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THE ZEROTH $\mathbb{P}^1$-STABLE HOMOTOPY SHEAF OF A MOTIVIC SPACE

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Abstract We establish a kind of 'degree 0 Freudenthal $\mathbb{G}_m$-suspension theorem' in motivic homotopy theory. From this we deduce results about the conservativity of the $\mathbb{P}^1$-stabilization functor.

In order to establish these results, we show how to compute certain pullbacks in the cohomology of a strictly homotopy-invariant sheaf in terms of the Rost–Schmid complex. This establishes the main conjecture of [2], which easily implies the aforementioned results.

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1. Introduction

After recalling some preliminaries in §2, this article has two main sections of very different flavors. In §3 we establish a technical result about Rost–Schmid complexes of strictly homotopy-invariant sheaves. Then in §4 we draw applications to the stabilization problem in motivic homotopy theory. We now describe these two main sections in reverse order, and then sketch their relation. For more background and motivation, the reader may wish to consult the introduction of [2].

1.1. $\mathbb{P}^1$-stabilization in motivic homotopy theory

Motivic homotopy theory is the universal homotopy theory of smooth algebraic varieties, say over a field k. It is built by freely adjoining homotopy colimits to the category of smooth k-varieties, and then enforcing Nisnevich descent and making $\mathbb{A}^1$ contractible [19]. Write $\mathcal{S}\mathrm{pc}(k)_*$ for the pointed version of this theory.[1] This is a symmetric monoidal category (the monoidal operation being given by the smash product), and every pointed smooth variety defines an object in it. Given a pointed motivic space $\mathcal{X} \in \mathcal{S}\mathrm{pc}(k)_*$, the classical *homotopy groups* upgrade to *homotopy sheaves*[2] $\underline{\pi}_i(\mathcal{X})$.

[1] We think of this as an ∞-category, but no information will be lost for the purposes of this introduction by just considering its homotopy 1-category.

[2] That is, Nisnevich sheaves on the site of smooth k-varieties.

The Riemann sphere $\mathbb{P}^1 := (\mathbb{P}^1,1) \in \mathcal{S}\mathrm{pc}(k)_*$ plays a similar role to the ordinary sphere in classical topology. *Stable* motivic homotopy theory is concerned with the category obtained by making $\Sigma_{\mathbb{P}^1} := \wedge\mathbb{P}^1$ into an equivalence. It is this context in which algebraic cycles and motivic cohomology naturally appear. We can take a more pedestrian approach. The functor $\Sigma_{\mathbb{P}^1}$ has a right adjoint $\Omega_{\mathbb{P}^1}$, and there is a directed diagram of endofunctors of $\mathcal{S}\mathrm{pc}(k)_*$:

$$\mathrm{id} \to \Omega_{\mathbb{P}^1}\Sigma_{\mathbb{P}^1} =: Q_1 \to \Omega^2_{\mathbb{P}^1}\Sigma^2_{\mathbb{P}^1} =: Q_2 \to \cdots \to \Omega^n_{\mathbb{P}^1}\Sigma^n_{\mathbb{P}^1} =: Q_n \to \cdots.$$

Denote by Q its homotopy colimit. Then $Q\mathcal{X}$ is the $\mathbb{P}^1$*-stabilization of* $\mathcal{X}$, and the homotopy sheaves of $Q\mathcal{X}$ are called the $\mathbb{P}^1$-stable homotopy sheaves of $\mathcal{X}$.

A simple form of our main application of our technical result is as follows. It is reminiscent of the fact that for an ordinary space X, the sequence of sets $\{\pi_0\Omega^i\Sigma^i X\}_{i\geq 0}$ is given by $\pi_0 X$, $F\pi_0 X$, $\mathbb{Z}(\pi_0 X)$, $\mathbb{Z}(\pi_0 X)$, ..., where for a pointed set A, FA denotes the free group on A (with identity given by the base point) and $\mathbb{Z}(A)$ denotes the free abelian group on A (with 0 given by the base point).

Theorem 1.1. *Let k be a perfect field and $n \geq 3$ (if char$(k) = 0$, $n = 2$ is also allowed). Then for $\mathcal{X} \in \mathcal{S}\mathrm{pc}(k)_*$, the canonical map*

$$\underline{\pi}_0 Q_n\mathcal{X} \to \underline{\pi}_0 Q\mathcal{X}$$

is an isomorphism.

Proof. This is an immediate consequence of Corollary 4.9 and, for example, Morel's Hurewicz theorem [18, Theorem 6.37]. □

Example 1.2. Morel's computations [18, Corollary 6.43] imply that for $\mathcal{X} = S^0$, already $\underline{\pi}_0 Q_2 S^0 \simeq \underline{GW} \simeq \underline{\pi}_0 QS^0$. Our result shows that this stabilization is not special to S^0, except that our results are not strong enough to establish stabilization at Q_2, only at Q_3. See also Remark 4.3.

We also obtain some conservativity results; here is a simple form. It is similar to the fact that stabilization is conservative on simply connected topological spaces. Write $\mathcal{S}\mathrm{pc}(k)_*(n) \subset \mathcal{S}\mathrm{pc}(k)_*$ for the subcategory generated under homotopy colimits by objects of the form $X_+ \wedge \mathbb{G}_m^{\wedge n}$, with $X \in \mathrm{Sm}_k$ (and $\mathbb{G}_m := (\mathbb{A}^1 \setminus 0, 1)$, $X_+ := X \coprod *$). Denote by $\mathcal{S}\mathrm{pc}(k)_{*,\geq 1} \subset \mathcal{S}\mathrm{pc}(k)_*$ the subcategory of $\mathbb{A}^1$-simply connected spaces.

Theorem 1.3. *Let k be perfect, and put $n = 1$ if char$(k) = 0$ and $n = 3$ if char$(k) > 0$. Then the stabilization functor*

$$Q : \mathcal{S}\mathrm{pc}(k)_{*,\geq 1} \cap \mathcal{S}\mathrm{pc}(k)_*(n) \to \mathcal{S}\mathrm{pc}(k)_*$$

is conservative (i.e., detects equivalences).

In particular, $\Sigma_{\mathbb{P}^1}$ and all of its iterates, and also $\Sigma^\infty_{\mathbb{P}^1}$, are conservative on the same subcategory.

Proof. This is an immediate consequence of Corollary 4.15 and, for example, [24, Corollary 2.23]. □

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The results in §4 are stronger than this sample; in fact, they are stated in terms of the stabilization functor from S^1-spectra to $\mathbb{P}^1$-spectra. The reader is encouraged to skip to this section directly. Our main results in the form of Corollary 4.9, Theorem 4.14, and Corollary 4.15 can be understood without reading the rest of the article (except perhaps for taking a glance at §4.1, where some notation is introduced).

1.2. Pullbacks and the Rost–Schmid complex

The results just sketched are obtained by combining the main results of [2] with a technical result that we describe now. Essentially, this establishes [2, Conjecture 6.10] (for $n \geq 3$); all our applications are a consequence of this and were already anticipated in writing [2].

Let M be a strictly homotopy-invariant sheaf (see §2 for this and related notions, and a more complete account of the following sketch) and X a smooth variety. Morel has proved [18, Corollary 5.43] that there is a very convenient complex, known as the *Rost–Schmid complex* $C^*(X,M)$, which can be used to compute the Nisnevich cohomology $H^*(X,M)$. This complex has the special property that $C^n(X,M)$ depends only on the n-fold contraction M_{-n}, and similarly so does the boundary map $C^n(X,M) \to C^{n+1}(X,M)$. Let $Z \subset X$ have codimension $\geq d$. An obvious modification $C^*_Z(X,M)$ of $C^*(X,M)$ can be used to compute $H^*_Z(X,M)$; by construction one has $C^n_Z(X,M) = 0$ for $n < d$. It follows that the group $H^d_Z(X,M)$ depends only on M_{-d} (in fact this holds for all groups $H^*_Z(X,M)$, but we are most interested in the lowest one). Now let $f : Y \to X$ be a morphism of smooth varieties with $f^{-1}(Z)$ also of codimension $\geq d$ on Y. Then the pullback map

$$f^* : H^d_Z(X,M) \to H^d_{f^{-1}(Z)}(Y,M) \tag{1}$$

is a morphism of abelian groups, both of which depend only on M_{-d}.

It is not difficult to show (using the results of [2]; see the proof of Theorem 4.6 in this article for details) that [2, Conjecture 6.10] is equivalent to the statement that the morphism (1) also depends only on M_{-d}, in an appropriate sense.[3] The main result of this article (Theorem 3.1) states that this is true.

We establish this by adapting an argument of Levine, using a variant of Gabber's presentation lemma to set up an induction on d. (The case $d = 0$ holds tautologically.)

1.3. From pullbacks to stabilization

This article brings to conclusion a program started in [2]. There we developed the following strategy for establishing stabilization results such as Theorem 1.1. First we note that S^1-stabilization is well understood and behaves largely as in topology; thus it suffices to prove the analogous result for motivic S^1-spectra. (For detailed definitions of this and the following notions, see §4.1.) Write $\mathcal{SH}^{S^1}(k)(d) \subset \mathcal{SH}^{S^1}(k)$ for the localizing

[3] In fact, our original plan for [2] was to establish [2, Conjecture 6.10] (and hence the results in §4) by proving that f^* depends only on M_{-d}. This turned out to be more difficult than we had anticipated.

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subcategory generated by d-fold $\mathbb{G}_m$-suspensions, and similarly $\mathcal{SH}(k)^{\mathrm{eff}}(d) \subset \mathcal{SH}(k)$ for the localizing subcategory generated by the image of $\mathcal{SH}^{S^1}(k)(d)$. These categories afford t-structures induced by the canonical generating sets, and hence the stabilization functor $\mathcal{SH}^{S^1}(k)(d) \to \mathcal{SH}(k)^{\mathrm{eff}}(d)$ is right-t-exact. One finds that in order to prove stabilization results, it will be enough to show that the induced functor on hearts $\mathcal{SH}^{S^1}(k)(d)^{\heartsuit} \to \mathcal{SH}(k)^{\mathrm{eff}}(d)^{\heartsuit}$ is an equivalence. Since the right-hand category is by now well understood, let us focus on the left-hand side. It is not difficult to show that the functor of d-fold $\mathbb{G}_m$-loops $\mathcal{SH}^{S^1}(k)(d)^{\heartsuit} \to \mathcal{SH}^{S^1}(k)^{\heartsuit} \simeq \mathbf{HI}(k)$ is monadic. In other words, we may think of objects of $\mathcal{SH}^{S^1}(k)(d)^{\heartsuit}$ as strictly homotopy-invariant sheaves with extra structure. One way of phrasing the main result of [2] is (see, e.g., [2, Remark 4.17]) that this extra structure is precisely the data of closed pullbacks on cohomology with support in codimension d. These are precisely the kinds of maps that we show depend only on M_{-d} 'in an appropriate sense'. To be more specific, the appropriate sense is that M_{-d} is a so-called sheaf with $\mathbb{A}^1$-transfers (see Remark 3.2 for details), and the pullback depends only on this additional structure.

