The Vibrations of a Particle about a Position of Equilibrium—Part III.

The Significance of the Divergence of the Series Solution.

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(*Read 10th November 1922. Received 27th August 1923.*)

§ 1. In the two parts of this investigation previously published * it has been shown that the solution in terms of elliptic functions represents the motion of the particular dynamical system under consideration throughout the whole range of values of s and g for which a real solution exists, except for those values for which $s = 2g$ and $k = 1$, but that, on the other hand, the series solution is convergent and represents the motion only so long as

$$\left| \frac{8\alpha^2}{s^2}(1 + \lambda - 4k_2^2) + \frac{16\alpha^4}{s^4}(2\lambda - 3\lambda^2) \right| < 1 \,; \dagger \quad \ldots\ldots\ldots\ldots(1)$$

for values of s and g for which the sign of this inequality is reversed the trigonometric series representing the solution are divergent. It is of importance to investigate what discontinuities, if any, of the system correspond to values of s and g which lie on the boundary of the region of convergence; the present part is concerned primarily with showing that under such circumstances no discontinuity of the system exists, thus confirming the suggestions made in Part I., § 12.

§ 2. *The Equation of the Boundary of the Region of Convergence.*

The first step is to obtain the analytical relation between s and g which corresponds to points on the boundary of the region of

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* Baker and Ross, "The Vibrations of a Particle about a Position of Equilibrium," *Proc. Edin. Math. Soc.*, XXXIX. (1920 21), pp. 34-57 (referred to in the sequel as Part I.); Baker, "The Vibrations of a Particle about a Position of Equilibrium — Part II.; The Relation between the Elliptic Function and Series Solutions," *Proc. Edin. Math. Soc.*, XL. (1921–22), pp. 34–49 (referred to in the sequel as Part II.).

† Part II., p. 48; it is evident that the sign of inequality in inequality (35) on that page should be reversed.

convergence. For such points the quantity under the modulus on the left hand side of inequality (1) must be equal either to $+1$ or to -1.

1°. Suppose

$$\frac{8\alpha^2}{s^2}(1+\lambda-4k_2^2)+\frac{16\alpha^4}{s^4}(2\lambda-3\lambda^2)=-1.$$

The quantity under the radical in the expressions for ν and μ * is then zero and we obtain

$$\nu=\mu=\frac{s^2}{8\alpha^2}+\frac{1}{2}-\frac{\lambda}{2}$$

But since $\nu=\mu$ and ν is the greatest root of the cubic, they must both equal $\frac{1}{2}$ for any real solution,† so that

$$\nu=\mu=\tfrac{1}{2}\text{ and therefore }\lambda=\frac{s^2}{4\alpha^2}.$$

The locus corresponding to these values of λ, μ, ν is the part of the double line, $s=2g$, between the points of contact with the curved branches of the discriminant curve; this part of the double line is therefore a boundary of the region of convergence (see the figure in Part II., p. 37).

2°. Suppose

$$\frac{8\alpha^2}{s^2}(1+\lambda-4k_2^2)+\frac{16\alpha^4}{s^4}(2\lambda-3\lambda^2)=+1;$$

then

$$\frac{s^4}{16\alpha^4}=\frac{s^2}{\alpha^2}\left(\frac{1+\lambda}{2}-2k_2^2\right)+(2\lambda-3\lambda^2)\quad\text{..................(2)}$$

Substituting this value in the expressions for ν and μ,‡ we obtain

$$\nu=\frac{s^2}{8\alpha^2}+\frac{1}{2}-\frac{\lambda}{2}+\frac{s^2}{4\sqrt{2}\alpha^2},$$

$$\mu=\frac{s^2}{8\alpha^2}+\frac{1}{2}-\frac{\lambda}{2}-\frac{s^2}{4\sqrt{2}\alpha^2},$$

$$\therefore\ \nu-\mu=\frac{s^2}{2\sqrt{2}\alpha^2}.\quad\text{..................................(3)}$$

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* Part II., § 11, p. 48.

† *Cf.* Part I., § 8, p. 48.

‡ Part II., § 11, p. 48.

