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NECESSARY AND SUFFICIENT CONDITIONS FOR THE EQUALITY OF $L(f)$ AND l^1

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Introduction. Let f be a modulus, $e_i = (\delta_{ij})$ and $E = \{e_i, i = 1, 2, \ldots\}$. The $L(f)$ spaces were created (to the best of our knowledge) by W. Ruckle in [**2**] in order to construct an example to answer a question of A. Wilansky. It turned out that these spaces are interesting spaces. For example l^p, $0 < p \leqq 1$ is an $L(f)$ space with $f(x) = x^p$, and every FK space contains an $L(f)$ space [**2**]. A natural question is: For which f is $L(f)$ a locally convex space? It is known that $L(f) \subseteq l^1$, for all f modulus (see [**2**]), and l^1 is the smallest locally convex FK space in which E is bounded (see [**1**]). Thus the question becomes: For which f does $L(f)$ equal l^1? In this paper we characterize such f. (An FK space need not be locally convex here.) We also characterize those f for which $L(f)$ contains a convex ball. The final result of this paper is to show that if f satisfies $f(x \cdot y) \leqq f(x) \cdot f(y)$ and $L(f) \neq l^1$ then $L(f)$ contains no infinite dimensional subspace isomorphic to a Banach space.

Throughout f will be a modulus and

$$B_a = \{X \in L(f) \colon |X|_f \leqq a\}.$$

LEMMA. *If for some $a > 0$, $B_{f(a)}$ is convex then for any finite collection of positive real numbers $\{c_1, \ldots, c_n\}$ with $\sum c_i = 1$ we have $f(a) = \sum f(c_i a)$.*

Proof. Let $X_m = ae_m$, $m = 1, 2, \ldots, n$ then $X_m \in B_{f(a)}$, for all m and $X = \sum c_i X_i$ is in $B_{f(a)}$, since $B_{f(a)}$ is convex. So $|X|_f \leqq f(a)$. But

$$|X|_f = \sum f(c_i a) \leqq f(a).$$

On the other hand

$$f(a) = f(\textstyle\sum c_i a) \leqq \sum f(c_i a),$$

so

$$f(a) = \sum f(c_i a).$$

THEOREM 1. *For f a modulus, $L(f) = l^1$ if and only if there exist two positive numbers r and ϵ such that $f(x) \leqq rx$ for all x in $[0, \epsilon)$.*

Proof. Assume that for every positive real number r and for every positive real number ϵ, there exists an x in $(0, \epsilon]$ such that $f(x) > rx$. So

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for every positive integer n there exists x_n in $(0, 1/n^2]$ such that $f(x_n) > nx_n$. Since f is continuous [**2**], we have for every n there exists an interval $I_n \subseteq (0, 1/n^2)$ such that $f(x) > nx$ for all $x \in I_n$. For each n choose a finite number of points $x_{n1}, x_{n2}, \ldots, x_{nt(n)}$ in I_n such that

$$1/n^2 \leqq \sum_{k=1}^{t(n)} x_{nk} \leqq 2/n^2.$$

This can be done because, for all x in I_n, $x_n \leqq 1/n^2$ so pick any point x_{n1} in I_n, then choose $x_{n2}, \ldots, x_{nt(n)}$ such that

$$\sum_{k=1}^{t(n)-1} x_{nk} \leqq 1/n^2 \quad \text{and} \quad \sum_{k=1}^{t(n)} x_{nk} \geqq 1/n^2.$$

Let

$$X = (x_{11}, x_{12}, \ldots, x_{1t(1)}, x_{22}, \ldots, x_{2t(2)}, \ldots)$$

then

$$|x|_f = \sum_{n=1}^{\infty} \sum_{k=1}^{t(n)} f(x_{nk}) \geqq \sum_{n=1}^{\infty} \sum_{k=1}^{t(n)} nx_{nk} = \sum_{n=1}^{\infty} n. \sum_{k=1}^{t(n)} x_{nk} \geqq \sum_{1}^{\infty} n. \frac{1}{n^2} = \sum \frac{1}{n}$$

so $X \notin L(f)$, while

$$||X|| = \sum_{n=1}^{\infty} \sum_{k=1}^{t(n)} x_{nk} \leqq \sum_{1}^{\infty} \frac{1}{n^2},$$

so $X \in l^1$ and $L(f) \neq l^1$.

Conversely, suppose $f(x) \leqq rx$ in $(0, \epsilon]$ for some positive real numbers r and ϵ, so $l^1 \subseteq L(f)$, but $L(f) \subseteq l^1$ for all f modulus (see [**2**]). So $L(f) = l^1$ as sets, and this with the theorem in [**4**, page 203] imply their equality as topological spaces.

THEOREM 2. *For f modulus, the following are equivalent:*

(1) *$B_{f(a)}$ is convex for some $a > 0$;*

(2) *there exists a positive real number a such that*

$$f(x) = \frac{f(a)}{a} x,$$

for all x in $[0, a]$;

(3) *there exists a positive real number b such that $B_{f(r)}$ is convex for all $r \leqq b$.*

Proof. (1) $\Rightarrow$ (2): Let n be any positive integer. By the lemma we have

$$f(a) = nf(a/n).$$

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Let m be a positive integer $m < n$; then

$$f(a) = f\left(\frac{m}{n}a + \frac{n-m}{n}a\right) = f\left(\frac{m}{n}a + \frac{1}{n}a + \frac{1}{n}a + \ldots + \frac{1}{n}a\right),$$

$(n - m)$ times

$$= f\left(\frac{m}{n}a\right) + (n-m)f\left(\frac{1}{n}a\right)$$

by the lemma. So

$$f(a) = f\left(\frac{m}{n}a\right) + \frac{n-m}{n}f(a).$$

Hence

$$\frac{m}{n}f(a) = f\left(\frac{m}{n}a\right).$$

So for any rational number $r < 1$ we have $f(ra) = rf(a)$. By the continuity of f we have

$$f(xa) = af(a) \text{ for any } x \in [0, 1].$$

Now for any $y \in [0, a]$, $y/a \leqq 1$. So $f(y) = y/a\, f(a)$.