All of this more or less implies[4] that $\mathcal{SH}^{S^1}(k)(d)^{\heartsuit}$ is equivalent to the full subcategory of the category of sheaves with $\mathbb{A}^1$-transfers on objects of the form M_{-d}. It follows from [2, Theorem 5.19] that for d big enough, this subcategory is equivalent to $\mathcal{SH}(k)^{\mathrm{eff}}(d)^{\heartsuit}$, as desired.

1.4. Notation and conventions

We fix throughout a field k. All nontrivial results will require k to be perfect.

Given a presheaf M on the category of smooth varieties over k, and an essentially smooth k-scheme X, we denote by $M(X)$ the evaluation at X of the canonical extension of M to pro-(smooth schemes), into which the category of essentially smooth schemes embeds by [8, Proposition 8.13.5]. In other words, if $X = \lim_i X_i$ is a cofiltered limit of smooth k-schemes with affine transition maps, then $M(X) = \operatorname{colim}_i M(X_i)$ (and this is known to be independent of the presentation of X).

Given a scheme X and a point $x \in X$, we identify x and $\operatorname{Spec}(k(x))$. In particular, if X is smooth and k is perfect (so that x is essentially smooth), then we write $M(x)$ for what is often denoted $M(k(x))$.

Given a scheme X and $d \geq 0$, we write $X^{(d)}$ for the set of points of X of codimension d on X; in other words, if $x \in X$, then $x \in X^{(d)}$ if and only if $\dim X_x = d$, where X_x denotes the localization of X in x. For example, $X^{(0)}$ is the set of generic points of X.

For a regular immersion $Y \hookrightarrow X$, we denote by $N_{Y/X}$ the normal bundle and by $\omega_{Y/X} = \det N_{Y/X}^{\vee}$ the determinant of the conormal bundle. More generally, for any morphism $Y \to X$ such that the cotangent complex $L_{Y/X}$ is perfect, we write $\omega_{Y/X} = \det L_{Y/X}$.

[4] Making these arguments precise requires some further effort; for this reason we follow a slightly different strategy in §4.2.

2. Preliminaries

We recall some well-known results from motivic homotopy theory.

2.1. Strictly homotopy-invariant sheaves

We write Sm_k for the category of smooth k-schemes. We make it into a site by endowing it with the Nisnevich topology [20]. This is the only topology we shall use; all cohomology will be with respect to it. Unless noted otherwise, by a (pre)sheaf we mean a (pre)sheaf of abelian groups on Sm_k.

Recall that a sheaf M is called *strictly homotopy invariant* if, for all $X \in \mathrm{Sm}_k$, the canonical map $H^*(X,M) \to H^*\left(\mathbb{A}^1 \times X, M\right)$ is an isomorphism. We denote the category of strictly homotopy-invariant sheaves by $\mathbf{HI}(k)$.

Example 2.1. For a commutative ring A, denote by $GW(A)$ its Grothendieck–Witt ring – that is, the additive group completion of the semiring of isometry classes of nondegenerate, symmetric bilinear forms on A [14]. Write $\underline{GW}$ for the associated Nisnevich sheaf on Sm_k. Then $\underline{GW}$ turns out to be strictly homotopy invariant (combine [21, Theorem A] and [18, §§2,3]).

Remark 2.2. As mentioned in the introduction, there exists a universal homotopy theory built out of (pointed) smooth varieties by enforcing $\mathbb{A}^1$-homotopy invariance and Nisnevich descent [19]; we denote it by $\mathcal{S}\mathrm{pc}(k)_*$. By construction, for $M \in \mathbf{HI}(k)$, the Eilenberg–MacLane spaces $K(M,i)$ define objects in $\mathcal{S}\mathrm{pc}(k)_*$. In this way, results about $\mathcal{S}\mathrm{pc}(k)_*$ translate into properties of the cohomology of strictly homotopy-invariant sheaves. For example, given $X \in \mathrm{Sm}_k$ and $Z \subset X$ closed, we have an isomorphism

$$H^i_Z(X,M) \simeq [X/X \setminus Z, K(M,i)]_{\mathcal{S}\mathrm{pc}(k)_*},$$

and given a morphism $f : Y \to X$ (resp., a closed subset $Z' \subset Z$) we have morphisms $Y/Y \setminus f^{-1}(Z) \to X/X \setminus Z$ (resp., $X/X \setminus Z \to X/X \setminus Z'$) inducing the pullback $H^i_Z(X,M) \to H^i_{f^{-1}(Z)}(Y,M)$ (resp., the extension of the support map $H^i_{Z'}(X,M) \to H^i_Z(X,M)$). We will use this correspondence freely in the sequel.

2.1.1. Unramifiedness. Let $X \in \mathrm{Sm}_k$ be connected and $\emptyset \neq U \subset X$ be open. Then for $M \in \mathbf{HI}(k)$, the canonical map $M(X) \to M(U)$ is an injection [17, Lemma 6.4.4]. It follows that if $\xi \in X$ is the generic point, then $M(X) \hookrightarrow M(\xi)$.

2.1.2. Contractions. For a presheaf M, write M_{-1} for the presheaf given by $M_{-1}(X) = \ker\left(M\left(\left(\mathbb{A}^1 \setminus 0\right) \times X\right) \xrightarrow{i_1^*} M(X)\right)$ and M_{-n} for the n-fold iteration of this construction. Here $i_1 : X \to \left(\mathbb{A}^1 \setminus 0\right) \times X$ denotes the inclusion at $1 \in \mathbb{A}^1$. Pullback along the structure map splits i_1^*, and hence M_{-1} is a summand of the internal mapping object $\underline{\mathrm{Hom}}\left(\mathbb{A}^1 \setminus 0, M\right)$. It follows that M_{-n} is a ((strictly) homotopy-invariant) (pre)sheaf if M is.

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Example 2.3. We have $H^1\left(\mathbb{P}^1_K, M\right) \simeq M_{-1}(K)$, for any finitely generated separable field extension K/k. Indeed, we can cover $\mathbb{P}^1_K$ by two copies of $\mathbb{A}^1_K$ with intersection $\left(\mathbb{A}^1 \setminus 0\right)_K$ and $H^i\left(\mathbb{A}^1_K, M\right) \simeq H^i(\mathrm{Spec}(K), M)$ (whence in particular $H^1\left(\mathbb{A}^1_K, M\right) = 0$), so the claim follows from the Mayer–Vietoris sequence for this covering.

2.1.3. ***GW*-module structure.** Set $X \in \mathrm{Sm}_k$ and $u \in \mathcal{O}(X)^\times$. Multiplication by u defines an endomorphism of $\left(\mathbb{A}^1 \setminus 0\right) \times X$ and hence of $\underline{\mathrm{Hom}}\left(\mathbb{A}^1 \setminus 0, M\right)(X)$; passing to the summand, we obtain $\langle u \rangle : M_{-1}(X) \to M_{-1}(X)$. Suppose that $M \in \mathbf{HI}(k)$. Since the map $\mathbb{Z}[\mathcal{O}^\times] \to \underline{GW}, u \mapsto \langle u \rangle$, is surjective on fields, unramifiedness implies that this construction extends in at most one way to a $\underline{GW}$-module structure on M_{-1}. It turns out that this $\underline{GW}$-module structure always exists [18, Lemma 3.49].

2.1.4. Twisting. Given a line bundle $\mathcal{L}$ on $X \in \mathrm{Sm}_k$, write $\mathcal{L}^\times$ for the sheaf of nonvanishing sections. For $M \in \mathbf{HI}(k)$ and $d > 0$, we put $M_{-d}(X, \mathcal{L}) = H^0\left(X, M_{-d} \times_{\mathcal{O}^\times} \mathcal{L}^\times\right)$; here the action of $\mathcal{O}^\times$ on M_{-d} is via $\mathcal{O}^\times \to \underline{GW}$, and the action on $\mathcal{L}^\times$ is given by multiplication. Note that since $\langle u \rangle = \left\langle u^{-1} \right\rangle$, we have $M_{-d}(X, \mathcal{L}) \simeq M_{-d}\left(X, \mathcal{L}^{-1}\right)$.

2.1.5. Thom spaces. For $X \in \mathrm{Sm}_k$ and V a vector bundle on X of rank d, we have $Th(V) := V/V \setminus 0_X \in \mathcal{S}\mathrm{pc}(k)_*$. For $M \in \mathbf{HI}(k)$, there are canonical isomorphisms [18, Lemma 5.35]

$$[V/V \setminus 0_X, K(M,d)]_{\mathcal{S}\mathrm{pc}(k)_*} \simeq H^d_{0_X}(V, M) \simeq M_{-d}(X, \det V).$$

2.1.6. Homotopy purity. Set $X \in \mathrm{Sm}_k$ and $U \subset X$ open, with reduced closed complement $Z = X \setminus U$ also smooth. Then in $\mathcal{S}\mathrm{pc}(k)_*$ there is a canonical equivalence [19, §3 Theorem 2.23]

$$X/X \setminus Z \simeq Th\left(N_{Z/X}\right).$$

2.1.7. Boundary maps. Set $X \in \mathrm{Sm}_k$ and $x \in X^{(d)}$. Then X_x is an essentially smooth scheme with closed point x. Homotopy purity supplies us with the collapse sequence[5]

$$X_x \to X_x/X_x \setminus x \simeq Th\left(N_{x/X}\right) \xrightarrow{\partial} \Sigma(X_x \setminus x).$$

Pullback along ∂ induces the boundary map in the long exact sequence of cohomology with support. We most commonly use the case where $d = 1$. Then $X_x \setminus x = \xi$, where ξ is the generic point of X (specializing to x), and the boundary map takes the familiar form

$$\partial : M(\xi) \to M_{-1}\left(x, \omega_{x/X}\right).$$

2.1.8. Monogeneic transfers. Let k be perfect and K/k be a finitely generated field extension, whence $X = \mathrm{Spec}(K)$ is an essentially smooth scheme. Let $K(x)/K$ be a finite, monogeneic field extension. We are supplied with an embedding $X' = \mathrm{Spec}(K(x)) \xhookrightarrow{x} \mathbb{A}^1_X \subset \mathbb{P}^1_X$, and thus homotopy purity provides us with a collapse map

[5] Here and in the sequel we view essentially smooth schemes as defining pro-objects in $\mathcal{S}\mathrm{pc}(k)$.

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$$\mathbb{P}^1_X \to \mathbb{P}^1_X/\mathbb{P}^1_X \setminus X' \simeq Th\left(N_{X'/\mathbb{A}^1_X}\right) \simeq Th(\mathcal{O}_{X'});$$

here the normal bundle is canonically trivialized by the minimal polynomial of x. Pullback along this collapse map induces the *monogeneic transfer*[6] [18, p.99]

$$\tau_x : M_{-1}(K(x)) \to M_{-1}(K).$$

Slightly more generally, suppose that $z \in \mathbb{P}^1_K$ is any closed point. Then we have the transfer map

$$\mathrm{tr}_z : M_{-1}\left(z, \omega_{z/\mathbb{P}^1_K}\right) \simeq H^1_z\left(\mathbb{P}^1_K, M\right) \to H^1\left(\mathbb{P}^1_K, M\right) \simeq M_{-1}(K).$$

This contains no new information: if $z \in \mathbb{A}^1_K$, then tr_z coincides up to isomorphism with τ_z, and the only other case is $z = \infty$, which is a rational point, so tr_z is isomorphic to the identity.