To obtain the equation of the boundary it is simplest to eliminate λ between equation (2) and the equation of the cubic which is satisfied by λ, viz.,

$$4\alpha^2 \lambda^3 - (4\alpha^2 + s^2)\lambda^2 + (\alpha^2 + 2sg)\lambda - g^2 = 0.$$

This elimination leads to the equation

$$\frac{1}{128}\left(\frac{s}{\alpha}\right)^{12} - \frac{1}{8}\left(\frac{s}{\alpha}\right)^{10} + \frac{7}{16}\left(\frac{s}{\alpha}\right)^{8} + \frac{1}{2}\left(\frac{s}{\alpha}\right)^{6} + \frac{1}{16}\left(\frac{s}{\alpha}\right)^{4} - \frac{1}{2}\left(\frac{s}{\alpha}\right)^{2}$$
$$+ \frac{g}{\alpha}\left[\frac{3}{8}\left(\frac{s}{\alpha}\right)^{9} - 3\left(\frac{s}{\alpha}\right)^{7} - \frac{7}{4}\left(\frac{s}{\alpha}\right)^{5} - 2\left(\frac{s}{\alpha}\right)^{3} + 2\,\frac{s}{\alpha}\right]$$
$$+ \left(\frac{g}{\alpha}\right)^{2}\left[\frac{9}{2}\left(\frac{s}{\alpha}\right)^{6} + \left(\frac{s}{\alpha}\right)^{9} + \frac{31}{2}\left(\frac{s}{\alpha}\right)^{2} - 2\right]$$
$$- \left(\frac{g}{\alpha}\right)^{3}\left[\left(\frac{s}{\alpha}\right)^{3} + 36\,\frac{s}{\alpha}\right] + 27\left(\frac{g}{\alpha}\right)^{4} = 0, \quad \text{..........................(4)}$$

which is the equation of the remainder of the boundary of the region of convergence of the series solution.

When $g = 0$, this equation reduces to the simple form

$$\frac{1}{128}\,\frac{s^2}{\alpha^{12}}\,(s^2 - 8\alpha^2)(s^8 - 8s^6\alpha^2 - 8s^4\alpha^4 + 8\alpha^8) = 0,$$

so that in this particular case we have the real values $s = 0$ and $s = \pm 2\sqrt{2}\alpha$, which agrees with the results previously obtained.*

In the general case no simple factors of the expression have been discovered.

Owing to the complicated form of equation (4) it seems impossible to discuss analytically the particular characteristics, if any, of those orbits for which the corresponding values of s and g lie on the curved part of the boundary of the region of convergence; recourse must therefore be had to the discussion of particular numerical cases.

Before proceeding to investigate such special cases another general result may be given, which has some bearing on the the question.

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* Part II., § 10 (ii), p. 47.

§ 3. *The Envelope of the Orbit.*

In the numerical cases considered in Part I. it will be observed that the whole orbit is contained in a certain bounded region of the plane of ϕ_1 and ϕ_2. It seems worth while to obtain the equation of the boundary of this region not only as a means of checking the numerical calculations,* but, what is more important from the point of view of the present investigation, because it is conceivable that, for points on the boundary of the region of convergence, these boundary curves may exhibit some peculiarities. For want of a better name we shall refer to the boundary curve as *the "envelope" of the orbit.*

To determine the equation of the "envelope" we employ a method similar to that used by Beth † in a particular simple case.

The position of a particle at any instant is given by the equations ‡

$$\phi_1 = \sqrt{\frac{2q_1}{s_1}} \cdot \cos p_1, \quad \phi_2 = \sqrt{\frac{2q_2}{s_2}} \cdot \cos p_2,$$

where p_1, p_2, q_1, q_2 have certain definite values which may be determined. Writing $2p_1 - p_2 = \phi$, the equations become

$$\phi_1 = \sqrt{\frac{2q_1}{s_1}} \cdot \cos p_1, \quad \phi_2 = \sqrt{\frac{2q_2}{s_2}} \cdot \cos (2p_1 - \phi), \quad \text{.........(5)}$$

the quantities q_1, q_2, p_1 and ϕ being determined at any instant.

