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ON THE FIRST EIGENCONE FOR THE FINSLER LAPLACIAN

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Abstract

In this paper, we characterise the structure of the eigencone for the Finsler Laplacian corresponding to the first Dirichlet eigenvalue on a compact Finsler manifold with a smooth boundary.

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1. Introduction

Let (M, F) be an n-dimensional *Finsler manifold*, that is, a connected smooth manifold equipped with a Finsler metric $F : TM \to [0, +\infty)$. A Finsler metric F on M induces a Finsler co-metric F^* on M, defined on $T^*M \setminus \{0\}$ by $F^*(x, \xi) := \sup_{F_x(y)=1} \xi(y)$. To consider the global analysis on a Finsler manifold M, we assume that (M, F) is orientable throughout the paper.

Given a smooth measure m on (M, F), for a weakly differentiable vector field $V : M \to TM$, we define its *divergence* $\mathrm{div}_m V : M \to R$ through the identity

$$\int_M \varphi \mathrm{div}_m V \, dm = -\int_M d\varphi(V) \, dm, \tag{1.1}$$

where $\varphi \in C_0^\infty(M)$ (that is, the set of smooth functions on M with a compact support). For any $u \in C^\infty(M)$, the *gradient* ∇u of u is defined to be the dual of the 1-form du under the Legendre transform $\mathcal{L} : T_xM \to T_x^*M$ for $x \in M$ (see Section 2). Note that the gradient vector field ∇u is not differentiable at points with $\nabla u(x) = 0$ even if (M, F) and u are smooth. However, it is continuous on M. The *Finsler Laplacian* Δ_m is formally defined by $\Delta_m u := \mathrm{div}_m(\nabla u)$, which is a nonlinear elliptic differential operator of the second order. To be more precise, $\Delta_m u$ is defined in a distributional sense through the identity

$$\int_M \varphi \Delta_m u \, dm = -\int_M d\varphi(\nabla u) \, dm \tag{1.2}$$

for all $\varphi \in C_0^\infty(M)$. For the sake of simplicity, we denote Δ_m as Δ in the following.

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Let (M, F, m) be a compact Finsler manifold (M, F) with or without a smooth boundary and equipped with a smooth measure m. Let $H^{1,2}(M)$ be the Sobolev space of L^2 functions on M such that $F^*(du) < \infty$ with respect to the norm $\|u\|_2 := \|u\|_{L^2} + \|F^*(du)\|_{L^2}$ and let H_0^1 be a space of functions $u \in H^{1,2}(M)$ with $\int u\, dm = 0$ if $\partial M = \emptyset$ and $u|_{\partial M} = 0$ if $\partial M \neq \emptyset$. For any nonzero function $u \in H_0^1 \setminus \{0\}$, define the energy of u by

$$E(u) := \frac{\int_M [F^*(x, du)]^2\, dm}{\int_M |u|^2\, dm}.$$

In [3], the authors proved that there is a function $u \in H_0^1 \setminus \{0\}$ with $\int u^2\, dm = 1$, which minimises the energy functional $E(u)$. Thus, $\lambda_1 := \inf_{u \in H_0^1 \setminus \{0\}} E(u)$ is a critical value of E, which is called *the first (nonzero) eigenvalue* for Δ and u is a critical point of E corresponding to λ_1, which is called *the eigenfunction* on (M, F, m) corresponding to λ_1. In this case, u satisfies

$$\Delta u = -\lambda_1 u \tag{1.3}$$

in a weak sense. It is known that the eigenfunction u is $C^{1,\alpha}$ for some $0 < \alpha < 1$ and C^∞ on an open subset $M_u := \{x \in M \mid du(x) \neq 0\}$ [3]. Further, some sharp lower bound estimates for the first eigenvalue of the Finsler Laplacian Δ were given in [9] and [10]. It is an interesting question to introduce and study the higher eigenvalues and the associated eigenfunctions for the Finsler Laplacian.

Denote by V_1 the union of the zero function and the set of all eigenfunctions corresponding to λ_1. In general, V_1 is a cone, not a linear subspace in H_0^1. Therefore, we will call V_1 the *eigencone* corresponding to λ_1. If F is Riemannian, then all eigenfunctions are C^∞ and the eigencone V_1 is a finite-dimensional subspace in H_0^1. In particular, the eigencone V_1 is one dimensional for the Dirichlet problem [2]. It is a natural question to study the structure of the first eigencone V_1 for a general Finsler metric. In this note, we give a characterisation of the structure of the first Dirichlet eigencone V_{λ_1}, consisting of the zero function and the Dirichlet eigenfunctions corresponding to λ_1 (that is, the weak solution of (1.3) with $u|_{\partial M} = 0$), for the Finsler Laplacian. In fact, we get the following result.

