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CONJUGACY CLASSES OF INVOLUTIONS IN COXETER GROUPS

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In this paper we give an elementary method for classifying conjugacy classes of involutions in a Coxeter group (W, S). The classification is in terms of W-equivalence classes of certain subsets of S.

Our basic reference for Coxeter groups is [2]. Let (W, S) be a Coxeter group and let $\sigma : W \to \mathrm{GL}(E)$ be the geometric realization of (W, S). If J is a subset of S, let W_J be the subgroup of W generated by J and let E_J be the subspace of E spanned by $\{e_s \mid s \in J\}$. We say that the subset J satisfies the *(-1)-condition* if there exists $c_J \in W_J$ such that $\sigma(c_J).x = -x$ for every $x \in E_J$. If J satisfies the (-1)-condition, then J is necessarily finite and the element c_J is uniquely determined and is an involution.

If J is a subset of S, let $J^* = \{e_s \mid s \in J\}$. We say that two subsets J and K of S are *W-equivalent* if there exists $w \in W$ such that $\sigma(w)(J^*) = K^*$.

Our main result is the following:

THEOREM A. *Let (W, S) be a Coxeter group and let $\mathcal{J}$ be the set of all subsets of S which satisfy the (-1)-condition.*

(a) Let $c \in W$ be an involution. Then there exists $J \in \mathcal{J}$

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such that c *is conjugate in* W *to* c_J .

(b) *Let* $J, K \in \mathcal{J}$. *Then the involutions* c_J *and* c_K *are conjugate in* W *if and only if the subsets* J *and* K *are* W*-equivalent.*

We see from Theorem A that the map which associates to each $J \in \mathcal{J}$ the involution c_J determines a bijection between W-equivalence classes of elements of $\mathcal{J}$ and conjugacy classes of involutions in W . Thus the conjugacy classification of involutions in W reduces to the classification of W-equivalence classes in $\mathcal{J}$. Fortunately there exists an easy algorithm for determining W-equivalence classes of arbitrary subsets of S . This algorithm is implicit in the papers of Howlett [5] (for finite Coxeter groups) and Deodhar [4], although it is not explicitly stated by either author. We discuss this algorithm in Section 3.

For the convenience of the reader who is not interested in Coxeter groups *per se*, but who is interested in the special case of Weyl groups, we discuss the case of Weyl groups in some detail. There already exists a complete classification of all conjugacy classes in Weyl groups (see [3] and the references therein). However the classification is quite complicated and it is difficult to extract from the classification a simple description of the conjugacy classes of involutions. In Section 4 we reformulate Theorem A for Weyl groups in terms of root systems and we indicate how one can easily obtain explicit representatives for W-equivalence classes of sets of simple roots.

1. Preliminaries

1.1. *The geometric realization of* (W, S) . (See [2, Chapter V, §4] and [6, §1].) Let (W, S) be a Coxeter group, let $E = E_S$ denote the real vector space $\mathbb{R}^{(S)}$ and let $\{e_s \mid s \in S\}$ be the canonical basis of $\mathbb{R}^{(S)}$. We define a symmetric bilinear form B on $E \times E$ by

$$B(e_s, e_{s'}) = -\cos \frac{\pi}{m(s,s')} ,$$

where $m(s, s')$ is the order of ss' . If $s \in S$, define $\sigma(s) \in \mathrm{GL}(E)$ by

$$\sigma(s).x = x - 2B(e_s, x)e_s \quad \text{for} \quad x \in E .$$

Then $\sigma(s)$ is a reflection, $\sigma(s).e_s = -e_s$ and $\sigma(s)$ is the identity on the hyperplane $H_s = \{x \in E \mid B(e_s, x) = 0\}$. The map $\sigma : S \to \mathrm{GL}(E)$ extends uniquely to a faithful representation $\sigma : W \to \mathrm{GL}(E)$; the representation $\sigma : W \to \mathrm{GL}(E)$ is the *geometric realization* of (W, S) . Each $\sigma(w)$, $w \in W$, leaves the bilinear form B invariant. If $w \in W$ and $x \in E$, then we frequently write $w.x$ for $\sigma(w).x$.

If J is a subset of S , let W_J be the subgroup of W generated by J and let E_J be the subspace of E generated by $\{e_s \mid s \in J\}$. Then (W_J, J) is a Coxeter group, the subspace E_J is W_J-stable, and the corresponding representation $\sigma_J : W_J \to \mathrm{GL}(E_J)$ can be canonically identified with the geometric realization of (W_J, J) .

The following results are known:

1.2. *W is finite if and only if B is positive definite.*

1.3. *Let H be a finite subgroup of W . Then there exists a subset J of S such that W_J is finite and H is conjugate to a subgroup of W_J .*

See [2, Chapter V, §4.8, Theorem 2] for 1.2 and [2, Chapter V, §4, Example 2 (d), p. 130] for 1.3.

1.4 (see [2, Chapter V, §4.8]). *Assume that W is finite. Then (E, B) is a finite-dimensional real hilbert space and $\sigma(s)$, $s \in S$, is the orthogonal reflection in the hyperplane H_s .*

In this case $\sigma(W)$ is a finite group generated by reflections and one can apply to $(\sigma(W), E)$ the results of [2, Chapter V]. If we let

$$C = \{x \in E \mid B(e_s, x) > 0 \text{ for every } s \in S\} ,$$

then C is a chamber of E with respect to the family of hyperplanes

$$\{w.H_s \mid w \in W \text{ and } s \in S\}$$

and the walls of the chamber C are precisely the hyperplanes H_s ,

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$s \in S$.