2.2. Cousin and Rost–Schmid complexes

Let M be a sheaf of abelian groups on X. The cohomology of M on X can be computed using the *coniveau spectral sequence* (see, e.g., [3, §1]). On the zero line of the E_1 page one finds the so-called *Cousin complex* [3, (1.3)]:

$$M(X) \to \bigoplus_{x \in X^{(0)}} M(x) \to \bigoplus_{x \in X^{(1)}} H^1_x(X, M) \to \cdots \to \bigoplus_{x \in X^{(d)}} H^d_x(X, M) =: C^d(X, M) \to \cdots. \tag{2}$$

Here

$$H^d_x(X, M) = \operatorname*{colim}_{V \ni x} H^d_{\overline{\{x\}} \cap V}(V, X), \tag{3}$$

where the colimit runs over open neighborhoods of x. The boundary maps in formula (2) are induced by certain boundary maps in long exact sequences of cohomology with support.

Now suppose that $M \in \mathbf{HI}(k)$. By the Bloch–Ogus–Gabber theorem [3, Theorem 6.2.1], the Cousin complex (2) is then *exact* when viewed as a complex of sheaves (i.e., for X local). Since it consists of flasque sheafes, it can thus be used to compute the Zariski cohomology of M. The terms also turn out to be Nisnevich-acyclic (see [3, Theorem 8.3.1] or [18, Lemma 5.42]), and hence the Cousin complex computes the Nisnevich cohomology of M as well (which thus turns out to coincide with the Zariski cohomology).

Remark 2.4. The Cousin complex can also be used to compute cohomology with support in a closed subscheme Z; just replace

$$C^d(X, M) = \bigoplus_{x \in X^{(d)}} H^d_x(X, M) \quad \text{by} \quad C^d_Z(X, M) = \bigoplus_{x \in X^{(d)} \cap Z} H^d_x(X, M).$$

This holds, since the resolving sheaves are flasque.

[6] Morel calls this the *geometric transfer*.

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Now let k be perfect. We would like to make this complex more explicit. As a first step, homotopy purity (see §§2.1.5 and 2.1.6) allows us to identify the groups (3) in the Cousin complex more explicitly as

$$H_x^d(X,M) \simeq M_{-d}\left(x,\omega_{x/X}\right).$$

Indeed, by generic smoothness, shrinking V if necessary, we may assume that $Z := \overline{\{x\}} \cap V$ is smooth of codimension d on V, and then

$$H_Z^d(V,M) \simeq [V/V \setminus Z, K(d,M)] \simeq \left[Th\left(N_{Z/V}\right), K(d,M)\right] \simeq M_{-d}\left(Z,\omega_{Z/V}\right);$$

the claim now follows by taking colimits. The boundary maps of the Cousin complex can also be identified. We only use the following weak form of this result:

Theorem 2.5. (Morel [18]). *Set $M \in \mathbf{HI}(k)$, $X \in \mathrm{Sm}_k$, and $d \geq 0$. Then the boundary map $C^d(X,M) \to C^{d+1}(X,M)$ in the Cousin complex depends only on the sheaf M_{-d}, together with (if $d > 0$) its structure as a $\underline{GW}$-module from §2.1.3.*

In fact, Morel proves this result by identifying the Cousin complex with another complex called the *Rost–Schmid* complex (which has the same terms but a priori different boundary maps). In other words, the boundary map in the Cousin complex admits an explicit formula, involving only the codimension 1 boundary of §2.1.7 and (composites of) the monogeneic transfer of §2.1.8.[7]

In the sequel, we will not distinguish between the Cousin and Rost–Schmid complexes.

3. A 'formula' for closed pullback

In this section we establish our main result.

Theorem 3.1. *Let k be a perfect field and set $M \in \mathbf{HI}(k)$, $f: Y \to X \in \mathrm{Sm}_k$, $d \geq 1$, and $Z \subset X$ closed of codimension $\geq d$ such that $f^{-1}(Z) \subset Y$ is also of codimension $\geq d$. Then the map*

$$f^*: H_Z^d(X,M) \to H_{f^{-1}(Z)}^d(Y,M)$$

depends only on $M_{-d} \in \mathbf{HI}(k)$, together with its GW-module structure and transfers along monogeneic field extensions (in the sense of §§2.1.3 and 2.1.8).

In the sequel, we shall say 'depends only on M_{-d}' to mean what is asserted in the theorem – that is, 'depends only on M_{-d} as a GW-module with transfers'.

Remark 3.2. Let us make precise the notion that 'f^* depends only on M_{-d}'.

For this, first recall from [2, §5.1] the notion of a presheaf with $\mathbb{A}^1$-transfers. This is just a presheaf F on Sm_k together with, for every finitely generated field K/k, a $GW(K)$-module structure on $F(K)$, and for every finite monogeneic extension $K(x)/K$, a transfer

[7] The monogeneic transfer on M_{-d} is extra structure, determined by the presentation of M_{-d} as a contraction. But the boundary $C^d \to C^{d+1}$ involves the monogeneic transfer only on M_{-d-1}, and so indeed depends only on the sheaf M_{-d}.

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$\tau_x : F(K(x)) \to F(K)$. A morphism of presheaves with $\mathbb{A}^1$-transfers is a morphisms of sheaves which commutes with the $GW(K)$-module structures and the transfers. If $M \in \mathbf{HI}(k)$ and $d \geq 1$, then M_{-d} acquires the structure of a presheaf with $\mathbb{A}^1$-transfers (see §§2.1.3 and 2.1.8 or [2, Example 5.2]).

Now suppose we are given $M, N \in \mathbf{HI}(k)$, and an isomorphism $M_{-d} \simeq N_{-d}$ of presheaves with $\mathbb{A}^1$-transfers. Then there is a canonical induced isomorphism $H^d_Z(X,M) \simeq H^d_Z(X,N)$ (and similarly for Y), by identifying the Rost–Schmid resolutions with support in Z (i.e., using Remark 2.4 and Theorem 2.5; this does not even depend on the identification of the transfers). The theorem asserts that this isomorphism is compatible with the pullback f^* (and for this we crucially need the compatibility of the transfers).

Remark 3.3. Note that the Rost–Schmid complex is functorial in smooth morphisms in an obvious way, so that the theorem is clear, for example, for f an open immersion. We will often use this in conjunction with the observation (which follows, e.g., from the form of the Rost–Schmid complex) that if Z has codimension $\geq d$ on X, then

$$H^d_Z(X,M) \hookrightarrow \bigoplus_z H^d_z(X_z, M),$$

where the sum is over the (finitely many) generic points of Z of codimension d on X.

If the support is smooth and the intersection is transverse, all is well.

Lemma 3.4. *Suppose that both Z and $f^{-1}(Z)$ (with its induced scheme structure as a pullback) are smooth. Then $f^* : H^d_Z(X,M) \to H^d_{f^{-1}(Z)}(Y,M)$ depends only on M_{-d}.*

Proof. Since Z is smooth (and so is X), we may write $Z = Z_0 \coprod Z_1$, where all components of Z_0 have codimension precisely d on X and all components of Z_1 have codimension $> d$. Consider the commutative diagram

$$\begin{array}{ccc} H^d_Z(X,M) & \xrightarrow{f^*} & H^d_{f^{-1}(Z)}(Y,M) \\ \uparrow & & \uparrow \\ H^d_{Z_0}(X,M) & \xrightarrow{f^*} & H^d_{f^{-1}(Z_0)}(Y,M). \end{array}$$

Here the vertical maps are extension of support, and hence depend only on M_{-d}. Moreover, by construction the left-hand vertical map is an isomorphism. We may thus replace Z by Z_0 – that is, assume that all components of Z have codimension precisely d on X.

Recall from Remark 2.2 that the pullback $f^* : H^d_Z(X,M) \to H^d_{f^{-1}(Z)}(Y,M)$ is induced by pullback along the map $Y/Y \setminus f^{-1}(Z) \to X/X \setminus Z \in \mathcal{S}\mathrm{pc}(k)_*$ on $K(d,M)$. Let η be a generic point of $f^{-1}(Z)$ (necessarily of codimension d on Y). Shrinking X around $f(\eta)$ using Remark 3.3, we may assume that the normal bundle $N_{Z/X}$ is trivial. Since $f : \big(Y, f^{-1}(Z)\big) \to (X,Z)$ is a morphism of smooth closed pairs [9, §3.5], the map $Y/Y \setminus f^{-1}(Z) \to X/X \setminus Z$ is equivalent to $Th(g)$, where $g : N_{f^{-1}(Z)/Y} \to N_{Z/X}$ is the map induced by f [9, Theorem 3.23]. Our assumptions on codimension imply that $f^*N_{Z/X} \simeq$

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$N_{f^{-1}(Z)/Y}$, whence $g \simeq f|_{f^{-1}(Z)} \times \mathrm{id}_{\mathbb{A}^d}$. Pullback along $Th(g) \simeq \left(f|_{f^{-1}(Z)}\right)_+ \wedge T^d$ thus depends only on M_{-d}, and the result follows. □

Recall that for (sets, say, and hence presheaves of sets) $A \subset X, B \subset Y$, we have a canonical isomorphism

$$X/A \wedge Y/B \simeq X \times Y/(A \times Y \cup X \times B). \tag{4}$$

Construction 3.5. Let $Z \subset X \times \mathbb{P}^1$ be closed with image Z' in X. Applying the isomorphism (4) with $Y = \mathbb{P}^1$, $A = X \setminus Z'$, $B = \emptyset$, we obtain the following equivalence:

$$X \times \mathbb{P}^1/\left(X \times \mathbb{P}^1 \setminus Z' \times \mathbb{P}^1\right) \simeq (X/X \setminus Z') \wedge \mathbb{P}^1_+.$$

Together with the stable splitting $\mathbb{P}^1_+ \simeq \mathbb{1} \vee \mathbb{P}^1$ and extension of support, this induces a map

$$\mathrm{tr}_Z : H^d_Z\left(X \times \mathbb{P}^1, M\right) \to H^d_{Z' \times \mathbb{P}^1}\left(X \times \mathbb{P}^1, M\right) \to H^{d-1}_{Z'}(X, M_{-1}).$$

Remark 3.6. This construction is clearly functorial in X.