THEOREM 1.1. *Let (M, F, m) be an n-dimensional compact Finsler manifold with a smooth boundary equipped with a smooth measure m. Assume that $u \in H_0^1$ is a Dirichlet eigenfunction corresponding to the first eigenvalue λ_1. Then either $u(x) > 0$ or $u(x) < 0$ in $M \setminus \partial M$. Furthermore, the first Dirichlet eigencone V_{λ_1} is a one-dimensional or two-dimensional cone. For the latter case, F must be nonreversible.*

In fact, if there are two linearly independent Dirichlet eigenfunctions with opposite signs in V_{λ_1}, then V_{λ_1} is a two-dimensional cone. Otherwise, V_{λ_1} is a one-dimensional cone (see the proof of Theorem 1.1 below). Example 3.1 below shows that V_{λ_1} is a two-dimensional cone if $a \neq 0$ and a one-dimensional cone if $a = 0$. Here we say that two eigenfunctions $f(x)$ and $h(x)$ in V_{λ_1} are *linearly dependent* if there is a nonzero constant c such that $f = ch$. Otherwise, we say that $f(x)$ and $h(x)$ are *linearly independent*. A nonzero function is always linearly independent.

In particular, if F is reversible, then all eigenfunctions in V_{λ_1} are linearly dependent. In this case, V_{λ_1} is one dimensional (cf. Remark 3.6) and Theorem 1.1 is reduced to Theorem 1.2 in [3]. Further, if F is Riemannian, then the Laplacian is a linear elliptic differential operator. Theorem 1.1 is reduced to the classical results (cf. Corollary 2 of Chapter I in [2], Theorem 8.38 in [4] or Theorem 1.3 in [5]).

2. Preliminaries

In this section, we briefly recall some fundamental concepts in Finsler geometry. For details, we refer to [1] or [7]. Moreover, we shall make some preparations to prove Theorem 1.1.

Let M be an n-dimensional connected smooth manifold. A *Finsler metric F on M* means a function $F : TM \to [0, \infty)$ with the following properties:

(1) F is C^∞ on $TM \setminus \{0\}$;
(2) $F(x, \lambda y) = \lambda F(x, y)$ for any $(x, y) \in TM$ and all $\lambda > 0$;
(3) the matrix $(g_{ij}) = ((1/2)(\partial^2 F(x, y)/\partial y^i \partial y^j))$ is positive.

Such a pair (M, F) is called a *Finsler manifold*. Given a smooth measure m, the triple (M, F, m) is called a *Finsler measure space*. It is easy to see that $F^2(x, y) = g_{ij}(x, y)y^i y^j$.

A Finsler metric F on M is said to be *reversible* if $F(x, -y) = F(x, y)$ for all $x \in M$ and $y \in T_xM$. Otherwise, F is said to be *nonreversible*. In this case, we can define the reverse Finsler metric $\overleftarrow{F}(x, y)$ by $\overleftarrow{F}(x, y) := F(x, -y)$.

Given a Finsler metric F on a manifold M, there is a dual Finsler metric F^* on the cotangent bundle T^*M given by

$$F^*(x, \xi_x) := \sup_{y \in T_xM \setminus \{0\}} \frac{\xi(y)}{F(x, y)} \quad \text{for all } \xi \in T_x^*M. \tag{2.1}$$

The *Legendre transformation* $\mathcal{L} : TM \to T^*M$ is defined by

$$\mathcal{L}(y) := \begin{cases} g_y(y, \cdot) & y \neq 0, \\ 0 & y = 0. \end{cases}$$

One can check that it is a diffeomorphism from $TM\setminus\{0\}$ onto $T^*M\setminus\{0\}$ and norm-preserving, namely, $F(y) = F^*(\mathcal{L}(y))$ for all $y \in TM$ (see [7, Section 3.1]). From (2.1), we have the Cauchy–Schwartz inequality

$$g_y(y, v) \leq F(y)F(v), \tag{2.2}$$

for $y, v \in T_xM$ and $y \neq 0$. Equality holds if and only if $v = cy$ for some $c = c(x) \geq 0$ [7, Lemma 1.2.3].

