REVIEWS

Numerical references occurring within the text of a review, or parenthetically in connection with items listed by title, are to previous reviews in this JOURNAL or to *A bibliography of symbolic logic* (this JOURNAL, vol. 1, pp. 121–218). Thus "Bertrand Russell (111*16*)" will refer to the entry so numbered in the *Bibliography*; "II 172" will refer to a review beginning on page 172 vol. 2 of this JOURNAL (or to the publication reviewed); "II 172(1)," "II 172(2)," and "II 172(3)" will refer respectively to the first, second, and third reviews beginning on page 172 of vol. 2 (or to the publication there reviewed).

HANS REICHENBACH. *Les fondements logiques du calcul des probabilités.* ***Annales de l'Institut Henri Poincaré,*** vol. 7 (1937), pp. 267–348.

In this paper, the author gives a survey of his theory of the foundations of probability and induction, which he has developed in detail in his ***Wahrscheinlichkeitslehre*** (439*4*) and supplemented recently in a book ***Experience and prediction.***

Reichenbach distinguishes two conceptions of the theory of probability, namely (1) as a purely formal mathematical theory, and (2) as an interpreted system which is applied in empirical science.

(1) The formal theory of probability admits—like geometry as a mathematical theory—of two different representations, namely, (1a) as an axiomatized system which contains, in Reichenbach's theory, just one primitive concept, that of probability-implication (of some degree p) between two classes A and B, and (1b) as a part of pure mathematics. The latter representation is based on defining the degree of probability of B with respect to A as the limit of the relative frequency of the B's among the A's in an infinite sequence of elements. Reichenbach does not, like von Mises and some other authors, impose any restricting condition of "irregularity" upon the sequences which are admitted in this context. By this interpretation, the formal theory of probability becomes a part of the mathematical theory of convergent infinite sequences; it is shown by Reichenbach to be a mathematical model of the axiom system mentioned before.

In both representations of Reichenbach's general theory, those theorems of the usual theory of probability whose validity presupposes a special kind of irregularity of the underlying sequences—such as the Bernoullian theorem—naturally possess only a conditional validity; Reichenbach sketches the outlines of a special theory of order which makes it possible to distinguish different kinds of irregularity in a sequence and to examine the probability properties connected with them. (These investigations are closely related to those of A. H. Copeland.)

(2) The theory of probability as an interpreted system is arrived at by relating probability as considered in (1b) to sequences of empirical events, such as throws of a die.

After a discussion of the difficulties connected with the question of the convergence of empirical sequences, Reichenbach extends the application of the concept of probability, which is originally restricted to classes (or properties) of events, to single events. The probability of a single event is defined as the probability of the narrowest class to which the event belongs and for which a reliable statistic is available. This probability is also called the *weight* of the sentence asserting the event in question.

The concept of weight (whose definition is yet in a rather unsatisfactory state) plays a very important rôle in Reichenbach's theory; in fact, Reichenbach claims that it represents a necessary generalization of the customary dichotomy "true-false," and that the logical structure of empirical science has to be represented by a many-valued logic rather than by the usual two-valued one. Reichenbach actually sets up truth-tables of a "probability logic" in which are admitted as truth-values all possible values of a weight, i.e. all real numbers r with $0 \leqq r \leqq 1$. The truth-tables are determined in accordance with the elementary theorems of the theory of probability; as a consequence, they differ from the truth-tables of other many-valued systems in that the truth-value of a connection—e.g. of a conjunction—appears as a function of the truth-values of the two connected members and of a third argument, the "degree of coupling" or the relative probability of the second member with respect to the first. It is, however, at least doubtful whether Reichenbach thus really establishes a language which is governed by a many-valued logic; for he seemingly wants to maintain in his language the rules of inference and the valid formulas of the sentential calculus. Thus it seems to be more adequate to say that by the establishment of the concept of weight Reichenbach introduces an important new semantical (or, perhaps, syntactical) concept, which, however, does not generalize the concept "true," but the concept "verified," and therefore does not conflict with the assumption of a two-valued logic.

Finally, Reichenbach expounds his theory of induction, which cannot be outlined here.

CARL G. HEMPEL

ANDRZEJ MOSTOWSKI. *O niezależności definicji skończoności w systemie logiki* (On the independence of the definitions of finiteness in a system of logic). ***Dodatek do Rocznika Polskiego Towarzystwa Matematycznego*** (supplement to the ***Annales de la Société Polonaise de Mathématique***), vol. 11 (1938), pp. 1–54.

This paper, which is the author's doctor's thesis, contains a succession of very valuable and interesting results from the domain of metalogic. As subject of his research the author has chosen the formalized system of ***Principia mathematica***, based on a simplified theory of types, and enlarged by adding the axiom of infinity; more precisely, he considers the system T_∞ outlined by the reviewer in 285*13*. However, all the results obtained are—according to the author—applicable also to other kindred formal systems, in particular to the formalized system of Zermelo (cf., e.g., Skolem 247*8* and Quine II 51.)

Mostowski's principal topic is the question of relationships of inference between certain definitions of the notion of a finite class. He shows the impossibility of proving the equivalence of different known definitions of finiteness without using Zermelo's axiom of choice. This concerns primarily the three following definitions (in which "reflexive" and "inductive" are used in the sense of ***Principia mathematica***):

(1) *A class is finite if and only if it is inductive.*

(2) *A class is finite if and only if it is not reflexive.*

(3) *A class is finite if and only if the class of all its subclasses is not reflexive.*

The author proves that no one of the sentences expressing the equivalence of two of these definitions is provable on the basis of the system T_∞ (provided, of course, this system is consistent). Since, however, the equivalence of these definitions can be proved after enlargement of the system T_∞ by addition of the axiom of choice, the author obtains as a side result *an exact proof of the independence of the axiom of choice from the axioms of the system* T_∞. This result can be considerably strengthened; it appears that even such a sentence as,

(4) *Every non-inductive class is a sum of two mutually exclusive non-inductive classes,* which is a very weak consequence of the axiom of choice, is not derivable from the axioms of the system T_∞.

The proof consists in applying with due care the classical method of (so called) proof by interpretation. The necessity of care is due to: (1) the fact that here the primitive concepts of logic itself and not as usual the primitive terms of an axiomatic system based on logic are interpreted; (2) the circumstance that the system dealt with is based on an infinite number of axioms. The most essential point of the proof is apparently the proper interpretation of the universal and existential quantifiers (i.e. of such expressions as "for every x" and "for certain x"), and therefore the author—following the reviewer—calls the method of proof used the *method of relativization of quantifiers*. The conception of this method comes from the reviewer, who applied it to different methodological problems in 285*19* and in a joint paper with Lindenbaum (I 115).

The results concerning the definitions (1)–(3) answer the questions put in the appendix to the reviewer's 285*5*; the problem of the independence of the sentence (4) is due to Chwistek (220*15*). These results, as well as the theorem concerning the independence of the axiom

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