Lemma 3.7. *In the foregoing notation, suppose that $Z \subset X \times \mathbb{P}^1$ has codimension $\geq d$ (so that $Z' \subset X$ has codimension $\geq d-1$). Then tr_Z only depends on M_{-d}.*

Proof. The transfer is given by pullback along the collapse map $\mathbb{P}^1_X/\mathbb{P}^1_X \setminus \mathbb{P}^1_{Z'} \to \mathbb{P}^1_X/\mathbb{P}^1_X \setminus Z$. Remarks 3.6 and 3.3 imply that the problem is local on X around generic points of Z' of codimension $d-1$; we may thus assume that Z' is smooth [22, Tag 0B8X] and $N_{Z'/X}$ is trivial. Lemma 3.9 identifies the transfer with the collapse map

$$N_{\mathbb{P}^1_{Z'}/\mathbb{P}^1_X}/N_{\mathbb{P}^1_{Z'}/\mathbb{P}^1_X} \setminus \mathbb{P}^1_{Z'} \to N_{\mathbb{P}^1_{Z'}/\mathbb{P}^1_X}/N_{\mathbb{P}^1_{Z'}/\mathbb{P}^1_X} \setminus Z.$$

By assumption, $N_{\mathbb{P}^1_{Z'}/\mathbb{P}^1_X}$ is trivial of rank $d-1$, so this map identifies with

$$\mathbb{A}^{d-1}_{\mathbb{P}^1_{Z'}}/\mathbb{A}^{d-1}_{\mathbb{P}^1_{Z'}} \setminus \mathbb{P}^1_{Z'} \to \mathbb{A}^{d-1}_{\mathbb{P}^1_{Z'}}/\mathbb{A}^{d-1}_{\mathbb{P}^1_{Z'}} \setminus Z.$$

Applying isomorphism (4) with $X = \mathbb{A}^{d-1}$, $Y = \mathbb{P}^1_{Z'}$, $A = \mathbb{A}^{d-1} \setminus 0$, and respectively $B = \emptyset$ or $B = \mathbb{P}^1_{Z'} \setminus Z$, this identifies with

$$\mathbb{A}^{d-1}/\mathbb{A}^{d-1} \setminus 0 \wedge \mathbb{P}^1_{Z'+} \xrightarrow{\mathrm{id} \wedge t} \mathbb{A}^{d-1}/\mathbb{A}^{d-1} \setminus 0 \wedge \mathbb{P}^1_{Z'}/\mathbb{P}^1_{Z'} \setminus Z.$$

$\Big($Use the fact that $\left(\mathbb{A}^{d-1} \setminus 0\right) \times \mathbb{P}^1_{Z'} = \mathbb{A}^{d-1}_{\mathbb{P}^1_{Z'}} \setminus \mathbb{P}^1_{Z'}$ and $\left(\mathbb{A}^{d-1} \setminus 0\right) \times \mathbb{P}^1_{Z'} \cup \mathbb{A}^{d-1} \times \left(\mathbb{P}^1_{Z'} \setminus Z\right) = \mathbb{A}^{d-1}_{\mathbb{P}^1_{Z'}} \setminus Z\Big)$. Pullback along t is the monogeneic transfer for Z/Z', essentially by definition (see §2.1.8). The result follows. □

Remark 3.8. The foregoing proof shows that on the level of the Rost–Schmid complex, the map tr_Z is given as follows. For $z \in Z$ of codimension e in $X \times \mathbb{P}^1$ and with image z' of codimension $e-1$ in X, the map is given in components by

$$\begin{aligned} &C^e\left(X \times \mathbb{P}^1, M\right) \supset M_{-e}\left(z, \omega_{z/X \times \mathbb{P}^1}\right) \\ &\simeq M_{-e}\left(z, \omega_{z/\mathbb{P}^1_{z'}} \otimes \omega_{\mathbb{P}^1_{z'}/\mathbb{P}^1_X}\right) \xrightarrow{\mathrm{tr}} M_{-e}\left(z', \omega_{z'/X}\right) \subset C^{e-1}(X, M_{-1}). \end{aligned}$$

Here tr is the monogeneic transfer coming from the embedding $z \in \mathbb{P}^1_{z'}$.

We have used the following form of the homotopy purity equivalence:

Lemma 3.9. *Let $Z \subset Y \subset X$ be closed immersions with X,Y smooth. Then the collapse map*

$$X/(X \setminus Y) \to X/(X \setminus Z)$$

is canonically homotopic to the collapse map

$$N_{Y/X}/N_{Y/X} \setminus Y \to N_{Y/X}/N_{Y/X} \setminus Z.$$

Proof. Write $X' = X \setminus Z$ and $Y' = Y \setminus Z$. We can write the collapse map as

$$\frac{X}{X \setminus Y} \to \frac{X/X \setminus Y}{X'/X' \setminus Y'}.$$

Since $(X',Y') \to (X,Y)$ is a morphism of smooth closed pairs, it is compatible with purity equivalences [9, after proof of Theorem 3.23], and so the collapse map identifies with

$$\frac{N_{Y/X}}{N_{Y/X} \setminus Y} \to \frac{N_{Y/X}/\left(N_{Y/X} \setminus Y\right)}{N_{Y'/X'}/\left(N_{Y'/X'} \setminus Y'\right)} \simeq \frac{N_{Y/X}}{\left(N_{Y/X} \setminus Y\right) \cup N_{Y'/X'}} = \frac{N_{Y/X}}{N_{Y/X} \setminus Z};$$

see also [9, top of p. 24]. This is the desired result. □

Lemma 3.10. *Let X be (essentially) smooth, $i: Y \hookrightarrow X$ closed of codimension 1 with Y essentially smooth, and $Z \subset X \times \mathbb{P}^1$ of codimension $\geq d$ such that $W := \left(Y \times \mathbb{P}^1\right) \cap Z$ also has codimension $\geq d$ on $Y \times \mathbb{P}^1$. Write Z',W' for the images of Z and W in X,Y, respectively. Let $\eta_1,\ldots,\eta_r$ be the generic points of W of codimension d. Suppose further that $Z \to Z'$ is quasi-finite and $W \to W'$ is birational at η_1. Then*

$$i^*: H^d_Z\left(X \times \mathbb{P}^1, M\right) \to H^d_W\left(Y \times \mathbb{P}^1, M\right)$$

depends only on M_{-d}, the map

$$i^*: H^{d-1}_{Z'}(X, M_{-1}) \to H^{d-1}_{W'}(Y, M_{-1}),$$

and the maps

$$i^*_{\eta_j}: H^d_Z\left(\left(X \times \mathbb{P}^1\right)_{\eta_j}, M\right) \to H^d_W\left(\left(Y \times \mathbb{P}^1\right)_{\eta_j}, M\right),$$

for $j > 1$.

In particular, the map i^* does *not* depend on i^*_1.

Proof. By Remarks 3.6 and 3.8, we have a commutative diagram

$$\begin{tikzcd}
H^d_Z\left(X \times \mathbb{P}^1, M\right) \arrow[r, "i^*"] \arrow[d, "\mathrm{tr}_Z"] & H^d_W\left(Y \times \mathbb{P}^1, M\right) \arrow[r, hook] \arrow[d, "\mathrm{tr}_W"] & \bigoplus_i C^d_{\eta_i}\left(Y \times \mathbb{P}^1, M\right) \arrow[d, "\mathrm{tr}_W"] \\
H^{d-1}_{Z'}(X, M_{-1}) \arrow[r, "i^*"] & H^{d-1}_{W'}(Y, M_{-1}) \arrow[r, hook] & \bigoplus_{\eta \in W' \cap Y^{(d-1)}} C^{d-1}_{\eta}(Y, M_{-1}).
\end{tikzcd}$$

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By Lemma 3.7, the vertical maps depend only on M_{-d}, and it follows from Remark 3.8 and our assumption that $W \to W'$ is birational at η_1 that the right-hand vertical map is injective on the component corresponding to η_1. Set $a \in H^d_Z\left(X \times \mathbb{P}^1, M\right)$. Write $i^*(a) = b_1 + \cdots + b_r$, where $b_i \in C^d_{\eta_i}\left(Y \times \mathbb{P}^1, M\right)$. For $j > 1$, we know $i^*_{\eta_j}$, hence we know b_j and thus we know $\mathrm{tr}_W(b_j)$. Since we know the bottom horizontal map, we know $i^*\mathrm{tr}_Z(a) = \mathrm{tr}_W(i^*(a))$. Consequently, we know $\mathrm{tr}_W(b_1) = \mathrm{tr}_W(i^*a) - \sum_{j>1} \mathrm{tr}_W(b_j)$, and hence b_1. This concludes the proof. □

Example 3.11. If $d = 1$, then the map $i^* : H^{d-1}_{Z'}(X, M_{-1}) \subset M_{-1}(X) \to H^{d-1}_{W'}(Y, M_{-1}) \subset M_{-1}(Y)$ clearly depends only on M_{-1}, as desired.

Example 3.12. If Z is smooth and transverse to Y at η_j for $j > 1$, then $i^*_{\eta_j}$ depends only on M_{-d} by Lemma 3.4, as desired.

The following is the key reduction. It is an adaptation of [11, Lemma 7.2].

Lemma 3.13. *Let X, Y be (essentially) smooth, $i : Y \hookrightarrow X$ closed of codimension 1, and $Z \subset X$ of codimension $\geq d$ such that $W = Y \cap Z$ is of codimension $\geq d$ in Y. Then*

$$i^* : H^d_Z(X, M) \to H^d_W(Y, M)$$

depends only on M_{-d}.

Proof. By a continuity argument, we may assume that X is smooth over k.

Using Remark 3.3, we may shrink X around a generic point of W. We may thus assume that W is smooth over k and connected. Pullback along the smooth map $X \times \mathbb{A}^1 \to X$ yields an understood isomorphism $H^d_Z(X, M) \to H^d_{Z \times \mathbb{A}^1}\left(X \times \mathbb{A}^1, M\right)$, functorial in X. It hence suffices to understand pullback along $i \times \mathbb{A}^1$. Let $w = \mathrm{Spec}(F)$ be a generic point of $W \times \mathbb{A}^1$ of codimension d on $Y \times \mathbb{A}^1$. Then w lies over the generic point of $\mathbb{A}^1$ [22, Tag 0CC1]. We may thus (using Remark 3.3 again) pass to the generic fiber over $\mathrm{Spec}(k(t)) \in \mathbb{A}^1$; essentially we have base-changed the entire problem to $k(t)/k$. Let us denote the base change of X by X_1, and so on. Since Z is geometrically reduced over k [22, Tag 020I], its base change Z_1 is geometrically reduced over $k(t)$ [22, Tag 0384]. Lemma 3.14 below supplies us with an étale neighborhood $X_2 \to X_1$ of w and a smooth map $X_2 \to W_1$ such that $Z_2 \to X_2 \to W_1$ is generically smooth and $Y_2 \to W_1$ is smooth. Let $X_3 = X_2 \times_{W_1} \{w\}$. Our base changes are illustrated in the following diagram:

$$\begin{tikzcd}
X_3 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_1 = X \otimes k(t) \arrow[r] & X \times \mathbb{A}^1 \arrow[r] & X \\
w \arrow[r] & W_1 = W \otimes k(t). & & &
\end{tikzcd}$$

By construction, $X_3 \to X_1$ is a pro-(étale neighborhood) of $w \in X \otimes k(t)$, and so (again using Remark 3.3) we may replace $X \otimes k(t)$ by X_3.