For a smooth function $u : M \to R$, we define the *gradient vector* $\nabla u(x)$ of u at x by $\nabla u(x) := \mathcal{L}^{-1}(du(x)) \in T_xM$. In a local coordinate system, we can re-express ∇u as

$$\nabla u(x) = \begin{cases} g^{ij}(x, \nabla u) \dfrac{\partial u}{\partial x^i} \dfrac{\partial}{\partial x^j} & x \in M_u, \\ 0 & x \in M \setminus M_u, \end{cases} \tag{2.3}$$

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where $M_u = \{x \in M \mid du(x) \neq 0\}$. Obviously, $\nabla u = 0$ if $du = 0$. In general, ∇u is only continuous on M, but smooth on M_u.

Given a smooth measure $dm = \sigma(x)\,dx$ and a weakly differentiable vector field V on M, the *divergence* $\mathrm{div}_m V : M \to R$ of V is defined by (1.1) and the *Finsler Laplacian* Δ_m acting on functions $u \in H^{1,2}(M)$ is formally defined by $\Delta_m u := \mathrm{div}_m(\nabla u)$. To be more precise, $\Delta_m u$ is the distributional Laplacian defined through the identity (1.2). We shall simply denote div_m and Δ_m by div and Δ in the following.

Let $\psi := (x^i) : U \subset M \to \mathbb{R}^n$ be a local coordinate system in M. It induces a standard local coordinate system (x^i, η_i) in T^*M by mapping $\eta = \eta_i\, dx^i|_x \to (x^i, \eta_i)$.

Set

$$A^i(x,\eta) := \frac{1}{2}\frac{\partial [F^*]^2}{\partial \eta_i}(x,\eta)$$

and

$$g^{*ij}(x,\eta) = \frac{\partial A^i}{\partial \eta_j}(x,\eta) = \frac{1}{2}\frac{\partial^2 [F^{*2}]}{\partial \eta_i \partial \eta_j}(x,\eta).$$

Since $F(x,y) = F^*(x,\eta)$, where $\eta \in T_x^*M$ is the dual of the vector $y \in T_xM$, we have $g^{ij}(x,y) = g^{*ij}(x,\eta)$. By choosing a smaller coordinate neighbourhood $U \subset M$ if necessary, one may assume that there exists a positive constant $C \geq 1$ such that, for $x \in \psi(U)$ and nonzero $\xi, \eta \in \mathbb{R}^n$,

$$|A^i(x,\eta)| \leq C|\eta|, \tag{2.4}$$

$$\left|\frac{\partial A^i}{\partial x^j}(x,\eta)\right| \leq C|\eta|, \tag{2.5}$$

$$C^{-1}|\xi|^2 \leq g^{*ij}(x,\eta)\xi_i\xi_j \leq C|\xi|^2, \tag{2.6}$$

which implies that

$$C^{-1}|\eta|^2 \leq F^{*2}(x,\eta) \leq C|\eta|^2, \tag{2.7}$$

$$|[A^i(x,\eta) - A^i(x,\zeta)]\xi_i| \leq C|\eta - \zeta||\xi|. \tag{2.8}$$

Note that $A(x,du) = (A^i)(x,du) = \nabla u$ on M_u for any function $u \in C^\infty(M)$. Then (1.3) is equivalent to

$$\int_{\psi(U)} \Big(A^i(x,du)\frac{\partial \varphi}{\partial x^i} - \lambda_1 u\varphi\Big)\sigma(x)\,dx = 0 \quad \text{for all } \varphi \in C_0^\infty(\psi(U)). \tag{2.9}$$

Thus, (2.9) is an elliptic quasilinear equation.

Consider the following quasilinear differential equation (in a weak sense):

$$\mathrm{div} A(x,u,du) + B(x,u,du) = 0 \tag{2.10}$$

defined on an open set Ω in $\mathbb{R}^n$. Assume that $A(x,u,\eta)$ and $B(x,u,\eta)$ satisfy the following assumptions: for all $K \geq 0$ and all $(x,u,\eta) \in \Omega \times (-K,K) \times \mathbb{R}^n$,

$$|A(x,u,\eta)| \leq a_0|\eta|^{\alpha-1} + |a_1(x)u|^{\alpha-1}, \tag{2.11}$$

$$\eta \cdot A(x,u,\eta) \geq |\eta|^\alpha - |a_2(x)u|^\alpha, \tag{2.12}$$

$$|B(x,u,\eta)| \leq b_0|\eta|^\alpha + b_1(x)|\eta|^{\alpha-1} + (b_2(x))^\alpha |u|^{\alpha-1}, \tag{2.13}$$

where $\alpha > 1$, a_0, b_0 are constants and $a_i(x), b_i(x)$ are nonnegative measurable functions with $|a_i(x)|, |b_j(x)| \leq \mu$, where μ is a constant. Trudinger proved the following Harnack inequality in [8].

