N$ (see (1.4)) is a regular value of $f|F_M \colon S^{m-1} \to S^{n-1}$.

Then,

$$C := C(f, f_0) = f^{-1}(s_{oN}(S^1))$$

is a closed, $(n-1)$-codimensional submanifold of M, equipped with a twisted framing, i.e. with a description of its normal bundle via the vector bundle isomorphism

$$\nu(C, M) \cong (f * (TF(p_N)))|C \cong p_M * ((\otimes^d \lambda) \oplus \underline{\mathbb{R}}^{n-2})|C$$

induced by the tangent map of f (see Figure 1 and (3.2); $TF(p_N)$ and $\underline{\mathbb{R}}^{n-2}$ denote the tangent bundle along the fibres of p_N and the trivial bundle, respectively).

Locally, over each subinterval J of S^1, we can identify $p_M^{-1}(J)$ with $J \times S^{m-1}$ and $p_N^{-1}(J)$ with $J \times S^{n-1}$ via diffeomorphisms (see Figure 1). We may assume that f takes the product form

$$\mathrm{id} \times (f^{\mathrm{v}}|F_M) \colon J \times S^{m-1} \to J \times S^{n-1}$$

over a small neighbourhood J of $[1] = [0] \in S^1 = I/(1 \sim 0)$ (see Figure 1). Now, $C \cap F_M = (f^{\mathrm{v}}|F_M)^{-1}(\{*\}) \subset F_M - \{*\}$ is a *framed* submanifold of codimension $n-1 \geqslant 2$. We can make it path connected via an embedded framed surgery (very much similar to, but somewhat simpler than, the procedure pictured in Figure 2). This corresponds, via the Pontryagin–Thom construction, to a deformation of $f^{\mathrm{v}}|F_M$, which we use to modify f near the fibre F_M over $[1] = [0] \in S^1 = I/(0 \sim 1)$.

Similarly, over $J = (0, 1) \subset S^1$, f takes the form

$$(p_M, f^{\mathrm{v}}) \colon J \times S^{m-1} \to J \times S^{n-1}.$$

Again, we can make the framed submanifold

$$C \cap p_M^{-1}(J) = (f^{\mathrm{v}})^{-1}(\{*\}) \subset M - s_{oM}(S^1)$$

path connected by an embedded surgery. This yields the required fibrewise homotopy of f. $\square$

Table 2. *A few non-stable dimension combinations (m, n), where $\deg_B$ cannot be injective when $d_N = 1$.*

	$m-n+1 =$	1	3	5	7	8	4,5,10,13
if $d_M = +1$:	$n =$	3	5	4 or 5	9	3,7,8,9,10	all
if $d_M = -1$:	$n =$	—	5	5	9	8,9,10	all

5. Conclusions and applications

In this section we discuss a few examples where the full obstruction condition, the Wecken property and the injectivity condition in Theorem 1.4 can be expressed entirely in terms of iterated Freudenthal suspensions.

If $d_N = 1$ (i.e. $N \cong S^1 \times S^{n-1}$ (see (1.8), (1.9) and (1.4))), then $\delta_{m-1} - \mathrm{id} = (d_M - 1)\,\mathrm{id}$. Thus, in view of (1.10) an obvious necessary condition for $\deg_B$ to be injective is that the suspension homomorphism

$$E_2^{\infty} = E^{\infty} \colon \pi_m(S^{n-1}) \to \pi^{\mathrm{S}}_{m-n+1} \tag{5.1}$$

($E_2^{\infty} \approx E^{\infty} \otimes \mathrm{id} \colon \pi_m(S^{n-1}) \otimes \mathbb{Z}_2 \to \pi^{\mathrm{S}}_{m-n+1} \otimes \mathbb{Z}_2$, respectively) is injective when $d_M = d_N = 1$ (when $(d_M, d_N) = (-1, 1)$, respectively). Often this possibility can be excluded by comparing the sizes of these homotopy groups if $d_M = 1$ (or by just counting their cyclic direct summands of even or infinite order if $d_N = -1$, respectively). An inspection of Toda's tables (see [**16**, Chapter XIV]) already yields strong injectivity restrictions. For a few examples in the non-stable dimension range $m \geqslant 2n - 3$, see Table 2.

If $d_N = -1$ (i.e. N is not orientable), then [**17**, Theorem XI, 8.5] (together with the obvious identity $((-\iota_{n-1}) + \iota_{n-1}) \circ \alpha = 0$) implies that

$$d_M \delta(\alpha) = (-\iota_{n-1}) \circ \alpha = -\alpha - \sum_{j=0}^{\infty} \omega_{j+3}(\iota_{n-1}, \iota_{n-1}) \circ h_j(\alpha) \tag{5.2}$$

for all $\alpha \in \pi_m(S^{n-1})$. Here the connection term to the right-hand side involves (iterated) basic Whitehead products of the generator $\iota_{n-1} = [\mathrm{id}] \in \pi_{n-1}(S^{n-1})$ as well as the Hopf–Hilton invariants of α.