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With these preparatory constructions out of the way, we rename X_3 to X, Y_3 to Y, and Z_3 to Z. We now have a smooth map $X \to \mathrm{Spec}(F)$, where F is infinite (since it contains $k(t)$), $Y \to \mathrm{Spec}(F)$ is smooth, and $Z \to \mathrm{Spec}(F)$ is generically smooth. Also, $W = \{w\}$ is an F-rational point of X, $\dim X = d+1$, $\dim Y = d$, and $\dim Z = 1$. Shrinking X if necessary, we may assume that $Y \subset X$ is principal, say cut out by $f \in \mathcal{O}(X)$, that every component of Z meets w, that Z is smooth away from w, and that X is affine.

Lemma 3.16 supplies us with $\bar{u} : Z \to \mathbb{A}^1$ with $\bar{u}(w) \neq 0$, $\bar{u}f : Z \to \mathbb{A}^1$ finite, and $\bar{u}$ having no double roots. Pick $u \in \mathcal{O}(X)$ reducing to $\bar{u} \in \mathcal{O}(Z)$. Then $\phi_1 := uf : X \to \mathbb{A}^1$ is finite when restricted to Z and satisfies $\phi_1(w) = 0$, and we claim that ϕ_1 is smooth at all points of $\phi_1^{-1}(0) \cap Z$. Note that by construction, (u,f) have no common root on Z (w being the only root of f on Z), and neither do (u,du) (u not having double roots on Z) or (f,df) ($Z(f)$ being smooth). It follows that $d(uf) = udf + fdu$ does not vanish at points $p \in Z$ with $(uf)(p) = 0$, proving the claim.

We may thus apply Lemma 3.17 to obtain $\phi_2,\dots,\phi_{d+1} : X \to \mathbb{A}^1$ such that $\phi = (\phi_1,\dots,\phi_{d+1}) : X \to \mathbb{A}^{d+1}$ is étale at all points of $\phi_1^{-1}(0) \cap Z$ and has $\phi(w) = 0$, and there exists an open neighborhood $0 \in U \subset \mathbb{A}^d_F$ such that $Z_U \xrightarrow{\phi} \mathbb{A}^1_U$ is a closed immersion (in fact, $U = U_1 \times \mathbb{A}^{d-1}$). The non-étale locus of ϕ meets Z in finitely many points (namely a closed subset not containing w, and hence no component of the curve Z), none of which map to 0 under ϕ_1. Shrinking U further, we may thus assume that Z_U is contained in the étale locus V of ϕ. Since $Z_U \simeq \phi(Z_U) \to \phi^{-1}(\phi(Z_U)) \cap V$ is a section of a separated unramified morphism, it is clopen [22, Tag 024T] – that is, we have $\phi^{-1}(\phi(Z_U)) \cap V = Z_U \coprod Z'$ with Z' closed in V.

Let $D = D(u)$; this is an open neighborhood of w in X. Note that $Z(uf) \cap D = Y \cap D$. Let

$$X' = \left(\phi^{-1}\left(\mathbb{A}^1_U\right) \cap V\right) \setminus Z',$$
$$U_0 = U \cap \left(\{0\} \times \mathbb{A}^{d-1}\right), \text{ and } Y' = \phi^{-1}\left(\mathbb{A}^1_{U_0}\right) \cap X' = Z(uf) \cap X'.$$

We have a commutative diagram of schemes

$$\begin{tikzcd}
X & Y \arrow[l, "i"'] & Y \cap D \cap X' \arrow[l] \\
X' \arrow[u] \arrow[d, "\phi"'] & Y' \arrow[l] \arrow[d, "\psi"'] & Y' \cap D \arrow[l] \arrow[u, equal] \\
\mathbb{A}^1_U & \mathbb{A}^1_{U_0}. \arrow[l, "j"'] &
\end{tikzcd}$$

Here j is the canonical closed immersion and ψ is the restriction (i.e., base change) of ϕ. In particular, ψ is étale and Y' is smooth. By construction, ϕ is an étale neighborhood of Z_U, and $Z_U \xrightarrow{\phi} \mathbb{A}^1_U \to U$ is finite. There is an induced commutative diagram

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$$\begin{tikzcd}
H^d_Z(X,M) \arrow[r, "i^*"] \arrow[d, "o"'] & H^d_W(Y,M) \arrow[r, "\simeq"'] & H^d_W(Y\cap D\cap X',M) \arrow[d, equal] \\
H^d_{Z_U}(X',M) \arrow[r] & H^d_{Z_U\cap Y'}(Y',M) \arrow[r, "o"] & H^d_W(Y'\cap D,M) \\
H^d_{Z_U}\left(\mathbb{A}^1_U,M\right) \arrow[u, "\phi^*", "\simeq"'] \arrow[r, "j^*"] & H^d_{Z_U\cap Z(uf)}\left(\mathbb{A}^1_{U_0},M\right). \arrow[u, "\psi^*", "\simeq"'] &
\end{tikzcd}$$

We need to understand the top left-hand horizontal map. All the labeled isomorphisms are pullback along étale maps, and isomorphisms by excision. The two maps labeled o are also pullback along étale morphisms (in fact open immersions), and hence understood.

We have thus reduced to understanding j^*; we rename Z_U to Z and $Z_U\cap Z(uf)$ to V. Since Z is finite over U, it remains closed in $\mathbb{P}^1_U$, and hence by a further excision argument it suffices to understand

$$\bar{j}^*: H^d_Z\left(\mathbb{P}^1_U,M\right) \to H^d_V\left(\mathbb{P}^1_{U_0},M\right).$$

We have $V=\{w,z_1,\dots,z_r\}$, and each z_i is a smooth point of Z (since w is the only singular point of Z). Moreover, z_i is a smooth point of V, since z_i is a simple root of u (since by construction u has no double roots on Z). By Lemma 3.4, the pullback

$$j^*_s: H^d_Z\left(\left(\mathbb{P}^1_U\right)_{z_s},M\right) \to H^d_V\left(\left(\mathbb{P}^1_{U_0}\right)_{z_s},M\right)$$

depends only on M_{-d}. Thus applying Lemma 3.10, it suffices to understand

$$k^*: H^{d-1}_{Z'}(U,M_{-1}) \to H^{d-1}_{V'}(U_0,M_{-1});$$

here Z' and V' are the images of Z and V in U. If $d=1$, we are done, by Example 3.11. The general case (i.e., $d>1$) now follows by induction (i.e., restart the argument with (U,U_0,Z') in place of (X,Y,Z)). □

Lemma 3.14. *Let X be a smooth scheme over an infinite field K, $W\subset X, Y\subset X$ smooth closed subschemes, $W\subset Z\subset X$ with $Z\subset X$ closed, and Z geometrically reduced over K (but not necessarily smooth). Set $w\in W\cap Y$ such that $\dim_w W\le \dim_w Y$. There exists an étale neighborhood $X'\to X$ of w together with a smooth morphism $X'\to W$ such that $Z\times_X X'\to W$ is generically smooth (that is, the smooth locus is dense in the source) and $Y\times_X X'\to W$ is smooth.*

Proof. We modify [4, Corollary 5.11]. Shrinking X around w, we may assume that X is affine and there exist $f_1,\dots,f_d\in\mathcal{O}(X)$ such that $W=Z(f_1,\dots,f_d)$ and W has codimension everywhere exactly d in X. Let $\{z_1,\dots,z_r\}\subset Z$ be a choice of smooth points in every component of Z, which exist because Z is geometrically reduced [22, Tag 056V]. Let $\dim X=d+n$. We claim that there exist $g_1,\dots,g_n\in\mathcal{O}(X)$ such that $dg_1,\dots,dg_n$ are linearly independent (over the respective residue fields) in $\Omega_w W$, $\Omega_w Y$, and $\Omega_{z_i} Z$ for every i; we shall prove this at the end. It follows that $df_1,\dots,df_d$ and $dg_1,\dots,dg_n$ are

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linearly independent in $\Omega_w X$. Consider the map $F = (f_1,\dots,f_d,g_1,\dots,g_n) : X \to \mathbb{A}^{d+n}$. Let $p : \mathbb{A}^{d+n} \to \mathbb{A}^n$ be the projection to the last n coordinates. By [8, 17.11.1], F is smooth at w, $pF|_W$ is smooth at w, $pF|_Y$ is smooth at w, and $pF|_Z$ is smooth at z_i. In particular, $pF|_Z$ is generically smooth. Shrinking X further around w, we may assume that F, $pF|_W$, and $pF|_Y$ are smooth (whence the former two are étale), and of course $pF|_Z$ remains generically smooth. Applying the construction of [4, §5.5, §5.9], we obtain a commutative diagram as follows:

$$\begin{tikzcd}
\Omega \arrow[r, "j"] & P \arrow[r] \arrow[d, "q"'] & \mathbb{A}^d_W \arrow[r] \arrow[d] & W \arrow[d, "pF|_W"] \\
& X \arrow[r, "F"] & \mathbb{A}^{d+n} \arrow[r, "p"] & \mathbb{A}^n.
\end{tikzcd}$$

Here both squares are cartesian by definition, j is an open immersion, and qj is an étale neighborhood of W (by construction of Ω). Since p, F, and j are smooth, so is $\Omega \to W$. By construction, $Z \to X \to \mathbb{A}^n$ is generically smooth, thus so is $Z \times_X P \to W$, and hence also $Z \times_X \Omega \to W$. Similarly, $Y \to X \to \mathbb{A}^n$ is smooth and hence so is $Y \times_X \Omega \to W$. Setting $X' = \Omega$, the result follows.

It remains to prove the claim. Embed X into $\mathbb{A}^N$, and let $g_1,\dots,g_n$ be linear projections. By Lemma 3.15 applied to $\left(\mathbb{A}^N\right)^* \otimes_K K(w) \subset \left(\Omega\mathbb{A}^N\right) \otimes_K K(w) \to \Omega_w W$, $dg_1,\dots,dg_n$ will be linearly independent in $\Omega_w W$ for general g_j, and similarly for $\Omega_w Y$ and $\Omega_{z_i} Z$. The result follows. □

For completeness, we include a proof of the following elementary fact:

Lemma 3.15. *Let K'/K be a finite field extension. Let V be a finite-dimensional K-vector space, V' a K'-vector space of dimension $\geq n$, and $V_{K'} \to V'$ a surjection. Write $\mathbb{A}(V)$ for the associated variety over K (isomorphic to $\mathbb{A}_K^{\dim V}$). There is a nonempty open subset U of $\mathbb{A}(V)^n$ such that any $(v_1,\dots,v_n) \in U(K)$ have K'-linearly independent images in V'. In particular, if F is infinite, then n general elements of V are linearly independent in V'.*

Proof. Replacing V' by a quotient of dimension n, we may assume that $\dim_{K'} V' = n$. There is a map $D : \mathbb{A}(V')^n \to \mathbb{A}^1_{K'}$ such that n elements of V' are linearly independent if and only if their image under D is nonzero (pick a basis of V' and let D be the determinant). By adjunction, the composite

$$\mathbb{A}(V)^n_{K'} \to \mathbb{A}(V')^n \xrightarrow{D} \mathbb{A}^1_{K'}$$

defines a map $D' : \mathbb{A}(V)^n \to R\left(\mathbb{A}^1_{K'}\right)$, where R denotes the Weil restriction along K'/K (see, e.g., [1, §7.6]). By construction, n elements of V have image in V' linearly independent over K' if and only if their image under D' is nonzero. Since D is not the zero map, neither is D', and hence $U = D'^{-1}\left(R\left(\mathbb{A}^1_{K'}\right) \setminus 0\right)$ is the desired nonempty open subset.