LEMMA 2.1 [8]. *Let $u(x)$ be a weak solution of (2.10) in a cube $\Omega = \Omega(3\rho) \subset \mathbb{R}^n$ with $0 \leq u \leq K$ in Ω, where $A(x, u, du)$ and $B(x, u, du)$ satisfy (2.11)–(2.13). Then*

$$\max_{\Omega(\rho)} u(x) \leq C' \min_{\Omega(\rho)} u(x),$$

where $C' = C'(\alpha, n, a_0, b_0K, \rho\mu)$ is a constant and $\Omega(\rho) := \Omega_{x_0}(\rho)$ means a cube in $\mathbb{R}^n$ of side ρ and centre x_0 whose sides are parallel to the coordinate axes.

From Lemma 2.1, one obtains the following result.

PROPOSITION 2.2. *Let $(M, F, d\mu)$ be a compact Finsler manifold with a smooth boundary and u be a nonnegative first Dirichlet eigenfunction on (M, F). Then u is positive on $M \setminus \partial M$.*

PROOF. Assume that there is a point $x_0 \in M \setminus \partial M$ such that $u(x_0) = 0$. Choose a coordinate neighbourhood (U, ψ) at x_0 such that $\psi(U) \subset \Omega_{x_0}(3\rho)$, a cube in $\mathbb{R}^n$. By the assumption, it is easy to see that, for a nonnegative function u, $\sigma A(x, du)$ and $B(x, u, du) = \lambda_1 \sigma u$ in (2.9) satisfy (2.11)–(2.13) with $\alpha = 2$ and $a_1 = a_2 = b_0 = b_1 = 0$ from (2.4)–(2.8). Thus, it follows from Lemma 2.1 that $u(x) \equiv 0$ on a sufficiently small cube $\Omega_{x_0}(\rho)$. Since M is connected and compact, we have $u(x) \equiv 0$ on M, which is impossible. □

Similarly, we can define the reverse gradient $\overleftarrow{\nabla}$ and the reverse Laplacian $\overleftarrow{\Delta}$ for the reverse Finsler metric $\overleftarrow{F}$. In fact, we have $\overleftarrow{g}(x, y) = g(x, -y)$, $\overleftarrow{\nabla} u = -\nabla(-u)$ and $\overleftarrow{\Delta} u = -\Delta(-u)$. Note that $\nabla(-u)$ and $-\nabla(u)$ are different. Obviously, if u is a weak solution of $\Delta u = -\lambda_1 u$, then $-u$ is a weak solution of $\overleftarrow{\Delta} u = -\lambda_1 u$ and vice versa. Let V_1 be the eigencone for Δ corresponding to λ_1 and

$$\overleftarrow{V}_1 := \{u \in H_0^1 \setminus \{0\} \mid \overleftarrow{\Delta} u = -\lambda_1 u \text{ in a weak sense}\} \cup \{0\},$$

which is the eigencone for $\overleftarrow{\Delta}$ corresponding to λ_1. Obviously, if F is reversible, then $V_1 = \overleftarrow{V}_1$. In general, $V_1 \neq \overleftarrow{V}_1$ and V_1, $\overleftarrow{V}_1$ are not subspaces in H_0^1. The following lemma is obvious.

LEMMA 2.3.

(1) *$u \in V_1$ if and only if $-u \in \overleftarrow{V}_1$.*

(2) *$u \in V_1$ if and only if $ku \in V_1$ for a nonnegative constant k, if and only if $ku \in \overleftarrow{V}_1$ for any nonpositive constant k.*

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Recall that V_{λ_1} is the first Dirichlet eigencone for the Dirichlet eigenvalue problem. As above, we can define $\overleftarrow{V}_{\lambda_1}$. In this case, $\lambda_1 > 0$ and the corresponding eigenfunction u cannot be constant. For any nonpositive function $u \in V_{\lambda_1}$, we have $0 \leq -u \in \overleftarrow{V}_{\lambda_1}$ from Lemma 2.3. By Proposition 2.2, we have the following theorem.

