Proceedings of the Edinburgh Mathematical Society (1981) **24**, 119–122 ©

ON THE FUNDAMENTAL GROUP OF AN ALMOST-ACYCLIC 2-COMPLEX

by JAMES HOWIE

(Received 15th October 1979)

A 2-complex K is called *almost-acyclic* if $H_2(K)=0$ and $H_1(K)$ is torsion-free. This class of complexes was introduced in a previous paper (**2**), and applied to a problem of J. H. C. Whitehead concerning aspherical 2-complexes. In this note, the methods developed in (**2**) are used to study the finitely-generated subgroups of the fundamental group of an almost-acyclic 2-complex.

Now a group can occur as the fundamental group of a *finite* almost-acyclic 2-complex if and only if it is an $\mathcal{M}$-group in the sense of Strebel (**8, 9**). Such groups have also been studied by Magnus (**5**) and by Stammbach (**7**). The class $\mathcal{M}$ contains all knot-like groups in the sense of Rapaport (**6**), in particular all knot groups.

We denote the class of all groups which occur as the fundamental groups of almost-acyclic 2-complexes by $\mathcal{N}$. From our point of view, $\mathcal{N}$ is a more convenient object of study than the subclass $\mathcal{M}$, because the methods used involve the passage from finite to infinite complexes via coverings. In (**8**) and (**9**), Strebel used homological methods to study a larger class $\mathscr{E}$ of groups. The condition defining $\mathcal{N}$ may be thought of as a topological analogue of the homological condition used to define $\mathscr{E}$, and indeed $\mathcal{N}$ is a subclass of $\mathscr{E}$. It seems possible that the results of this paper extend to $\mathscr{E}$, but the combinatorial methods used here apply only to $\mathcal{N}$.

The following is our main result.

Theorem. *Suppose $H\in\mathcal{N}$, and G is a finitely-generated subgroup of H. Then there is a finitely-generated subgroup G_1 of H such that $G\subseteq G_1$, G_1^{ab} is free abelian, and the inclusion-induced map $G^{ab}\to G_1^{ab}$ has finite cokernel.*

Here G^{ab} denotes the commutator quotient group $G/[G, G]$.

Note that the rank of the free abelian group G_1^{ab} can be no greater than the torsion-free rank of G^{ab}. In particular, if G^{ab} is finite, then G_1 is perfect. Hence this result generalises Theorem B of (**2**).

A group is *locally indicable* if every non-trivial finitely-generated subgroup has infinite abelianisation. Such groups are of interest in connection with the problem of the existence of zero-divisors or non-trivial units in group rings (**1**). A consequence of the above result is that a group in $\mathcal{N}$ is locally indicable precisely if it has no (non-trivial) finitely-generated perfect subgroups.

This work was done at the Forschungsinstitut für Mathematik der ETH, Zürich, and was supported by a European Scientific Exchange Programme fellowship from the Royal Society of London. I am grateful to the referee for several useful suggestions.

1. Preliminaries

We first recall some definitions from (**2**).

Definition. A 2-complex K is *almost-acyclic* if $H_2(K)=0$ and $H_1(K)$ is torsion-free.

Definition. A cellular map $f: K \to L$ between CW-complexes is *combinatorial* if it maps (the interior of) each cell of K homeomorphically onto (the interior of) a cell of L.

Definition. A $\mathbb{Z}$*-cover* is a connected regular covering $p: K \to L$ of CW-complexes whose group of covering transformations is infinite cyclic.

The next result is an easy generalisation of Theorem D of (**2**), and we merely sketch a proof.

Proposition 1. *Suppose S is a finite, connected 2-complex, L is an almost-acyclic 2-complex, and $f: S \to L$ is a combinatorial map. Then there exists a commutative triangle of combinatorial maps*

$$\begin{array}{ccc} S & \xrightarrow{f'} & K \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle g} \\ & & L \end{array}$$

such that:

(i) *K is a finite, connected, almost-acyclic 2-complex;*
(ii) *$H_1(f'): H_1(S) \to H_1(K)$ has finite cokernel;*
(iii) *g is a composite of inclusion maps and $\mathbb{Z}$-covers.*

Sketch of Proof. Let $K_0 \subseteq L$ denote the image of f. If the cokernel of $H_1(S) \to H_1(K_0)$ is finite, we may set $K=K_0$ and g the inclusion map $K \to L$ to obtain the desired result. Otherwise f may be lifted over some $\mathbb{Z}$-cover $L_1 \to K_0$ to a map $f_1: S \to L_1$, say. Repeating this argument gives a sequence of $\mathbb{Z}$-covers $L_{r+1} \to K_r$ and lifts f_{r+1} of f_r, such that $K_r \subset L_r$ is the image of f_r, in the same way as the proof of Theorem D in (**2**). As in (**2**), a counting argument shows that the sequence cannot continue indefinitely. In other words, some map $H_1(f_n): H_1(S) \to H_1(K_n)$ has finite cokernel.