Now suppose we keep q_1, q_2 and ϕ fixed and vary p_1; the corresponding values of ϕ_1 and ϕ_2 determine a certain Lissajous figure, which is the curve instantaneously described at the moment. We may call such a curve an "osculatory curve," using a term

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* The plotting of the boundary curve disclosed a numerical slip in the calculations on which the orbit represented in figure 2 of Part I., p. 45, was based; the correct orbit for this case is shown in figure 3 of the present paper. This emphasises the value of finding the envelope as a check on the long and intricate calculations.

† *Archives Néerlandaises des Sc. ex. et nat.* (2), 15 (1910), pp. 246–283. The system discussed by Beth corresponds to the case in which $s = 0$.

‡ Part I., § 2, p. 35, eqns. (1).

employed by Poincaré in a similar connection.* The envelope of all such osculatory curves will be the required "envelope" of the orbit.

Write $q_1 = \zeta$, so that $q_2 = \frac{1-\zeta}{2}$†; then from equations (5)

$$\sqrt{\zeta}\cos p_1 = \phi_1 \sqrt{\frac{s_1}{2}} = x \text{ (say)}, \quad \text{.................. (6)}$$

$$\sqrt{1-\zeta}\cos(2p_1 - \phi) = \phi \sqrt{s_2} = 2y \text{ (say)} \quad \text{.......... (7)}$$

Further

$$h = 1 - g = s_1 q_1 + s_2 q_2 + \alpha q_1 q_2^{\frac{1}{2}} \cos\phi, \ddagger$$

or

$$\zeta\sqrt{1-\zeta}\cos\phi = \frac{s-2g}{\sqrt{2}\,\alpha} - \frac{s}{\sqrt{2}\,\alpha}\zeta.$$

Writing $k = \frac{s-2g}{\sqrt{2}\,\alpha}$, $l = \frac{s}{\sqrt{2}\,\alpha}$, this equation becomes

$$\zeta\sqrt{1-\zeta}\cos\phi = k - l\zeta. \quad \text{.............................. (8)}$$

Eliminating ϕ and p_1 between equations (6), (7) and (8) we obtain

$$\zeta^2\left(x^2+y^2-ly+\frac{l^2}{4}\right)+\zeta\left(yk-x^2-x^4-\frac{kl}{2}+2x^2yl\right) + \frac{k^2}{4} - 2x^2yk + x^4 = 0. \quad \text{...(9)}$$

Differentiating this equation with respect to ζ and eliminating ζ between the resulting equation and equation (9) we have finally

$$\left(yk-x^2-x^4-\frac{kl}{2}+2x^2yl\right)^2 = 4\left(x^2+y^2-ly+\frac{l^2}{4}\right)\left(\frac{k^2}{4}-2x^2yk+x^4\right),$$

or, expanding and removing the factor x^2,

$$x^6 - 2x^4 + x^2(1+kl-l^2) - 4lx^4y + 6kx^2y + x^2y^2(4l^2-4) - 2ky - 4kly^2 + 8ky^3 + kl - k^2 = 0, \quad \text{...................... (10)}$$

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* See Poincaré, *Leçons de la Mécanique céleste*, I., p. 90.

† Part I., § 2, p. 36, eqn. (6).

‡ Part I., § 2, p. 36, eqn. (5).

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which is the required equation of the envelope in terms of the coordinates x and y.

If we seek to identify equation (10) with the equation

$$(x^2 + A_1 y + B_1)(x^2 + A_2 y + B_2)(x^2 + A_3 y + B_3) = 0,$$

we find that A_1, A_2, A_3 are the roots of the cubic

$$\xi^3 + 4l\xi^2 + 4(l^2 - 1)\xi - 8k = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