THEOREM 2.4. *If u is a nonnegative (respectively nonpositive) first Dirichlet eigenfunction on a compact Finsler manifold (M, F) with a smooth boundary, then u is positive (respectively negative) on $M \setminus \partial M$.*

3. Proof of Theorem 1.1

In this section, we are going to study the eigenfunctions corresponding to the first eigenvalue for the Dirichlet problem and prove Theorem 1.1. First of all, we give an example.

EXAMPLE 3.1. Let $F(y) = |y| + \langle a, y\rangle$ be a Minkowski norm on $\mathbb{R}^n$, where a is a constant vector with $|a| < 1$, $\langle \cdot\,, \cdot\rangle$ is a usual Euclidean inner product and $|\cdot|$ is a Euclidean norm. With respect to the Busemann Hausdorff measure, the volume form of F is given by

$$dV = (1 - |a|^2)^{(n+1)/2}\,dx$$

(see [7, Example 2.2.2]). Let $u(y) = f(F(y))$ for some nondecreasing C^2 function f on $\mathbb{R}^+$. Then

$$\nabla u = \frac{f'(F(y))}{F(y)}y, \quad \Delta u = \frac{(n-1)f'(F(y))}{F(y)} + f''(F(y)). \tag{3.1}$$

If f is nonincreasing, then analogous expressions hold for $v(x) = f(F(-y))$, that is,

$$\nabla v = \frac{f'(F(-y))}{F(-y)}y, \quad \Delta v = \frac{(n-1)f'(F(-y))}{F(y)} + f''(F(-y)). \tag{3.2}$$

For nonincreasing f, ∇ and Δ in (3.1) are replaced by $\overleftarrow{\nabla}$ and $\overleftarrow{\Delta}$. Similarly, ∇ and Δ in (3.2) are replaced by $\overleftarrow{\nabla}$ and $\overleftarrow{\Delta}$ for nondecreasing f.

Assume that λ_1 is the first Dirichlet eigenvalue of Δ and f is a strictly increasing function satisfying the following ordinary differential equation on the ball $B^n(1) := \{y \in \mathbb{R}^n \mid F(y) < 1\}$:

$$f''(t) + \frac{n-1}{t}f'(t) + \lambda_1 f(t) = 0, \quad f(1) = 0.$$

Then both $u(y) = f(F(y))$ and $v(y) = -f(F(-y))$ are the first Dirichlet eigenfunctions of Δ corresponding to λ_1. In fact, $\lambda_1(B^n(1)) = \lambda_1(\mathbb{B}^n(1))$, where $\lambda_1(\mathbb{B}^n(1))$ is the first eigenvalue of the Euclidean Laplacian Δ_0 with the first eigenfunction $u_0(y) = f(|y|)$ defined on an Euclidean ball $\mathbb{B}^n(1)$ in $\mathbb{R}^n$ (cf. [6, Example]).

Further, $u(y)$ and $v(y)$ are linearly independent unless F is reversible. In fact, assume that there is a constant c such that $u = cv$, that is, $f(F(y)) = -cf(\overleftarrow{F}(y))$. Differentiating

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this with respect to y yields $F_{y^i} = -c\overleftarrow{F}_{y^i}$, where we use $f' > 0$. By contracting it with y^i, we have $F(y) = -c\overleftarrow{F}(y)$. Hence, $\overleftarrow{F}(y) = c^2\overleftarrow{F}(y)$, which means that $c^2 = 1$. Since $F > 0$ and $\overleftarrow{F} > 0$, we have $c = -1$. Thus, $F(y) = \overleftarrow{F}(-y)$, that is, F is reversible. Observe that $F(y) = |y| + \langle a, y\rangle$ is nonreversible if and only if $a \neq 0$. Thus, if $a \neq 0$, then $u(y)$ and $v(y)$ are linearly independent.

The above example shows that the first Dirichlet eigencone for the Finsler Laplacian may not be one dimensional. In the following, we always assume that (M, F) is a compact Finsler manifold with a smooth boundary.