The last statement follows because nonempty open subsets of affine space over an infinite field have rational points. □

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Lemma 3.16. *Let Z be an affine curve over the field F, smooth away from a rational point w, and let $f : Z \to \mathbb{A}^1$ be nowhere constant. There exists $u : Z \to \mathbb{A}^1$ such that $u(w) \neq 0$ and $fu : Z \to \mathbb{A}^1$ is finite. If F is infinite, it can be arranged that u has no double zeros.*

Proof. Let $\bar{Z}$ be a compactification of Z which is smooth away from w, and $\bar{Z} \setminus Z = \{z_1, \ldots, z_r\}$. Since Z is smooth at infinity, any map $Z \to \mathbb{A}^1$ extends to $\bar{Z} \to \mathbb{P}^1$. It suffices to find $u_i : Z \to \mathbb{A}^1$ such that $\mathrm{ord}_{z_i}(u_i) + \mathrm{ord}_{z_i}(f) < 0$, for every i. Then $u = \sum_i u_i^{e_i}$ for suitably big e_i will satisfy the same condition, but for all i at once. Now $uf : \bar{Z} \to \mathbb{P}^1$ is proper and $\bar{Z} \setminus Z \supset (uf)^{-1}(\infty) \supset \{z_1, \ldots, z_r\}$, which implies that $uf : Z \to \mathbb{A}^1$ is finite (being proper and affine). If $u(w) = 0$, then replace u by $u+1$; the first claim follows. Replacing u by $u^n + g$ for suitable g and n large, we may assume that du has only finitely many zeros. Then $u + c$ for general c has no double zeros (away from w) and satisfies $u(w) \neq 0$, so that if F is infinite we may arrange the second claim.

Let $\tilde{\bar{Z}}$ be the normalization of $\bar{Z}$, and $\tilde{Z} \subset \tilde{\bar{Z}}$ the open subset over Z – that is, the normalization of Z. Note that $\tilde{\bar{Z}} \to \bar{Z}$ is an isomorphism near z_i. By Riemann–Roch, we can find $v : \tilde{\bar{Z}} \to \mathbb{P}^1$ with an arbitrarily large pole at z_i and no poles away from z_i, so in particular no poles on $\tilde{Z}$. Since $\tilde{Z} \to Z$ is integral, there exists an equation $v^n + a_1 v^{n-1} + \cdots + a_n = 0$, with $a_i : Z \to \mathbb{A}^1$. It follows that (at least) one of the a_i must have a pole at z_i (at least) as large as v. This concludes the proof. □

We have used the following variant of the second part of Gabber's lemma [6, Lemma 3.1(b)]; our proof is heavily inspired by Gabber's:

Lemma 3.17. *Let F be an infinite field, X smooth and affine of dimension d over F, $Z \subset X$ closed, $w \in Z$ a rational point, $e < d$, and $\phi' = (\phi_1, \ldots, \phi_e) : X \to \mathbb{A}^e$ such that $\phi'(w) = 0$, $\phi'|_Z : Z \to \mathbb{A}^e$ is finite, and ϕ' is smooth at all points of $\phi'^{-1}(0) \cap Z$.*

Then there exist $\phi_{e+1}, \ldots, \phi_d : X \to \mathbb{A}^1$ such that $\phi = (\phi_1, \ldots, \phi_d) : X \to \mathbb{A}^d$ is étale at all points of $\phi'^{-1}(0) \cap Z$, $\phi(w) = 0$, and there exists an open neighborhood $0 \in W \subset \mathbb{A}^e$ such that $Z_W \to \mathbb{A}^{d-e}_W$ is a closed immersion.

Note that the new map ϕ is also finite when restricted to Z.

Proof. Set $X \hookrightarrow \mathbb{A}^N$ with w mapped to 0. We claim that general linear projections $\phi_{e+1}, \ldots, \phi_d$ have the desired properties. They vanish on w by definition.

In order for ϕ to be smooth at some point $x \in \phi'^{-1}(0) \cap Z$, we need only ensure that $d\phi_1, \ldots, d\phi_d \in \Omega_x X$ are linearly independent [8, 17.11.1]. Since ϕ' is smooth at x, the $d\phi_1, \ldots, d\phi_e$ are linearly independent at x, and then the other $d\phi_i$ are linearly independent, for general ϕ_i; this follows from Lemma 3.15 applied to $V' = \Omega_x X / \langle d\phi_1, \ldots, d\phi_e \rangle$. Since $\phi'^{-1}(0) \cap Z$ is a finite set of points, the étaleness claim holds for general ϕ_i.

It remains to prove the claim about the closed immersion. Note that by Nakayama's lemma, if $f : X \to Y$ is a morphism of affine S-schemes with X finite over S and S Noetherian, and there exists $s \in S$ such that $f_s : X_s \to Y_s$ is a closed immersion, then there exists an open neighborhood U of s such that $f_U : X_U \to Y_U$ is a closed immersion. Let $\psi : Z \to \mathbb{A}^d$ be the restriction of ϕ, which we view as a morphism over $S = \mathbb{A}^e$ via ϕ' (and the projection $\mathbb{A}^d \to \mathbb{A}^e$ to the first e coordinates). It is thus enough to show that $\psi_0 : \phi'^{-1}(0) \cap Z \to \mathbb{A}^{d-e}$ is a closed immersion (for general ϕ_i). Since ϕ is étale at all

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points of $\phi'^{-1}(0)\cap Z$ (for general ϕ_i) and $Z \to X$ is a closed immersion, ψ is unramified at all points above 0 and so ψ_0 is unramified (for general ϕ_i). Since $\phi : Z \to \mathbb{A}^d$ is finite, so is ψ; in fact, $\phi'^{-1}(0)\cap Z$ is finite over F. By [22, Tags 04XV and 01S4], a morphism is a closed immersion if and only if it is proper, unramified, and radicial; we already know that ψ_0 is finite (hence proper) and unramified. Being radicial is fpqc local on the target [22, Tag 02KW], so may be checked after geometric base change. In other words (using that $\phi'^{-1}(0)\cap Z$ is finite over F), we need the ϕ_i to separate a finite number of specified geometric points. This clearly holds for general ϕ_i. □

Proof of Theorem 3.1. If the theorem holds for composable maps f and g, then it holds for fg. Given $f : Y \to X$, we factor it as

$$Y \xrightarrow{i} \Gamma_f \xrightarrow{p} X;$$

here $i : Y \to \Gamma_f$ is the graph of f. Then i is a regular immersion and p is smooth. It follows that $p^{-1}(Z) \subset \Gamma_f$ has codimension $\geq d$ (see, e.g., [8, Corollary 6.1.4]). Hence it suffices to prove the result for i and p separately – that is, we may assume that f is either a regular immersion or a smooth morphism. The case of smooth maps was already explained in Remark 3.3, so assume that $f : Y \hookrightarrow X$ is a regular immersion. As usual, we may localize in a generic point of $Z\cap Y$ of codimension d on Y; hence we may assume that $Z\cap Y = \{z\}$ is a closed point, X is local, and $\dim Y = d$. It follows (e.g., from [22, Tag 00NQ]) that there exists a sequence of codimension 1 embeddings of essentially smooth schemes

$$Y = Y_d \hookrightarrow Y_{d+1} \hookrightarrow \cdots \hookrightarrow Y_n = X.$$

Let $Z_i = Z\cap Y_i$. Then $\dim Z_i \geq \dim Z_{i+1} - 1$ [22, Tag 00KW], but $\dim Z_n = n-d = \dim Z_d + n - d$, so that $\dim Z_i = i-d$ for all i. It follows that we may prove the result for each $Y_i \hookrightarrow Y_{i+1}$ separately; this is Lemma 3.13. □

4. Applications

After introducing some notation in §4.1, we identify the heart of $\mathcal{SH}^{S^1}(k)(d)$ (for $d \geq 3$) in §4.2. This establishes [2, Conjecture 6.10]. Finally, in §4.3 we study convergence of the resolution of an S^1-spectrum by infinite $\mathbb{P}^1$-loop spectra arising from the adjunction $\mathcal{SH}^{S^1}(k) \leftrightarrows \mathcal{SH}(k)$ and deduce some conservativity results.

We also assume that k is perfect; we will restate this assumption with the most important results only.

4.1. Notation and hypotheses

We write $\mathcal{SH}^{S^1}(k)$ for the category of motivic S^1-spectra [16, §4] (i.e., the motivic localization of the category of spectral presheaves on Sm_k), and $\mathcal{SH}(k) = \mathcal{SH}^{S^1}(k)\left[\mathbb{G}_m^{\wedge -1}\right]$ for the category of motivic spectra [16, §5]. For the stabilization functors, we use the notation

$$\mathrm{Sm}_{k*} \xrightarrow{\Sigma^\infty_{S^1}} \mathcal{SH}^{S^1}(k) \xrightarrow{\sigma^\infty} \mathcal{SH}(k)^{\mathrm{eff}} \subset \mathcal{SH}(k),$$

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and we denote by ω^∞ the right adjoint of σ^∞. Here $\mathcal{SH}(k)^{\mathrm{eff}}$ is the localizing subcategory generated by the image of σ^∞.

There are localizing subcategories

$$\mathcal{SH}^{S^1}(k) \supset \mathcal{SH}^{S^1}(k)(1) \supset \cdots \supset \mathcal{SH}^{S^1}(k)(d) \supset \cdots;$$

here $\mathcal{SH}^{S^1}(k)(d)$ is generated by $\Sigma^\infty_{S^1} X_+ \wedge \mathbb{G}_m^{\wedge d}$ for $X \in \mathrm{Sm}_k$. The inclusion $\mathcal{SH}^{S^1}(k)(d) \subset \mathcal{SH}^{S^1}(k)$ has a right adjoint which we denote by f_d.[8] There are canonical cofiber sequences $f_{d+1} \to f_d \to s_d$ defining the functors s_d. There is a similar filtration of $\mathcal{SH}(k)^{\mathrm{eff}}$, given by $\mathcal{SH}(k)^{\mathrm{eff}}(d) := \mathcal{SH}(k)^{\mathrm{eff}} \wedge \mathbb{G}_m^{\wedge d}$, and the right adjoints (resp., cofibers) are again denoted by f_d (resp., s_d). See [10, 23] or [2, §6.1] for more details on these functors.

Recall that $\mathcal{SH}^{S^1}(k)$ has a t-structure with nonnegative part generated by $\Sigma^\infty_{S^1} X_+$ for $X \in \mathrm{Sm}_k$; its heart canonically identifies with $\mathbf{HI}(k)$ [16, Lemma 4.3.7(2)]. We denote by $E_{\geq 0}$, $E_{\leq 0}$, and $\underline{\pi}_0 E$ the truncations and homotopy sheaves, respectively. The categories $\mathcal{SH}^{S^1}(k)(d)$, $\mathcal{SH}(k)$, and $\mathcal{SH}(k)^{\mathrm{eff}}(d)$ have related t-structures, with nonnegative parts generated by $X_+ \wedge \mathbb{G}_m^{\wedge d}$.