Now the cubic of Part I., § 9, p. 49, eqn (20), may be written in the form

$$q_2^3 - \left(1 + \frac{l^2}{2}\right) q_2^2 + \left\{\tfrac{1}{4} + l(l-k)\right\} q_2 - \frac{(l-k)^2}{2} = 0\ ;$$

this is the cubic whose roots are λ, μ, ν. Transforming this cubic into the corresponding cubic for q_1 by putting $q_1 = 1 - 2q_2$, we obtain

$$q_1^3 - (1 - l^2) q_1^2 - 2lkq_1 + k^2 = 0,$$

which, on putting $q_1 = -\dfrac{2k}{z}$, becomes

$$z^3 + 4lz^2 + 4(l^2 - 1)z - 8k = 0.$$

The values of A_1, A_2, A_3 are therefore given by

$$-\frac{2k}{1-2\lambda}, \quad -\frac{2k}{1-2\mu}, \quad -\frac{2k}{1-2\nu}.$$

The expressions for B_1, B_2 and B_3 may be found without difficulty. The "envelope" of the orbit thus breaks up into the three coaxial parabolas

$$x^2 - \frac{2k}{1-2\lambda} y + \frac{(1-2\nu)(1-2\mu)\{k(1-6\lambda) - l(1-2\lambda)^2\}}{4k(\nu-\lambda)(\mu-\lambda)} = 0, \quad \ldots(12)$$

$$x^2 - \frac{2k}{1-2\mu} y + \frac{(1-2\lambda)(1-2\nu)\{k(1-6\mu) - l(1-2\mu)^2\}}{4k(\lambda-\mu)(\nu-\mu)} = 0, \quad \ldots(13)$$

$$x^2 - \frac{2k}{1-2\nu} y + \frac{(1-2\lambda)(1-2\mu)\{k(1-6\nu) - l(1-2\nu)^2\}}{4k(\lambda-\nu)(\mu-\nu)} = 0. \quad \ldots(14)$$

Transforming back to the coordinates ϕ_1 and ϕ_2 by means of equations (6) and (7) we obtain three similar parabolas in the plane of ϕ_1 and ϕ_2.

Now in all cases in which there is a real solution

$$\lambda<\tfrac{1}{2},\ \mu<\tfrac{1}{2} \text{ and } \nu>\tfrac{1}{2};$$

thus :—

when $k>0$, *i.e.* $s>2g$, the parabolas (12) and (13) extend to infinity in the positive direction of the axis of ϕ_2, whereas the parabola (14) extends to infinity in the opposite direction;

when $k<0$, *i.e.* $s<2g$, the parabola (14) extends to infinity in the positive direction of the axis of ϕ_2, whereas the parabolas (12) and (13) extend to infinity in the opposite direction.

From a consideration of the equations (12), (13) and (14) it is evident that the "envelope" of the orbit possesses no essential peculiarities when the values of s and g correspond to a point on the boundary of the region of convergence of the series solution; this is further demonstrated by the following numerical cases.

§ 4. *Orbits on the Boundary of the Region of Convergence.*

All analytical attempts to discover peculiarities of the orbits on the boundary of the region of convergence having failed, we are led to the consideration of particular numerical cases in which s and g satisfy the relation (2).

1°. The simplest case is where

$$g=0,\quad s=2\sqrt{2}\alpha.$$

We take as before $\alpha=0{\cdot}1$ and obtain as the result of the calculations

$$q_2=0{\cdot}0432{,}1362+\frac{0{\cdot}0428{,}9083 \text{ sn } u}{1+0{\cdot}0074{,}6967 \text{ sn } u},$$

$$\text{sn } u=1{\cdot}0000{,}1822\left[\frac{\sin v-q^2\sin 3v}{1-2q\cos 2v}\right],$$

$$v=0.9999{,}8605\,u,\quad u=1{\cdot}0000{,}1395\,v,$$

$$q=0{\cdot}0000{,}0348{,}75,$$

$$q_1=1-2q_2$$

$$p_1=2{\cdot}9921{,}9499\,v+\tfrac{1}{2}\tan^{-1}\{11{\cdot}5704{,}2647 \text{ dc } u\}$$
$$-0{\cdot}0009{,}3373\sin 2v+0{\cdot}0000{,}0174\sin 4v+\eta_1{}^*$$

$$p_2=5{\cdot}4843{,}8992\,v+\eta_2{}^*$$

* η_1 and η_2 are arbitrary constants of integration which may be added to p_1 and p_2. They are determined so that q_1, q_2, p_1, and p_2 satisfy the integral of energy

$$1-g=h=s_1q_1+s_2q_2+\alpha q_1q_2^{\frac{1}{2}}\cos(2p_1-p_2).$$

Their importance was not recognised in the calculations given in Part I.