Lemma 3.2. *Let u be a Dirichlet eigenfunction on (M, F) corresponding to the first eigenvalue λ_1. Assume that $u(x_0) \geq 0$ for some point $x_0 \in M$. Then $u(x) \geq 0$ on M.*

Proof. Since $u(x_0) \geq 0$, there is a coordinate neighbourhood (U, ψ) such that $\psi(U) = \mathbb{B}_r(x_0) \subset \mathbb{R}^n$ and $u(x) \geq 0$ on U. Note that u satisfies (2.9) on $\psi(U)$. By [3, Lemma 3.1], $u \in H^{2,2}$. Thus, we write (2.9) in a nondivergence form:

$$Lu := a^{ij}\frac{\partial^2 u}{\partial x^i \partial x^j} + b^i\frac{\partial u}{\partial x^i} = -\lambda_1 u, \quad x \in \mathbb{B}_r(x_0),$$

where

$$a^{ij}(x) := g^{*ij}(x, du(x)), \quad b^i(x) := \frac{\partial \ln\sigma}{\partial x^j} g^{*ij}(x, du(x)) + \frac{\partial g^{*ij}(x, du(x))}{\partial x^j}.$$

Obviously, L is an elliptic operator and $b^i \in L^\infty(\mathbb{B}_r(x_0))$. Let $\varepsilon > 0$ and $\alpha > 0$. For any $x = (x^1, \cdots, x^n) \in \mathbb{B}_r(x_0)$, we assume that $0 \leq x^1 - x_0^1 \leq r$ without loss of generality. Define

$$v(x) := \sup_{\partial\mathbb{B}_r(x_0)} u^+ + \lambda_1(e^{\alpha r} - e^{\alpha(x^1 - x_0^1)}) \sup_{\mathbb{B}_r(x_0)} u,$$

where $u^+ = \max\{u, 0\}$. Assume that $\sum_{i=1}^n |b^i|^2 \leq b^2$. For $x \in \mathbb{B}_r(x_0)$,

$$\begin{aligned} Lv &= a^{ij}\frac{\partial^2 v}{\partial x^i \partial x^j} + b^i\frac{\partial v}{\partial x^i} \\ &= \lambda_1\alpha(-\alpha a^{11} - b^1)e^{\alpha(x^1 - x_0^1)} \sup_{\mathbb{B}_r(x_0)} u \\ &\leq \lambda_1\alpha\Big(-\frac{\alpha}{C} + b\Big)e^{\alpha(x^1 - x_0^1)} \sup_{\mathbb{B}_r(x_0)} u, \end{aligned}$$

where we use (2.6). Choose $\alpha = C(b+1) > 1$ such that $Lv \leq -\lambda_1 \sup_{\mathbb{B}_r(x_0)} u$. Thus,

$$L(v - u) = Lv - Lu \leq -\lambda_1 \sup_{\mathbb{B}_r(x_0)} u + \lambda_1 u \leq 0.$$

Obviously, $v - u \geq 0$ on $\partial\mathbb{B}_r(x_0)$. By the maximum principle,

$$\inf_{\mathbb{B}_r(x_0)} (v - u) \geq \inf_{\partial\mathbb{B}_r(x_0)} (v - u)^- = 0.$$

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Hence,

$$\sup_{\partial\mathbb{B}_r(x_0)} u^+ = \inf_{\mathbb{B}_r(x_0)} v \geq \sup_{\mathbb{B}_r(x_0)} u \geq 0,$$

which means that $u(x) \geq 0$ on $\partial\mathbb{B}_r(x_0)$. Otherwise, we have $u \equiv 0$ in $\mathbb{B}_r(x_0)$. Since M is compact and connected, u is identically zero on M, which is impossible. Thus, $u(x) \geq 0$ on M by the compactness and connectivity of M. □

If $u \in V_{\lambda_1}$, then $-u \in \overleftarrow{V}_{\lambda_1}$. So, we have the following counterpart to Lemma 3.2.

Lemma 3.2′. *Let u be a Dirichlet eigenfunction on (M, F) corresponding to λ_1. Assume that $u(x_0) \leq 0$ for some point $x_0 \in M$. Then $u(x) \leq 0$ on M.*

The following proposition follows directly from Lemmas 3.2–3.2′ and Theorem 2.4.