Recall from [2, §5.1] the notion of a presheaf with $\mathbb{A}^1$-transfers. This is just a presheaf F on Sm_k together with, for every finitely generated field K/k, a $GW(K)$-module structure on $F(K)$, and for every finite monogeneic extension $K(x)/K$, a transfer $\tau_x : F(K(x)) \to F(K)$. The category $\mathcal{SH}(k)^{\mathrm{eff}\heartsuit}$ embeds fully faithfully into the category of presheaves with $\mathbb{A}^1$-transfers [2, Corollary 5.17] (morphisms in this category are given by morphisms of presheaves compatible with the GW-module structures and transfers). Given a presheaf with $\mathbb{A}^1$-transfers M, we say that *the transfers extend to framed transfers* if M is in the essential image of this embedding. Recall also that Morel has shown that if $M \in \mathbf{HI}(k)$ and $d > 0$, then M_{-d} canonically extends to a presheaf with $\mathbb{A}^1$-transfers (see §2.1.8 or [2, Example 5.2]).

Definition 4.1. Let k be a perfect field and $d > 0$.

(1) Set $M \in \mathbf{HI}(k)$. We shall say that hypothesis $T_d(M)$ holds if the canonical $\mathbb{A}^1$-transfers on M_{-d} extend to framed transfers. We shall say that hypothesis $T_d(k)$ holds if $T_d(M)$ holds for all $M \in \mathbf{HI}(k)$.

(2) We shall say that hypothesis $S_d(k)$ holds if for any $E \in \mathcal{SH}^{S^1}(k)$ and $i \in \mathbb{Z}$, the spectrum $f_d \underline{\pi}_i s_d E$ is in the essential image of $\omega^\infty : \mathcal{SH}(k) \to \mathcal{SH}^{S^1}(k)$.

Remark 4.2. If k is perfect, then $T_d(k)$ holds for any $d \geq 3$, and if $\mathit{char}(k) = 0$, then $T_2(k)$ also holds [2, Theorem 5.19].

Remark 4.3. We speculate that $T_1(k)$ holds for any perfect field.

Remark 4.4. If $f : \mathrm{Spec}(l) \to \mathrm{Spec}(k)$ is an algebraic extension (automatically separable) and $M \in \mathbf{HI}(k)$ such that $T_d(M)$ holds, then $T_d(f^* M)$ also holds. This is obvious for f finite, and the general case follows by continuity and essentially smooth base change.

[8] We abuse notation somewhat and view this as a functor $\mathcal{SH}^{S^1}(k) \to \mathcal{SH}^{S^1}(k)$.

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Theorem 4.5. (Levine [11]). *Let $char(k)=0$. Then $S_d(k)$ holds for any $d>0$.*

Proof. This is essentially [11, Theorem 2]; we just have to show that

$$s_{p,n}E \simeq f_n\Sigma^{p+n}\underline{\pi}_{p+n}s_nE.$$

By definition [11, main construction (9.2)], $s_{p,n}E(X)$ is the realization of (a rectification of) the simplicial object $\Sigma^p\pi_p(s_nE)^{(n)}(X,\bullet)$; here $(s_nE)^{(n)}(X,\bullet)$ is the homotopy coniveau tower model of $f_ns_nE \simeq s_nE$, and π_p just means taking the pth Eilenberg–MacLane spectrum of the (levelwise) ordinary spectrum $(s_nE)^{(n)}(X,\bullet)$.

The map $(s_nE)_{\geq n+p} \to s_nE$ of spectral sheaves induces a map

$$\alpha_p : \left((s_nE)_{\geq p+n}\right)^{(n)}(-,\bullet) \to (s_nE)^{(n)}(-,\bullet)$$

of simplicial spectral presheaves. We claim that α_p induces an isomorphism on π_i for $i\geq p$, and that the source has $\pi_i=0$ for $i<p$. This yields an equivalence

$$\left((s_nE)_{\geq p+n}\right)^{(n)}(-,\bullet) \simeq \tau_{\geq p}(s_nE)^{(n)}(-,\bullet),$$

where $\tau_{\geq p}$ just means levelwise truncation of the simplicial presheaf of spectra. Taking cofibers, we obtain

$$\left(\Sigma^{p+n}\underline{\pi}_{p+n}s_nE\right)^{(n)}(-,\bullet) \simeq \Sigma^p\pi_p(s_nE)^{(n)}(-,\bullet),$$

which is what we set out to prove (using [10, Theorem 7.1.1]).

It is hence enough to prove the claim. Thus let $X\in \mathrm{Sm}_k$ and $W\subset \mathbb{A}^m_X$ have codimension $\geq n$. The definition of the homotopy coniveau tower (recalled, e.g., in [11, §9]) implies that it is enough to show that

$$H_W^{-i}\left(\mathbb{A}^m_X, s_nE\right) \simeq H_W^{-i}\left(\mathbb{A}^m_X,(s_nE)_{\geq n+p}\right) \text{ for } i\geq p$$

and

$$H_W^{-i}\left(\mathbb{A}^m_X,(s_nE)_{\geq n+p}\right)=0 \text{ for } i<p.$$

Considering the (strongly convergent) descent spectral sequence (for $F\in \mathcal{SH}^{S^1}(k)$)

$$H^p_W\left(\mathbb{A}^m_X,\underline{\pi}_qF\right) \Rightarrow H_W^{p-q}\left(\mathbb{A}^m_X,F\right),$$

for this it suffices to show that for $j\in\mathbb{Z}$ we have

$$H^i_W\left(\mathbb{A}^m_X,\underline{\pi}_js_nE\right)=0 \text{ for } i\neq n.$$

We can compute this cohomology group using the Rost–Schmid resolution; since W has codimension $\geq n$, the vanishing follows from the observation that $\underline{\pi}_j(s_nE)_{-i}=0$ for $i>n$, which holds because $\Omega^i_{\mathbb{G}_m}s_nE\simeq 0$ (for $i>n$) by definition. □

4.2. The heart of $\mathcal{SH}^{S^1}(k)(d)$

Consider the adjunction

$$\sigma^\infty : \mathcal{SH}^{S^1}(k) \leftrightarrows \mathcal{SH}(k) : \omega^\infty.$$

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Then $\sigma^\infty\left(\mathcal{SH}^{S^1}(k)(d)\right) \subset \mathcal{SH}(k)^{\mathrm{eff}}(d)$ and $\sigma^\infty\left(\mathcal{SH}^{S^1}(k)(d)_{\geq 0}\right) \subset \mathcal{SH}(k)^{\mathrm{eff}}(d)_{\geq 0}$, for any $d \geq 0$. Moreover, it follows from [2, Lemmas 6.1(2) and 6.2(1,2)] that $\omega^\infty\left(\mathcal{SH}(k)^{\mathrm{eff}}(d)\right) \subset \mathcal{SH}^{S^1}(k)(d)$ and $\omega^\infty\left(\mathcal{SH}(k)^{\mathrm{eff}}(d)_{\geq 0}\right) \subset \mathcal{SH}^{S^1}(k)(d)_{\geq 0}$. This implies that there is an induced adjunction

$$\pi_0^d\sigma^\infty : \mathcal{SH}^{S^1}(k)(d)^{\heartsuit} \leftrightarrows \mathcal{SH}(k)^{\mathrm{eff}}(d)^{\heartsuit} : \omega^\infty,$$

where π_0^d denotes the truncation functor in the t-structure on $\mathcal{SH}(k)^{\mathrm{eff}}(d)$.

Theorem 4.6. *Let k be a perfect field such that $T_d(k)$ holds. Then the functor $\omega^\infty : \mathcal{SH}(k)^{\mathrm{eff}}(d)^{\heartsuit} \to \mathcal{SH}^{S^1}(k)(d)^{\heartsuit}$ is an equivalence of categories.*

This establishes [2, Conjecture 6.10] (for $n = d$).

Proof. The functor is fully faithful by [2, Theorem 6.9]; it hence suffices to prove essential surjectivity. We shall prove the following more precise statement: if $M \in \mathbf{HI}(k)$ and $T_d(M)$ holds, then M is in the essential image of ω^∞.

We first prove this assuming that k is infinite. We have $\underline{\pi}_i(M)_{-d} = 0$ for $i \neq 0$ [2, Lemma 6.2(3)], and hence the canonical map $M \to f_d\underline{\pi}_0 M$ is an equivalence (indeed, it induces an equivalence on $\underline{\pi}_i(-)_{-d}$ for every i, and this detects equivalence in $\mathcal{SH}^{S^1}(k)(d)$ by [2, Lemma 6.1(1)]). By assumption, the $\mathbb{A}^1$-transfers on $\underline{\pi}_0(M)_{-d}$ extend to framed transfers; hence there exists $\tilde{M} \in \mathcal{SH}(k)^{\mathrm{eff}\heartsuit}$ such that $\omega^\infty\left(\tilde{M}\right)_{-d} \simeq \underline{\pi}_0(M)_{-d}$ as presheaves with $\mathbb{A}^1$-transfers. By Lemma 4.7 (this is where we use the assumption that k is infinite), this implies that $f_d\omega^\infty\left(\tilde{M}\right) \simeq f_d\underline{\pi}_0(M)$. It follows from [10, Theorem 9.0.3] that f_d commutes with ω^∞; we thus find that

$$M \simeq f_d\underline{\pi}_0 M \simeq f_d\omega^\infty\left(\tilde{M}\right) \simeq \omega^\infty f_d\tilde{M}.$$

The claim is thus proved for k infinite.

Now let k be finite and $M \in \mathbf{HI}(k)$ such that $T_d(M)$ holds. Since ω^∞ is fully faithful, M is in the essential image of ω^∞ if and only if the canonical map $M \to \omega^\infty\sigma^{\infty\heartsuit}M$ is an isomorphism. The functors ω^∞ and $\sigma^{\infty\heartsuit}$ commute with essentially smooth base change. Let $f : \mathrm{Spec}(l) \to \mathrm{Spec}(k)$ be an infinite algebraic p-extension of k, for some prime p. Using Lemma 4.8 we reduce to proving that f^*M is in the essential image of ω^∞. By Remark 4.4, $T_d(f^*M)$ holds, and thus we are reduced to what was already established.