$$\text{dc}\, u = 0\cdot9999,8177,7 \left(\frac{1 + 2q \cos 2v}{\cos v + q^2 \cos 3v}\right),$$

$$s_1 = 1, \quad s_2 = 1\cdot7171,5728,8,$$

$$\phi_1 = \sqrt{2q_1} \,.\, \cos p_1, \quad \phi_2 = \sqrt{\frac{2q_2}{1\cdot7171,5728,8}} \,.\, \cos p_2,$$

$$\eta_1 = 0, \quad \eta_2 = 135^\circ\ 0'\cdot42.$$

The orbit for values of v from 0° to 366° is shown in figure 1.

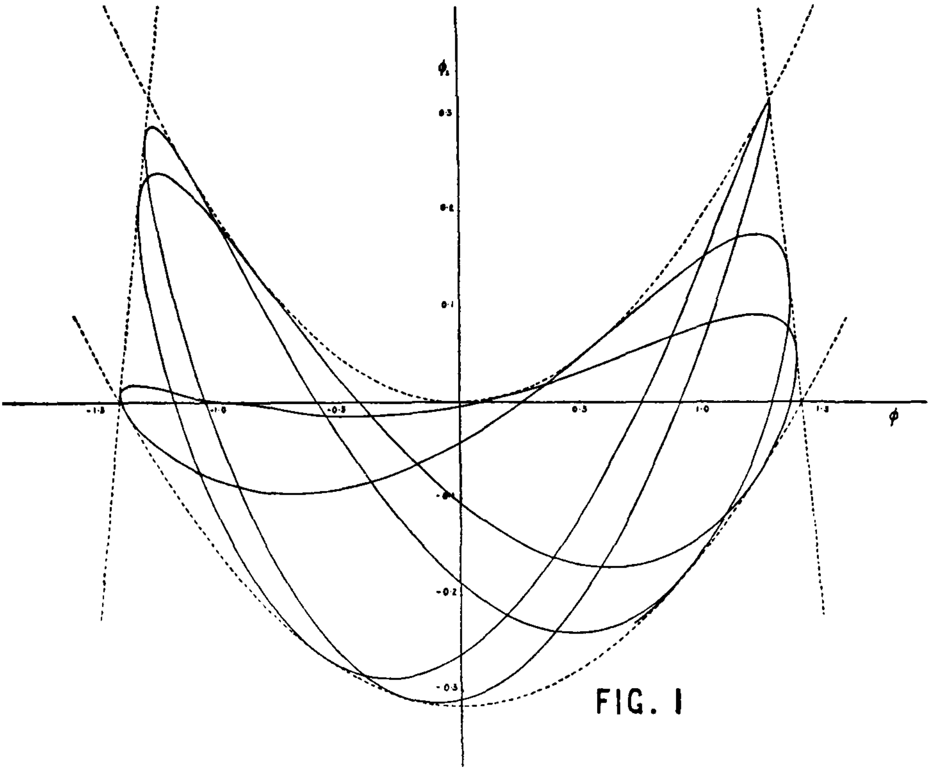

FIG. I

The parabolas forming the envelope of the orbit have the equations

$$\phi_1^2 = 5\cdot2416,141\ \phi_2; \quad \phi_2 = (\phi_1^2 - 2)\, 0\cdot1580,4812;$$

$$\phi_2 = \left(1 - \frac{\phi_1^2}{2}\right) 1\cdot8423,4370;$$

the envelope is represented by the broken lines in the figure.

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The characteristic features of this orbit are that for a certain value of the time, q_2 becomes zero, and that one of the parabolas forming the envelope of the orbit passes through the origin of coordinates. These peculiarities are readily seen to be due to the fact that $g=0$, and are not associated with the condition that the values of s and g under consideration correspond to a point on the boundary of the region of convergence. For q_2 is zero when $\text{sn}\, u = -1$, provided that $mk=l$; but if this condition is satisfied we must have $\lambda=0$ and therefore $g=0$.