Remark. If $H_1(S)$ is finite in Proposition 1, then the 2-complex K will be acyclic. Theorem D of (**2**) deals with the case $H_1(S)=0$.

2. The main result

Theorem 2. *Suppose $H \in \mathcal{N}$, and G is a finitely-generated subgroup of H. Then there is a finitely-generated subgroup G_1 of H such that $G \subseteq G_1$, G_1^{ab} is free abelian, and the inclusion-induced map $G^{ab} \to G_1^{ab}$ has finite cokernel.*

Proof. We may suppose $H=\pi_1(L,z)$, where L is an almost-acyclic 2-complex and z is a 0-cell of L. Since G is finitely-generated, so is G^{ab}. Let r denote the torsion-free rank of G^{ab}. Then there exists a finite generating set $\{g_1,\dots,g_n\}$ for G such that $\{g_1\cdot[G,G],\dots,g_r\cdot[G,G]\}$ is a basis for the torsion-free part of G^{ab}, and $g_i\cdot[G,G]$ has finite order in G^{ab} for $r+1\leqq i\leqq n$. Thus there exists a positive integer m such that $g_i^m\in[G,G]$ for $r+1\leqq i\leqq n$.

Let $L^{(1)}$ denote the 1-skeleton of L. Then the inclusion-induced map $\eta\colon \pi_1(L^{(1)},z)\to\pi_1(L,z)$ is surjective. Hence there are elements b_i $(1\leqq i\leqq n)$ in $\pi_1(L^{(1)},z)$ such that $\eta(b_i)=g_i$. Thus η maps the subgroup F of $\pi_1(L^{(1)},z)$ generated by $b_1,\dots,b_n$ onto G. Hence there are elements $w_i(r+1\leqq i\leqq n)$ in $[F,F]$ such that $\eta(w_i)=g_i^m=\eta(b_i^m)$.

Take S to be a wedge of complexes S_i $(1\leqq i\leqq n)$ and $f\colon S\to L$ a map chosen as follows. For $1\leqq i\leqq r$, S_i is taken to be a subdivided circle, and f maps S_i to a closed path in $L^{(1)}$ which represents the element b_i of $\pi_1(L^{(1)},z)$. The subdivision of the circle and the map f are chosen so that each 1-cell of S_i is mapped to a 1-cell of L. For $r+1\leqq i\leqq n$, S_i is chosen to be a simply-connected planar 2-complex, mapped to L in such a way that every cell is mapped to a cell of the same dimension, and the boundary of S_i in the plane is mapped to a closed path in $L^{(1)}$ representing the element $b_i^m\cdot w_i^{-1}$ of Ker η. (See van Kampen (**3**, Lemma 1)).

Then S is a finite 2-complex with fundamental group $\pi_1(S)$ free of rank r, and $f\colon S\to L$ is a combinatorial map. By Proposition 1, we can express f as a composite

$$S\xrightarrow{f'}K\xrightarrow{g}L$$

Such that K is finite and almost-acyclic; $H_1(f')$ has finite cokernel; and g is a composite of inclusion maps and $\mathbb{Z}$-covers. Now let $G_0=g_*(\pi_1(K,f'(x))\subseteq\pi_1(L,x)$. We claim that $G\subseteq G_0$. We state this as a lemma, and postpone the proof for the moment.

Lemma 2.1. $G\subseteq G_0$.

In the commutative square

$$\begin{array}{ccc}\mathbb{Z}^r=H_1(S) & \xrightarrow{f'_*} & H_1(K)\\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle g_*}\\ G^{ab} & \xrightarrow[h]{} & G_0^{ab}\end{array}$$

the map g_* is onto, and f'_* has finite cokernel. It follows that h has finite cokernel.

Also, since K is finite and almost-acyclic, $H_1(K)$ is free abelian. Since f'_* has finite cokernel, the rank of $H_1(K)$ is at most r. Thus G_0^{ab} is generated by at most r elements. In particular, if $r=0$ then G_0^{ab} is free abelian of rank 0, so we may set $G_1=G_0$.