This concludes the proof. □

Lemma 4.7. *Let k be an infinite perfect field, $M, N \in \mathbf{HI}(k)$, and $d > 0$. Suppose that $T_d(M)$ holds. Any isomorphism $M_{-d} \simeq N_{-d}$ respecting the $\mathbb{A}^1$-transfers yields an equivalence $f_dM \simeq f_dN$.*

Proof. We have $f_dM \simeq M^{(d)}$ [10, Theorem 7.1.1] (this is where we use the assumption that k is infinite). As explained in [2, Remark 4.17], the (truncated) BLRS complex of M provides a model of $M^{(d)}$ which depends only on M_{-d} as a presheaf of GW-modules together with the maps $f^* : H^d_Z(X, M) \to H^d_{f^{-1}(Z)}(Y, M)$ for $f : Y \to X \in \mathrm{Sm}_k$, with $Z \subset X$

closed of codimension $\geq d$ such that $f^{-1}(Z)$ also has codimension $\geq d$. (In order to apply this remark, we need to know that M_{-d} has framed transfers – see, e.g., [2, Proposition 4.14]); this is the only reason for assuming $T_d(k)$.) Theorem 3.1 shows that f^* depends only on M_{-d} as a presheaf with $\mathbb{A}^1$-transfers. The result follows. □

Lemma 4.8. *Let k be a perfect field and k_p/k (resp., k_q/k) a separable algebraic p-extension (resp., q-extension), for primes $p \neq q$. Set $\alpha : E \to F \in \mathcal{SH}^{S^1}(k)(d)$ such that $T_d(M)$ holds for any homotopy sheaf M of E or F. If the image of α in $\mathcal{SH}^{S^1}(k_p)(d) \times \mathcal{SH}^{S^1}(k_q)(d)$ is an equivalence, then so is α.*

Proof. It suffices to prove that $\Omega^d_{\mathbb{G}_m}(\alpha)$ is an equivalence [2, Lemma 6.1(1)] – that is, that $\underline{\pi}_i(\alpha)_{-d}$ is an isomorphism for all i. By assumption, this is a morphism between sheaves admitting framed transfers, and by [2, Corollary 5.17] the morphism preserves the transfers. The result thus follows from [5, Corollary B.2.5] (using the fact that essentially smooth base change commutes with $\Omega^d_{\mathbb{G}_m}$). □

The following is our degree 0 $\mathbb{G}_m$-Freudenthal theorem:

Corollary 4.9. *Suppose that $T_d(k)$ holds.*

(1) *Set $E \in \mathcal{SH}^{S^1}(k)_{\geq 0} \cap \mathcal{SH}^{S^1}(k)(d)$. Then*

$$\underline{\pi}_0(E) \simeq \underline{\pi}_0(\omega^\infty \sigma^\infty E).$$

(2) *Set $E \in \mathcal{SH}^{S^1}(k)_{\geq 0}$. Then*

$$\underline{\pi}_0(\omega^\infty \sigma^\infty E) \simeq \underline{\pi}_0\left(\mathbb{G}_m^{\wedge d} \wedge E\right)_{-d}.$$

In particular, this holds for $d \geq 3$ if k is perfect, and for $d \geq 2$ if $char(k) = 0$.

Proof. (1) We have $E \in \mathcal{SH}^{S^1}(k)(d)_{\geq 0}$ [2, Lemma 6.2(3)]. Write π_0^d for the homotopy object in the t-structure on $\mathcal{SH}^{S^1}(k)(d)$. Then since σ^∞ and ω^∞ are both right-t-exact [2, Lemma 6.2(1,2)], we learn from Theorem 4.6 that

$$\pi_0^d E \simeq \pi_0^d \omega^\infty \sigma^\infty E. \tag{*}$$

Since $\mathcal{SH}^{S^1}(k)(d)_{\geq 0} \subset \mathcal{SH}^{S^1}(k)_{\geq 0}$ (by construction), we have $\underline{\pi}_0 \pi_0^d \simeq \underline{\pi}_0$ (when applied to objects in $\mathcal{SH}^{S^1}(k)(d)_{\geq 0}$), and hence the result follows by applying $\underline{\pi}_0$ to ($*$).

(2) We have

$$\begin{aligned}
\underline{\pi}_0(\omega^\infty \sigma^\infty E) &\simeq \underline{\pi}_0\left(\omega^\infty \Omega^d_{\mathbb{G}_m} \Sigma^d_{\mathbb{G}_m} \sigma^\infty E\right) \\
&\simeq \underline{\pi}_0\left(\Omega^d_{\mathbb{G}_m} \omega^\infty \sigma^\infty \Sigma^d_{\mathbb{G}_m} E\right) \\
&\simeq \underline{\pi}_0\left(\omega^\infty \sigma^\infty \mathbb{G}_m^{\wedge d} \wedge E\right)_{-d} \\
&\overset{(1)}{\simeq} \underline{\pi}_0\left(\mathbb{G}_m^{\wedge d} \wedge E\right)_{-d}.
\end{aligned}$$

□

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4.3. Canonical resolutions

In this section we will freely use the language of ∞-categories as set out in [13, 12].

Given any adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ of ∞-categories, there is a monad structure on GF [12, Proposition 4.7.4.3]. Hence for $E \in \mathcal{C}$ there is a canonical 'triple resolution' $E \to E^\bullet$, where $E^\bullet$ denotes a cosimplicial object with $E^n = (GF)^{\circ(n+1)}(E)$. In more detail, by the definition of a monad, GF promotes to an $\mathcal{E}_1$-algebra in $\mathrm{Fun}(\mathcal{C},\mathcal{C})$ under the composition monoidal structure, and then [15, Construction 2.7] yields an augmented cosimplical object in $\mathrm{Fun}(\mathcal{C},\mathcal{C})$; the triple resolution is obtained by applying this cosimplicial endofunctor to E. This also makes it clear that the triple resolution is functorial in E.

Applying this to the stabilization adjunction $\sigma^\infty : \mathcal{SH}^{S^1}(k) \leftrightarrows \mathcal{SH}(k) : \omega^\infty$, we obtain the *canonical resolution*

$$E \to E^\wedge := \lim_{\Delta} \left[(\omega^\infty \sigma^\infty)^{\circ(\bullet+1)} E \right],$$

functorially in E.

Definition 4.10. We say that *the canonical resolution converges for E* if this morphism is an equivalence.

Example 4.11. The canonical resolution converges if E is in the essential image of ω^∞, since then the cosimplicial object is split.

Example 4.12. Given a cofiber sequence $E_1 \to E_2 \to E_3$, if the canonical resolution converges for any two of the three terms, then it converges for the third. This holds because all the functors involved are stable.

The following result clearly holds in much greater generality, but for simplicity we state it in our restricted context:

Lemma 4.13. *Set $E \in \mathcal{SH}^{S^1}(k)$ and suppose we are given a tower*

$$\cdots \to E_2 \to E_1 \to E_0 := E$$

and a sequence $n_i \in \mathbb{Z}$ such that

(1) $\lim_i n_i = \infty$,

(2) $E_i \in \mathcal{SH}^{S^1}(k)_{\geq n_i}$ *(for all i), and*

(3) *the canonical resolution converges for $cof(E_{i+1} \to E_i)$ (for all i).*

Then the canonical resolution converges for E.

Proof. Define an endofunctor F of $\mathcal{SH}^{S^1}(k)$ by $F(X) = fib(X \to X^\wedge)$. By construction, this is a stable functor such that $F(X) \simeq 0$ if and only if the canonical resolution converges for X. Since $F(cof(E_{i+1} \to E_i)) \simeq 0$ for all i, we find that the tower

$$\cdots \to F(E_1) \to F(E_0) = F(E)$$

is constant. In order to prove that $F(E) = 0$, it thus suffices to show that $\lim_i F(E_i) \simeq 0$. Commuting the limits, we find that

$$\lim_i F(E_i) \simeq \lim_{i\geq 0, n\in\Delta} fib\left(E_i \to (\omega^\infty\sigma^\infty)^{\circ(n+1)}E_i\right) \simeq \lim_{n\in\Delta}\lim_i fib\left(E_i \to (\omega^\infty\sigma^\infty)^{\circ(n+1)}E_i\right);$$

it thus suffices to show that the inner limit over i vanishes. Since σ^∞ and ω^∞ are both right-t-exact [2, Lemma 6.2(1,2)], we have $fib(\dots) \in \mathcal{SH}^{S^1}(k)_{\geq n_i - 1}$, and hence it is enough to show that if X_i is a sequence of spectra with $X_i \in \mathcal{SH}^{S^1}(k)_{\geq n_i}$ then $\lim_i X_i \simeq 0$. Since $\mathcal{SH}^{S^1}(k)$ is generated as a localizing subcategory by objects of the form $\Sigma^\infty_{S^1} U_+$ for $U \in \mathrm{Sm}_k$, considering the Milnor exact sequence [7, Proposition VI.2.15] it suffices to show that if $n > \dim U$, then $\left[\Sigma^\infty_{S^1}U_+, \mathcal{SH}^{S^1}(k)_{\geq n}\right] = 0$. This follows from the descent spectral sequence. □

Theorem 4.14. *Let k be a perfect field such that $T_d(k)$ holds.*

(1) *The canonical resolution converges for all $E \in \mathcal{SH}^{S^1}(k)_{\geq 0} \cap \mathcal{SH}^{S^1}(k)(d)$.*
(2) *Suppose in addition that $S_j(k)$ holds for all $1 \leq j < d$. Then the canonical resolution converges for all $E \in \mathcal{SH}^{S^1}(k)_{\geq 0} \cap \mathcal{SH}^{S^1}(k)(1)$.*

Proof. (1) Write $\tau^d_{\geq i}$ for the truncation in the t-structure on $\mathcal{SH}^{S^1}(k)(d)$. We have $E \in \mathcal{SH}^{S^1}(k)(d)_{\geq 0}$ [2, Lemma 6.2(3)]. Apply Lemma 4.13 with $E_i = \tau^d_{\geq i}E$ and $n_i = i$; assumption (i) is clear, and (ii) holds by [2, Lemma 6.2(1)]. For (iii), it suffices by Example 4.11 to show that $\pi^d_i E$ is in the essential image of ω^∞. This follows from Theorem 4.6.

(2) By Example 4.12, part (1) of the theorem, and induction, it suffices to show that the canonical resolution converges for s_jE, for $1 \leq j < d$. Since $f_j : \mathcal{SH}^{S^1}(k) \to \mathcal{SH}^{S^1}(k)$ is right-t-exact [2, Lemma 6.2], we have $s_jE \in \mathcal{SH}^{S^1}(k)_{\geq 0}$. Apply Lemma 4.13 with $E_i = f_j\left[(s_jE)_{\geq i}\right]$ and $n_i = i$. Assumption (i) is clear, (ii) holds by the right-t-exactness of f_j, and (iii) holds by the definition of $S_j(k)$ and Example 4.11. □

Corollary 4.15. (1) *Suppose that $T_d(k)$ holds. Then*

$$\sigma^\infty : \mathcal{SH}^{S^1}(k)_{\geq 0} \cap \mathcal{SH}^{S^1}(k)(d) \to \mathcal{SH}(k)$$

is conservative.

(2) *Suppose that additionally $S_j(k)$ holds, for $1 \leq j < d$. Then*

$$\sigma^\infty : \mathcal{SH}^{S^1}(k)_{\geq 0} \cap \mathcal{SH}^{S^1}(k)(1) \to \mathcal{SH}(k)$$

is conservative.

In particular, (1) holds for $d = 3$ if k is perfect, and (2) holds if $char(k) = 0$.

Proof. Clearly the convergence of canonical resolutions implies conservativity (which is equivalent to detecting zero objects, by considering cofibers), so this is immediate from Theorem 4.14. □

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