The orbit therefore presents no peculiarities which can be attributed to the fact that the values of s and g correspond to a point on the boundary of the region of convergence.

2°. As an additional case we will assume $s=0{\cdot}2$ and the corresponding value of g, which satisfies equation (2) is found to be $g=0{\cdot}0297{,}0372{,}3$, taking, as before, $\alpha=0{\cdot}1$. We obtain

$$q_2 = 0{\cdot}1626{,}1533 + \frac{0{\cdot}1096{,}8478\ \text{sn}\, u}{1+0{\cdot}0360{,}7865\ \text{sn}\, u},$$

$$q_1 = 1 - 2q_2,$$

$$\text{sn}\, u = \frac{1{\cdot}0001{,}6216 \sin v - 0{\cdot}0000{,}0001 \sin 3v}{1 - 0{\cdot}0001{,}6281 \cos 2v}$$

$$p_1 = 4{\cdot}0922{,}3122\, v + \tfrac{1}{2}\tan^{-1}\{3{\cdot}3383{,}6429\ \text{dc}\, u\} - \left[\frac{v}{2} - \tfrac{1}{2}\tan^{-1}\{0{\cdot}9576{,}341 \tan v\}\right] + \eta_1$$

$$p_2 = 7{\cdot}1844{,}120\, v + \tfrac{1}{2}\tan^{-1}\{0{\cdot}9914{,}7001\ \text{dc}\, u\} + \left[\frac{v}{2} - \tfrac{1}{2}\tan^{-1}\{0{\cdot}7037{,}413 \tan v\}\right] + \eta_2$$

$$\text{dc}\, u = \frac{0{\cdot}9998{,}3787\,(1+0{\cdot}0001{,}6281 \cos 2v)}{\cos v},$$

$$\eta_1 - 0,\quad \eta_2 = 134^\circ\, 59'.91,$$

$$\phi_1 = \sqrt{2q_1}\,.\cos p_1,\quad \phi_2 = \sqrt{\frac{q_2}{0.9}}\,.\cos p_2.$$

The orbit for values of v from 0° to 360° is shown in figure 2.