Proposition 3.3. *Let u be a Dirichlet eigenfunction on (M, F) corresponding to λ_1. Then u is nonnegative or nonpositive on M. Consequently, u is positive or negative on $M \setminus \partial M$.*

Lemma 3.4. *Let u and v be two nonnegative Dirichlet eigenfunctions on (M, F) corresponding to λ_1. Then there exists a positive constant c such that $v = cu$.*

Proof. Since $0 \leq u \in V_{\lambda_1}$, u is a nonnegative solution of the following equation:

$$\int_M d\phi(\nabla u)\, dm = \lambda_1 \int_M u\phi\, dm \tag{3.3}$$

for any $\phi \in C_0^\infty(M)$. From the definition of the eigenfunction, u, v are not constant on M. Consider the functions $u_\epsilon := u + \epsilon > 0$ and $v_\epsilon := v + \epsilon > 0$, where ϵ is a sufficiently small positive number. Obviously, $du_\epsilon = du$. Hence, we have $M_{u_\epsilon} = M_u$, $\nabla u_\epsilon = \nabla u$ and $\Delta u_\epsilon = \Delta u$. Similar equalities hold for v and v_ϵ.

Let

$$\phi_1 = \frac{u_\epsilon^2 - v_\epsilon^2}{u_\epsilon} \quad \text{and} \quad \phi_2 = \frac{v_\epsilon^2 - u_\epsilon^2}{v_\epsilon}$$

be test functions in $C_0^1(M)$. From (3.3),

$$\lambda_1 \int_M \Big(\frac{u}{u_\epsilon} - \frac{v}{v_\epsilon}\Big)(u_\epsilon^2 - v_\epsilon^2) = \int_M d\phi_1(\nabla u) + \int_M d\phi_2(\nabla v). \tag{3.4}$$

Note that

$$d\phi_1 = \Big(1 + \frac{v_\epsilon^2}{u_\epsilon^2}\Big)du_\epsilon - \frac{2v_\epsilon}{u_\epsilon}dv_\epsilon, \quad d\phi_2 = \Big(1 + \frac{u_\epsilon^2}{v_\epsilon^2}\Big)dv_\epsilon - \frac{2u_\epsilon}{v_\epsilon}du_\epsilon. \tag{3.5}$$

Substituting (3.5) into (3.4) yields

$$\begin{aligned}\lambda_1 \int_M \Big(\frac{u}{u_\epsilon} - \frac{v}{v_\epsilon}\Big)(u_\epsilon^2 - v_\epsilon^2) &= \int_M \frac{u_\epsilon^2 + v_\epsilon^2}{u_\epsilon^2}F^2(\nabla u_\epsilon) + \int_M \frac{u_\epsilon^2 + v_\epsilon^2}{v_\epsilon^2}F^2(\nabla v_\epsilon) \\ &\quad - \int_M \frac{2v_\epsilon}{u_\epsilon}dv_\epsilon(\nabla u_\epsilon) - \int_M \frac{2u_\epsilon}{v_\epsilon}du_\epsilon(\nabla v_\epsilon).\end{aligned} \tag{3.6}$$

Since $dv_\epsilon(\nabla u_\epsilon) = g_{\nabla v_\epsilon}(\nabla v_\epsilon, \nabla u_\epsilon) \le F(\nabla v_\epsilon)F(\nabla u_\epsilon)$ by (2.2), from (3.6),

$$\lambda_1 \int_M \Big(\frac{u}{u_\epsilon} - \frac{v}{v_\epsilon}\Big)(u_\epsilon^2 - v_\epsilon^2) \ge \int_M (u_\epsilon^2 + v_\epsilon^2)\Big[\frac{F(\nabla u_\epsilon)}{u_\epsilon} - \frac{F(\nabla v_\epsilon)}{v_\epsilon}\Big]^2 \ge 0. \tag{3.7}$$

Note that u and v are positive on $M \setminus \partial M$ by Theorem 2.4. Moreover, it is obvious that

$$\lim_{\epsilon \to 0^+} \int_M \Big(\frac{u}{u_\epsilon} - \frac{v}{v_\epsilon}\Big)(u_\epsilon^2 - v_\epsilon^2) = 0.$$

By taking a limit $\epsilon \to 0^+$ on both sides of (3.7) and using Fatou's lemma, there is a nonnegative function $c = c(x)$ such that $\nabla v = c\nabla u$ from the Cauchy–Schwartz inequality (2.2) and

$$\frac{F(\nabla u)}{u} = \frac{F(\nabla v)}{v} \tag{3.8}$$

almost everywhere on M. Thus, $v = cu$ almost everywhere on M from (3.8). By continuity, $v = cu$ holds at every point on M.