Suppose then that $r>0$. If G_0^{ab} has torsion-free rank r then it is necessarily free abelian of rank r, so again we may set $G_1=G_0$. Otherwise the torsion-free rank of G_0^{ab} is strictly less than r.

The proof is completed by induction on r.

Proof of Lemma 2.1. It is sufficient to prove that $F\subseteq g_*(\pi_1(K^{(1)}))$ as subgroups of $\pi_1(L^{(1)})$. We use the fact that g is a composite of $\mathbb{Z}$-covers and inclusions. From this it follows that $g_*: \pi_1(K^{(1)}) \to \pi_1(L^{(1)})$ is injective, and there is a chain of subgroups

$$g_*(\pi_1(K^{(1)}) = F_0 \subset F_1 \subset \ldots \subset F_k = \pi_1(L^{(1)})$$

of the free group $\pi_1(L^{(1)})$ such that, for $1 \leqq j \leqq k$, the subgroup F_{j-1} is either

(i) normal in F_j with infinite cyclic quotient; or

(ii) a free factor of F_j.

It follows from the construction of f that $b_i \in f_*(\pi_1(S^{(1)}) \subseteq F_0$ for $1 \leqq i \leqq r$, and similarly $b_i^m \cdot w_i^{-1} \in F_0$ for $r+1 \leqq i \leqq n$.

Clearly $F \subseteq F_k$. Suppose inductively that $1 \leqq j \leqq k$ and $F \subseteq F_j$.

(i) If $F_{j-1} \lhd F_j$ with F_j/F_{j-1} infinite cyclic, then $w_i \in [F, F] \subset F_{j-1}$ for $r+1 \leqq i \leqq n$. Thus $b_i^m \in F_{j-1}$, so $b_i \in F_{j-1}$ for $r+1 \leqq i \leqq n$.

(ii) If F_{j-1} is a free factor of F_j, then by the Kurosch subgroup theorem (**4**, p. 17) F has a free product decomposition $F = F' * F''$, where $F' = F \cap F_{j-1}$. Hence $F^{ab} \cong (F')^{ab} \oplus (F'')^{ab}$. Since $b_i \in F'$ for $1 \leqq i \leqq r$ and $b_i^m \cdot w_i^{-1} \in F'$ for $r+1 \leqq i \leqq n$, it follows that $(F'')^{ab}$ is finite of order dividing $m^{(n-r)}$. But F'' is a free group, so $F'' = 1$.

In either case $F \subseteq F_{j-1}$. It follows by induction on j that $F \subseteq F_0$, so $G \subseteq G_0$ as claimed.

Corollary. Suppose H is a group in $\mathcal{N}$ with no non-trivial finitely-generated perfect subgroups, and R is an integral domain. Then the group ring RH has no non-trivial zero-divisors, and no non-trivial units.

Proof. By the theorem, it follows that every non-trivial finitely-generated subgroup of H has infinite abelianisation. That is, H is locally indicable. Hence Higman's results (**1**, Theorems 12, 13) apply.

REFERENCES

(**1**) G. Higman, The units of group rings, *Proc. London Math. Soc.* (2) **46** (1940), 231–248.

(**2**) J. Howie, Aspherical and acyclic 2-complexes, *J. London Math. Soc.* (2) **20** (1979), 549–558.

(**3**) E. R. van Kampen, On some lemmas in the theory of groups, *American J. Math.* **55** (1933), 268–273.

(**4**) A. Kurosch, *The theory of groups, Volume II* (Chelsea Publishing Co., 1955).

(**5**) W. Magnus, Ueber freie Faktorgruppen und freie Untergruppen gegebener Gruppen, *Monatshefte f. Math. u. Phys.* **47** (1939), 307–313.

(**6**) E. S. Rapaport, Knot-like groups, *Knots, groups, and 3-manifolds*, ed. L. P. Neuwirth (Ann. Math. Studies 84, Princeton Univ. Press, 1975).

(**7**) U. Stammbach, Ueber freie Untergruppen gegebener Gruppen, *Comment. Math. Helv.* **43** (1968), 132–136.

(**8**) R. Strebel, *Die Reihe der Derivierten von E-Gruppen* (Diss. 5148, ETH Zürich, 1973).

(**9**) R. Strebel, Homological methods applied to the derived series of groups, *Comment. Math. Helv.* **49** (1974), 302–332.

Department of Mathematics
University of Edinburgh
Edinburgh EH9 3JZ