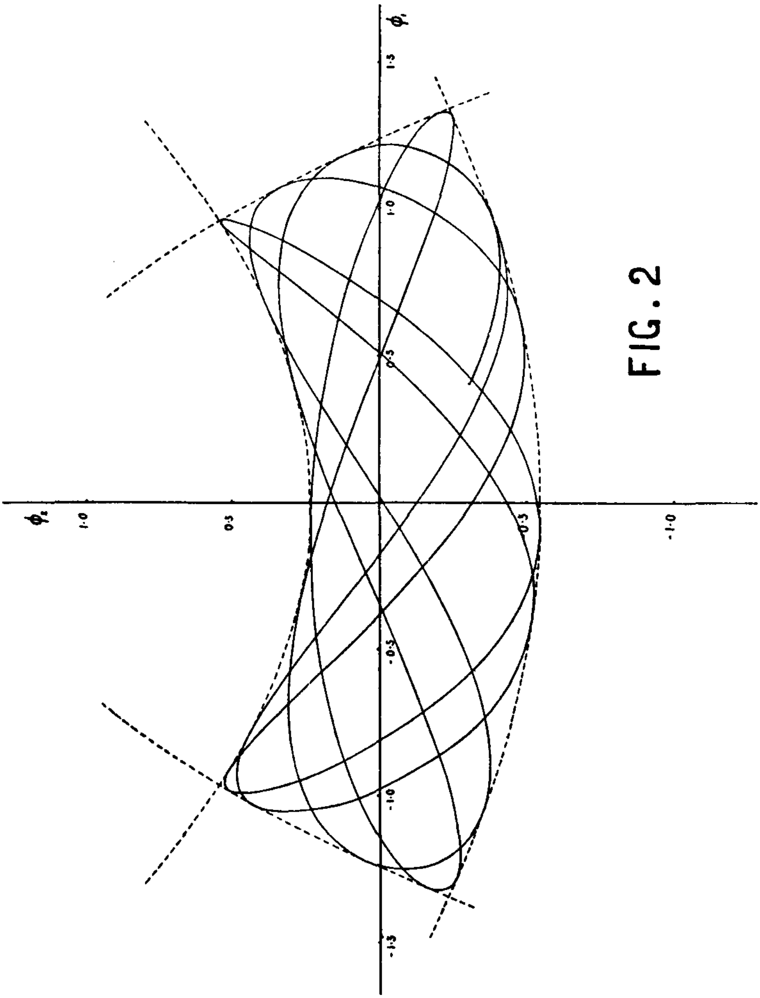

FIG. 2

The equations of the parabolas forming the envelope of the orbit are found to be

$$\phi_2 = 0{\cdot}6765{,}3662\left\{\frac{\phi_1^2}{2} + 0{\cdot}3442{,}7800\right\},$$

$$\phi_2 = 0{\cdot}3471{,}6324\left\{\frac{\phi_1^2}{2} - 1\ 5732{,}6368\right\},$$

$$\phi_2 = 1{\cdot}7734{,}4980\left\{0{\cdot}7710{,}1434 - \frac{\phi_1^2}{2}\right\};$$

these curves are shown by broken lines in the figure.

3°. To show that the orbit just discussed exhibits no peculiarities it may be compared with the orbit of Part I., fig. 1 or with the *correct* solution of the case discussed in Part I., § 7, pp. 44–45.* In this case $s = 0$, $g = 0{\cdot}02$, $\alpha = 0{\cdot}1$, and we find

$$q_2 = \frac{0{\cdot}1676{,}1353\ \mathrm{sn}\, u + 0{\cdot}2089{,}2106}{0{\cdot}1605{,}8204\ \mathrm{sn}\, u + 1}$$

$$q_1 = 1 - 2q_2,$$

$$\mathrm{sn}\, u = \frac{1.0032{,}6295 \sin v}{1 - 0{\cdot}0032{,}6560 \cos 2v},\quad u = 1{\cdot}0065{,}4186\, v,$$

$$p_1 = 7{\cdot}3564{,}8496\, u - \tfrac{1}{2}\tan^{-1}\{3{\cdot}7645{,}8405\ \mathrm{dc}\, u\} + 0{\cdot}0101{,}5432 \sin 2v$$
$$- 0{\cdot}0001{,}0177 \sin 4v + 0{\cdot}0000{,}0137 \sin 6v - 0{\cdot}0000{,}0002 \sin 8v + \eta_1,$$

$$p_2 = 14{\cdot}7129{,}6994\, u + \tfrac{1}{2}\tan^{-1}\{0{\cdot}7594{,}2794\ \mathrm{dc}\, u\}$$
$$+ \left[\frac{v}{2} - \tfrac{1}{2}\tan^{-1}\{0{\cdot}6008{,}7091 \tan v\}\right] - 0{\cdot}0000{,}0501 \sin 2v + \eta_2,$$

$$\mathrm{dc}\, u = \frac{0{\cdot}9967{,}4766\,(1 + 0{\cdot}0032{,}6560 \cos 2v)}{\cos v},$$

$$\eta_1 = 157°30'{\cdot}00,\quad \eta_2 = 0.$$

The orbit for values of v from 0° to 360° is shown in figure 3.

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* As previously mentioned the work given in that place contained a numerical slip.