Obviously, $c(x) > 0$ on $M_u \cap M_v$. Thus, $g_{ij}(x, \nabla v(x)) = g_{ij}(x, \nabla u(x))$ on $M_u \cap M_v$. From this and (2.3), one obtains that $v_{x^j} = cu_{x^j}$, which means that c is a constant on $M_u \cap M_v$. Since M is connected and compact, and $c(x)$ is continuous, c is a positive constant on M. This completes the proof. □

If $u \in V_{\lambda_1}$ and $v \in V_{\lambda_1}$ are two nonpositive eigenfunctions corresponding to λ_1, then $-u \in \overleftarrow{V}_{\lambda_1}$ and $-v \in \overleftarrow{V}_{\lambda_1}$ are two nonnegative eigenfunctions from Lemma 2.1. Thus, from Lemma 3.4, one obtains the following result.

Lemma 3.4′. *Let u and v be two nonpositive Dirichlet eigenfunctions on (M, F) corresponding to λ_1. Then there exists a positive constant c such that $v = cu$.*

For the case when u and v have opposite signs on M, we have the following result.

Lemma 3.5. *Let u and v be two Dirichlet eigenfunctions on (M, F) corresponding to λ_1. Suppose that $u \ge 0$ and $v \le 0$ on M. Then one of the following holds.*

(1) $V_{\lambda_1} = \overleftarrow{V}_{\lambda_1}$. *Equivalently, u and v are linearly dependent on M.*
(2) *u and v are linearly independent. In this case, F must be nonreversible.*

Proof. Note that both u and v are not zero identically on M. Assume that $-v = cu$ for some constant $c > 0$. Then $-v, -u \in V_{\lambda_1}$, which implies $u, v \in \overleftarrow{V}_{\lambda_1}$ by Lemma 2.1. Thus, $u, v \in V_{\lambda_1} \cap \overleftarrow{V}_{\lambda_1}$ and hence $-u, -v \in V_{\lambda_1} \cap \overleftarrow{V}_{\lambda_1}$. For any other eigenfunction $w \in V_{\lambda_1}$, w must be nonnegative or nonpositive on M by Proposition 3.3. Thus, either w, u are linearly dependent or w, v are linearly dependent. In any case, $w \in \overleftarrow{V}_{\lambda_1}$ and vice versa. Hence, $V_{\lambda_1} = \overleftarrow{V}_{\lambda_1}$. Conversely, if $V_{\lambda_1} = \overleftarrow{V}_{\lambda_1}$, then $|u|, |v| \in V_{\lambda_1}$ for any u, $v \in V_{\lambda_1}$. Thus, $|u| = c|v|$ for some constant $c > 0$ by Lemma 3.4. Obviously, u and v are linearly dependent.

If u and v are not linearly dependent, then they are linearly independent and hence $V_{\lambda_1} \ne \overleftarrow{V}_{\lambda_1}$, which implies that F is nonreversible. This finishes the proof. □

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Remark 3.6. If F is reversible, then $|u|, |v| \in V_{\lambda_1}$ for any $u, v \in V_{\lambda_1}$. Thus, $|u| = c|v|$ for some positive constant c by Lemma 3.4. In this case, $V_{\lambda_1} = \overleftarrow{V}_{\lambda_1}$ and all eigenfunctions in V_{λ_1} are linearly dependent. Consequently, V_{λ_1} is a one-dimensional cone, which was proved in [3] in a different way.

Proof of Theorem 1.1. The first-half result follows from Proposition 3.3. Further, if all eigenfunctions in V_{λ_1} have the same sign, then these eigenfunctions are linearly dependent by Lemmas 3.4–3.4′. Thus, V_{λ_1} is one dimensional.

If there exist two eigenfunctions $u, v \in V_{\lambda_1}$ for which $u \geq 0$ and $v \leq 0$, then u and v are linearly independent unless $V_{\lambda_1} = \overleftarrow{V}_{\lambda_1}$ by Lemma 3.5. If $V_{\lambda_1} = \overleftarrow{V}_{\lambda_1}$, then V_{λ_1} is one dimensional by Remark 3.6. If $V_{\lambda_1} \neq \overleftarrow{V}_{\lambda_1}$, then u, v are linearly independent. In this case, V_{λ_1} is a two-dimensional cone and F is nonreversible by Lemma 3.5 again. □

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