(2) $\Rightarrow$ (3):

$$f(x) = \frac{f(a)}{a}x \quad \text{for all } x \in [0, a],$$

so $l^1 = L(f)$ and for any $r \leqq a$

$$\begin{aligned} B_r &= \{X \in L(f): |X|_f \leqq r\} \\ &= \{X \in L(f): \|X\|_1 = |X|_f/\alpha \leqq r/\alpha\}, \alpha = f(a)/a \\ &= \{X \in l^1: \|X\|_1 \leqq r/\alpha\}. \end{aligned}$$

So B_r is a convex set for all $r \leqq a$.

(3) $\Rightarrow$ (1) is trivial.

Remark. The equality of $L(f)$ and l^1 does not guarantee the existence of convex balls in $L(f)$. Take for example $f(x) = x/(1 + x)$. f is a modulus. Since $f(x) < 2f(x/2)$ for all x, no ball is convex. And it is clear that $L(f) = l^1$.

The final theorem is a generalization (in the method of the proof and the conclusion) of the one given by Stiles [**3**], for the l^p spaces $0 < p < 1$. In the proof we will use his terminology.

THEOREM 3. *If $L(f) \neq l^1$ and f satisfy $f(xy) \leqq f(x)f(y)$ then $L(f)$ contains no infinite-dimensional subspace isomorphic to a Banach space.*

Proof. First we will show that if B is a closed infinite dimensional subspace of $L(f)$, then B contains a subspace isomorphic to $L(f)$.

Now if B is infinite dimensional, it contains a sequence $\{b_n\}$ such that $|b_n|_f = 1$ where b_n is of the form

$$b_n = (0, \dots, 0, b^n_{k_n}, b^n_{k_n+1}, 0, \dots)$$

where k_n is chosen arbitrarily large. Select b_n such that

$$\sum_{k=k_n+1}^{\infty} f|b_{k_n}{}^n| < 1/2^{n+1}.$$

Let

$$C_n = (0, \dots, 0, b^n_{k_n}, \dots, b^n_{k_{n+1}-1}, 0, \dots), \; n = 1, 2, \dots$$

$\{C_n\}$ is the basic sequence equivalent to $\{e_n\}$ in $L(f)$ for

$$\left|\sum_{n=1}^{\infty} \lambda_n C_n\right|_f = \sum_{n=1}^{\infty} \sum_{k_n}^{k_{n+1}-1} f|b_k{}^n \lambda_n| \geqq \sum_{n=1}^{\infty} f\left(\sum_{k_n}^{k_{n+1}-1} |\lambda_n b_k{}^n|\right)$$

$$= \sum_{n=1}^{\infty} f\left(\sum_{k_n}^{\infty} |\lambda_n b_k{}^n| - \sum_{k_{n+1}}^{\infty} |\lambda_n b_k{}^n|\right) = \sum_{n=1}^{\infty} f\left(|\lambda_n|\left(\sum_{k_n}^{\infty} |b_k{}^n| - \sum_{k_{n+1}}^{\infty} |b_k{}^n|\right)\right)$$

$$\geqq \sum_{n=1}^{\infty} f\left(|\lambda_n|\left(1 - \frac{1}{2^{n+1}}\right)\right) \geqq \sum_{n=1}^{\infty} f(\tfrac{1}{2}|\lambda_n|) \geqq \frac{1}{2}\sum_{n=1}^{\infty} f|\lambda_n| \dots (*)$$

On the other hand

$$\left|\sum_{n=1}^{\infty} \lambda_n C_n\right|_f = \sum_{n=1}^{\infty} \sum_{k_n}^{k_{n+1}-1} f|\lambda_n b_k{}^n| \leqq \sum_{n=1}^{\infty} f|\lambda_n| \sum_{k_n}^{k_{n+1}-1} f|b_k{}^n|$$

$$\leqq \sum_{n=1}^{\infty} f|\lambda_n| \cdot |b_n|_f \leqq |\lambda|_f.$$

We also have $\{C_n\}$ equivalent to $\{b_n\}$, for if $\sum \lambda_n b_n$ converges then $\sum \lambda_n C_n$ converges from the definition of $\{C_n\}$. On the other hand,

$$\left|\sum_{n=1}^{m} \lambda_n (b_n - C_n)\right|_f = \left|\sum_{n=1}^{m} \lambda_n (0, \dots, 0, b^n_{k_{n+1}}, \dots)\right| \leqq \sum_{n=1}^{m} \sum_{k_{n+1}}^{\infty} f|\lambda_n b_k{}^n|$$

$$\leqq \sum_{n=1}^{m} f|\lambda_n| \cdot \sum_{k_{n+1}}^{\infty} f|b_k{}^n| \leqq \sum_{n=1}^{m} f|\lambda_n| \cdot \frac{1}{2^{n+1}} \leqq \frac{1}{2}\sum_{n=1}^{m} f|\lambda_n| \leqq \left|\sum_{n=1}^{m} \lambda_n C_n\right|,$$

the last inequality coming from $(*)$. So $\{b_n\}$ is a basis for a subspace of B which is isomorphic to $L(f)$.

Now if $L(f)$ contains an infinite dimensional subspace isomorphic to a Banach space S then by the above result $L(f)$ is isomorphic to a subspace of S. But $L(f) \neq l^1$ so by Theorem 1, $L(f)$ contains no convex neighbourhood, which is a contradiction.

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