This can make computations rather complicated. Note, however, that

$$d_M \delta_m([\iota_{n-1}, \iota_{n-1}]) = [d_N \iota_{n-1}, d_N \iota_{n-1}] = [\iota_{n-1}, \iota_{n-1}], \tag{5.3}$$

where d_N is equal to $+1$ or -1.

The full claim of Theorem 1.10 is proved by standard diagram chasing in (1.10). When E_2^{∞} is onto, we even have a (partial) inverse: $\deg_B$ is injective if and only if E_1^{∞} and E_2^{∞} are injective.

In the stable dimension range $2 \leqslant m \leqslant 2n - 4$, both E_1^{∞} and E_2^{∞} are bijective and $\deg_B$ is an isomorphism of groups.

So consider the first non-stable dimension setting $m = 2n - 3 \geqslant 2$. Here, E_1^{∞} is an isomorphism and E_2^{∞} is onto. Therefore (and since π^{∞}_{n-2} is finite), the following conditions

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are equivalent:

(i) $\deg_B$ is injective,

(ii) $\deg_B$ is a group isomorphism,

(iii) E_2^∞ is injective,

(iv) the cokernels of $(\delta_{2n-3} - \mathrm{id})\colon \pi_{2n-3}(S^{n-1})\circlearrowleft$ and of $(d-1)\cdot\colon \pi_{n-2}(\mathrm{S})\circlearrowleft$ have the same (finite) cardinality.

Proof of Theorem 1.13. We assume first that $n \geqslant 3$ is odd. Then,

$$\pi_{2n-3}(S^{n-1}) = \mathbb{Z} \oplus \text{torsion}. \tag{5.4}$$

If $n \neq 3, 5$ or 9, the $\mathbb{Z}$-summand is generated by the Whitehead product $[\iota_{n-1}, \iota_{n-1}]$, which is killed by suspensions (see, for example, [**17**, X, 8.20]). On the other hand, $[\iota_{n-1}, \iota_{n-1}] = \pm\delta([\iota_{n-1}, \iota_{n-1}])$ (see (5.3)) cannot lie in

$$(\delta_{2n-3} - \mathrm{id})(\pi_{2n-3}(S^{n-1})) \subset 2\mathbb{Z} \oplus \text{torsion}$$

and, hence, gives rise to a non-trivial element in the kernel of E_2^∞.

We consider the remaining case where $n = 3, 5$ or 9. An element H with Hopf invariant 1 generates the direct summand $\mathbb{Z}$ in (5.3). Since the Hopf invariant takes the value 2 on $[\iota_{n-1}, \iota_{n-1}]$, we conclude from (5.3) that $\delta(H) - d_M H$ is a torsion element. Therefore, *we* may *assume* that $d_M = -1$ because otherwise $\mathrm{coker}(\delta - \mathrm{id})$ fails to be finite and E_2^∞ cannot be injective.

If $n = 3$, then the cokernels of $\delta_3 - \mathrm{id} = -2\,\mathrm{id}\colon (\pi_3(S^2) \cong \mathbb{Z})\circlearrowleft$ and of $(d-1)\cdot \equiv 0\colon (\pi_1^{\mathrm{S}} \cong \mathbb{Z}_2)\circlearrowleft$ each have two elements, and E_2^∞ is injective whether d_N equals $+1$ or -1.

If $n = 5$ or 9, then $\pi_{2n-3}(S^{n-1})$ is the direct sum of $\mathbb{Z}$ (generated by H) and $\mathbb{Z}_q$ (with some generator v), where $q = 12$ (or 120, respectively) when $n = 5$ (or 9, respectively). According to Table 2, $\deg_B$ can be injective only if $d_M = d_N = -1$. In this case, it follows from [**4**, Propositions 4.3 and 4.7] that, for all $r, s \in \mathbb{Z}$,

$$(\delta - \mathrm{id})(rH + sv) = -2rH + rv.$$

This implies that the cardinalities of $\mathrm{coker}(\delta - \mathrm{id})$ and of π_{n-2}^{S} are both equal to $2q$. It remains only to prove the claim in Theorem 1.13 (ii). Thus, assume that $n > 2$ is even. If $d_N = -1$, then, for all $\alpha \in \pi_{2n-3}(S^{n-1})$,

$$\delta_{2n-3}(\alpha) = d(\alpha + [\iota_{n-1}, \iota_{n-1}] \circ h_0(\alpha)) = d\alpha$$

(see (5.2)); indeed, according to [**17**, Theorem XI, 8.6],

$$2h_0\colon \pi_{2n-3}(S^{n-1}) \to \pi_{2n-3}(S^{2n-3}) \cong \mathbb{Z}$$

vanishes, and, hence, so does h_0. Therefore, whether d_N equals $+1$ or -1, we see that $\delta_{2n-3} - \mathrm{id} = (d-1)\,\mathrm{id}$. It is now easy to check criterion (iv) above by again inspecting Toda's tables (see also (5.1)). This completes the proof of Theorem 1.13. $\square$

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