SOME REMARKS IN THE FOURIER ANALYSIS

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To KIYOSHI NOSHIRO on his 60th Birthday

In this paper, I should like to add some remarks to my previous note titled "Some results in the Fourier analysis" (Nagoya Math. Journal, Vol. 27, 1966). At first, we shall show that the orthnormal set $\exp[2\pi i(m_1x_1+\cdots+m_nx_n)]$ is complete in the Hilbert space L^2 over the unit cube $E=\{(x_1,\ldots,x_n);\ 0\leqq x_j\leqq 1\ (1\leqq j\leqq n)\}$, where the inner product $\langle f, g\rangle$ for $f, g\in L^2$ over E is defined by

$$\int\underset{E}{\cdots}\int f(x_1,\ldots,x_n)\bar{g}(x_1,\ldots,x_n)dx_1\cdots dx_n.$$

This means that if $f\in L^2$ over E and

$$\langle f, \exp[2\pi i(m_1x_1+\cdots+m_nx_n)]\rangle=0$$

for any integral values $m_1,\ldots,m_n$, then $f=0$ almost everywhere. To prove this, we define the set function

$$F(S)=\int\underset{S}{\cdots}\int f(u_1,\ldots,u_n)\,du_1\cdots du_n.$$

If S is any interval in E, then clearly $F(S)=0$, in virtue of Lemma 2 in the paper cited above. Hence, if S is a closed set in E, accordingly if S is a measurable set in E, then $F(S)=0$. Let $E_n=\left\{(x_1,\ldots,x_n)\ ;\ (x_1,\ldots,x_n)\in E, f(x_1,\ldots,x_n)\geqq\frac{1}{n}\right\}$ and $E'_n=\left\{(x_1,\ldots,x_n); (x_1,\ldots,x_n)\in E, f(x_1,\ldots,x_n)\leqq-\frac{1}{n}\right\}$. Since $0=F(E_n)\geqq\frac{1}{n}m(E_n)$ and $0=F(E'_n)\leqq-\frac{1}{n}m(E'_n)$, we obtain $m(E_n)=m(E'_n)=0$. From this consideration, we have $m\{(x_1,\ldots,x_n);(x_1,\ldots,x_n)\in E, f(x_1,\ldots,x_n)\neq 0\}=0$. We get, therefore, in the usual way, the following

THEOREM 3. *Let $f(x_1,\ldots,x_n)$, $g(x_1,\ldots,x_n)$ be L^2-integrable in the unit cube E and set* $a(m_1,\ldots,m_n)=\langle f, \exp[2\pi i(m_1x_1+\cdots+m_nx_n)]\rangle$, $b(m_1,\ldots,$

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$m_n) = \langle g, \exp[2\pi i(m_1x_1 + \cdots + m_nx_n)]\rangle$. *Then we have*

$$\int \underset{E}{\cdots} \int f(x_1, \ldots, x_n)\bar{g}(x_1, \ldots, x_n)\,dx_1 \cdots dx_n =$$
$$\sum_{m_1} \cdots \sum_{m_n=-\infty}^{\infty} a(m_1, \ldots, m_n)\bar{b}(m_1, \ldots, m_n).$$

Next, we note that if

$$\frac{\partial^{p_1+\cdots+p_n}}{\partial^{p_1}x_1 \cdots \partial^{p_n}x_n} f(x_1, \ldots, x_n) \qquad (0 \leqq p_j \leqq 2)$$

are continuous and L-integrable over X, the whole n-dimensional Euclidean space, then

$$g(v_1, \ldots, v_n) = \int \underset{X}{\cdots} \int e^{2\pi i(v_1u_1+\cdots+v_nu_n)} f(u_1, \ldots, u_n)\,du_1 \cdots du_n$$

is L-integrable over X. Consequently, by Theorem 2 of the paper cited above, the Fourier-transform formula

$$f(x_1, \ldots, x_n) = \int \underset{X}{\cdots} \int e^{-2\pi i(v_1x_1+\cdots+v_nx_n)} g(v_1, \ldots, v_n)\,dv_1 \cdots dv_n$$

holds.

Proof. We take a_m such that

$$a_1 < a_2 < \cdots < a_m \to \infty \qquad (\text{as } m \to \infty),$$

and define

$$l_m(t) = a_m^2 - t)^{2n}\left(t - \frac{a_m^2}{2}\right)^{2n}$$

$$k_m(r) = \frac{1}{C}\int_{a^2m/2}^{r} l_m(t)\,dt, \text{ where } C = \int_{a^2m/2}^{a^2m} l_m(t)\,dt,$$

$$h_m(x_1, \ldots, x_n) = \begin{cases} 1 & x_1^2 + \cdots + x_n^2 \leqq \dfrac{a_m^2}{2} \\ 1 - k_m(x_1^2 + \cdots + x_n^2) & \dfrac{a_m^2}{2} \leqq x_1^2 + \cdots + x_n^2 \leqq a_m^2 \\ 0 & a_m^2 \leqq x_1^2 + \cdots + x_n^2, \end{cases}$$

and set

$$f_m(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)h_m(x_1, \ldots, x_n).$$

By partial integration,

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$$g_m(v_1, \ldots, v_n) = \int \underset{X}{\cdots} \int e^{2\pi i(v_1u_1+\cdots+v_nu_n)} f_m(u_1, \ldots, u_n)\, du_1 \cdots du_n$$

$$= \int_{-a_m}^{a_m} \cdots \int_{-a_m}^{a_m} \Big\{ \Big[\frac{\exp[2\pi i(v_1u_1 + \cdots + v_nu_n)]}{2\pi i v_1} f_m(u_1, \ldots, u_n) \Big]_{-a_m}^{a_m}$$

$$- \int_{-a_m}^{a_m} \frac{\exp[2\pi i(v_1u_1 + \cdots + v_nu_n)]}{2\pi i v_1} \frac{\partial}{\partial u_1} f_m(u_1, \ldots, u_n) \Big\} du_2 \cdots du_n$$

$$= \cdots$$

$$= \frac{(-1)^{q_1+\cdots+q_n}}{(2\pi i)^{q_1+\cdots+q_n}} \int_{-a_m}^{a_m} \cdots \int_{-a_m}^{a_m} \frac{\exp[2\pi i(v_1u_1 + \cdots + v_nu_n)]}{v_1^{q_1} \cdots v_n^{q_n}} \cdot \frac{\partial^{q_1+\cdots+q_n}}{\partial u_1^{q_1} \cdots \partial u_n^{q_n}} f_m(u_1, \ldots, u_n)\, du_1 \cdots du_n,$$

where q_j are taken such that $q_j = 2$ if $|v_j| \geqq 1$ and $q_j = 0$ if $|v_j| < 1$. Since

$$\frac{\partial^{p_1+\cdots+p_n}}{\partial x_1^{p_1} \cdots \partial x_n^{p_n}} h_m(x_1, \ldots, x_n)$$

are uniformly bounded for all m, we obtain

$$g_m(v_1, \ldots, v_n) = 0\left(\frac{1}{|v_1|^{q_1} \cdots |v_n|^{q_n}}\right)$$

By the Lebesgue dominated convergence theorem, sending $m \to \infty$,

$$g(v_1, \ldots, v_n) = 0\Big\{\mathrm{Min}\left(1, \frac{1}{v_1^2}\right) \cdots \mathrm{Min}\left(1, \frac{1}{v_n^2}\right)\Big\},$$

from which we can infer that $g(v_1, \ldots, v_n)$ is L-integrable over X.

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