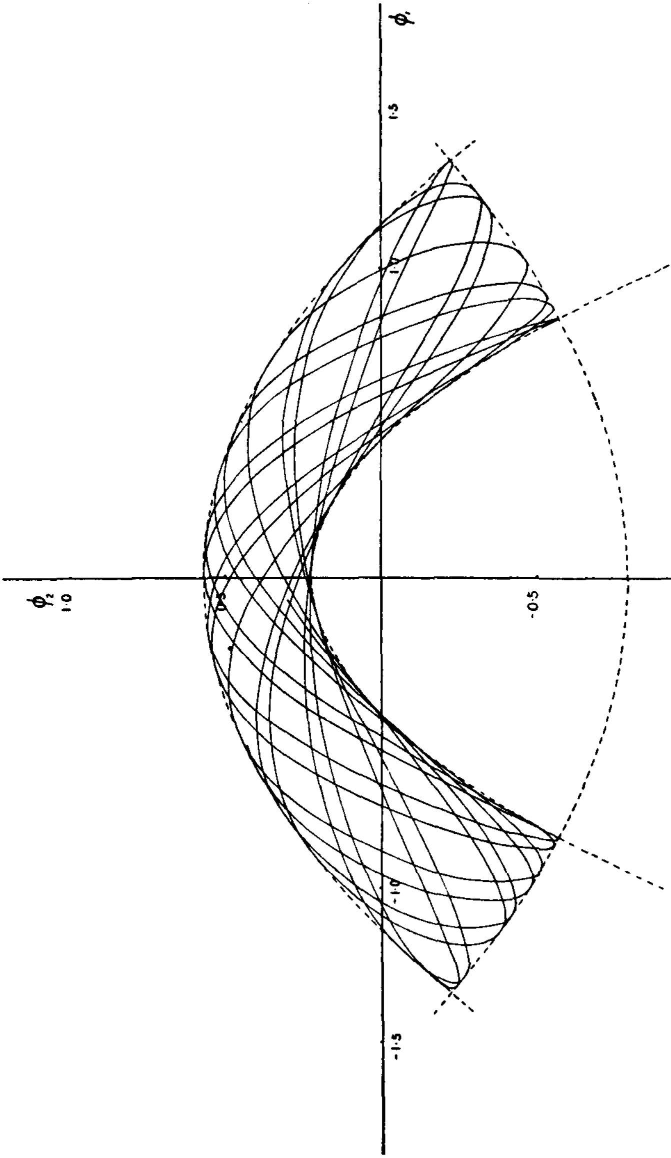

FIG. 3

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The equations of the parabolas forming the envelope of the orbit are found to be

$$\phi_2 = 1\cdot 12699\,(0\cdot 19686 - \phi_1^2),$$
$$\phi_2 = 0\cdot 43890\,(1\cdot 29788 - \phi_1^2),$$
$$\phi_2 = 0\cdot 31588\,(\phi_1^2 - 2\cdot 50546)\,;$$

these curves are shown by broken lines in the figure.*

5 It has therefore been demonstrated adequately, that there is no discontinuity in the system corresponding to a passage from values of s and g for which the series solution is convergent to values for which it is divergent. In other words, the divergence of the series solution represents no discontinuity in the system but merely the failure of the series solution to represent the motion for such values of s and g. On the other hand, it has been shown that the same remarks do not apply to that part of the boundary of the region of convergence which consists of the double line $s = 2g$; for values of s and g on this line the system does possess a discontinuity.

To sum up the results obtained for the particular dynamical system which has been considered:—

It has been shown that real solutions of the problem exist for a range of values of s and g determined from the discriminant of the cubic

$$4\,\alpha^2 x^3 - (4\,\alpha^2 + s^2)\,x^2 + (\alpha^2 + 2\,sg)\,x - g^2 = 0.$$

A solution has been obtained in terms of elliptic functions which is valid throughout the region of real solutions, except on the double line $s = 2g$, where the solution degenerates into a certain asymptotic form.

The solution in trigonometric series, however, is only valid in a part of this region, corresponding to values of s and g for which the roots of the cubic are expressible in the form of infinite series of positive powers of $\dfrac{\alpha}{s}$; the boundary of the region of convergence of the series solution is thus defined.

It has further been shown that the divergence of the series solution represents no discontinuity in the system.

In Part IV. it will be shown how these results may be extended to any dynamical system whatever.

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* The complete calculations for this and the previous Parts are given in the Thesis submitted by the Author for the degree of D.Sc. in the University of Edinburgh, entitled "The Convergence of the Trigonometric Series of Dynamics," 2 vols., which may be found in the General Library of the University of Edinburgh.