We let $\Phi = \{w.e_s \mid w \in W \text{ and } s \in S\}$. The results 1.5-1.7 below are proved in [6, §1].

1.5. *Each* $\varphi \in \Phi$ *can be written in the form*

$$\varphi = \sum_{s \in S} a_s(\varphi) e_s ,$$

where the coefficients $a_s(\varphi) \in \mathbb{R}$ *are either all greater than or equal to* 0 *or all less than or equal to* 0 .

We write $\varphi > 0$ (respectively $\varphi < 0$) if all coefficients $a_s(\varphi)$ are greater than or equal to 0 (respectively less than or equal to 0). We set $\Phi^+ = \{\varphi \in \Phi \mid \varphi > 0\}$. If $w \in W$, let $\Phi_w = \{\varphi \in \Phi^+ \mid w.\varphi < 0\}$.

1.6. $\ell(w) = |\Phi_w|$.

1.7. *Let* $w = s_1 \ldots s_k$ *be a reduced expression for* w . *Then*

$$\Phi_w = \{e_{s_k}, s_k(e_{s_{k-1}}), \ldots, s_k \ldots s_2(e_{s_1})\} .$$

If $J \subset S$, let

$$\Phi_J = \{w(e_s) \mid w \in W_J \text{ and } s \in J\} .$$

1.8. *Let* $J \subset S$ *and* $w \in W_J$. *Then* $\Phi_w \subset \Phi_J$.

This follows easily from 1.7 and [2, Chapter IV, §1.8, Proposition 7].

We say that (W, S) satisfies the *(-1)-condition* if there exists $c \in W$ such that $c.x = -x$ for every $x \in E$. A subset J of S satisfies the (-1)-condition if the Coxeter group (W_J, J) satisfies the (-1)-condition.

1.9. *(a) If* (W, S) *satisfies the (-1)-condition, then* W *is finite.*

(b) Assume that (W, S) *is irreducible. Then* (W, S) *satisfies the (-1)-condition if and only if the center of* W *is not equal to* $\{1\}$.

(c) Let S *be finite and let* $(W_1, S_1), \ldots, (W_r, S_r)$ *be the irreducible components of* (W, S) . *Then* (W, S) *satisfies the (-1)-*

condition if and only if each (W_i, S_i) *satisfies the* (-1)*-condition.*

The proofs of *(a)* and *(b)* follow from Exercises 2 (b) and 3 (b) of [2, Chapter V, §4, p. 130]. The proof of *(c)* is trivial.

If $J \subset S$ satisfies the (-1)-condition, let c_J be the unique element of W_J such that $c_J.x = -x$ for every $x \in E_J$. Clearly c_J is an involution and E_J is the (-1)-eigenspace of c_J on E.

1.10. *Let* $J \subset S$ *satisfy the* (-1)*-condition. Then* $\Phi_J = E_J \cap \Phi$.

Proof. It is clear from the definitions that $\Phi_J \subset E_J \cap \Phi$. For the reverse inclusion it will suffice to show that $E_J \cap \Phi^+ \subset \Phi_J$. Let $\varphi \in E_J \cap \Phi^+$. Then $c_J.\varphi = -\varphi < 0$. Hence $\varphi \in \Phi_{c_J}$. But, by 1.8, $\Phi_{c_J} \subset \Phi_J$. This proves 1.10.

If J is a subset of S, by the Coxeter graph of J we mean the Coxeter graph of the Coxeter group (W_J, J). We say that a subset J_1 of J is a connected component of J if J_1 is the set of vertices of a connected component of the Coxeter graph of J.

1.11. The finite Coxeter groups are classified in [2, Chapter VI, §4.1, Theorem 1]. It follows from the classification that an irreducible Coxeter group (W, S) is finite if and only if its Coxeter graph is of one of the following types:

A_n, B_n $(n \geq 2)$, D_n $(n \geq 4)$, E_6, E_7, E_8, F_4, G_2, H_3, H_4, or
$I_2(p)$ $(p = 5$ or $p \geq 7)$.

1.12. *Let* (W, S) *be an irreducible Coxeter group. Then* (W, S) *satisfies the* (-1)*-condition if and only if its Coxeter graph is of one of the following types:* A_1, B_n, D_{2n}, E_7, E_8, G_2, F_4, H_3, H_4, *or* $I_2(2p)$.

Proof. By 1.9 we may assume that W is finite, hence that the Coxeter graph of (W, S) is of one of the types listed in 1.11. For each of these graphs, one must determine whether $-1 \in \sigma(W) \subset GL(E)$. For the Weyl groups $A_n, \ldots, G_2$, this information is given in [2, Planches I-IX].

For types H_3, H_4 and $I_2(p)$ it can be checked directly.

1.13. *Let L be a subset of S such that W_L is finite. Then there exists a unique element $w_L \in W_L$ such that if $\varphi \in \Phi_L \cap \Phi^+$, then $w_L(\varphi) < 0$. The element w_L is the longest element of W_L and is an involution. Moreover w_L maps $L^* = \{e_s \mid s \in L\}$ onto $-L^*$.*

See [2, Chapter V, §4, Exercise 2, p. 130] for these results.

If W_L is finite, we define a permutation $\iota_L : L \to L$ as follows: if $s \in L$, then $w_L(e_s) = -e_{s'}$, for some $s' \in L$; we set $\iota_L(s) = s'$. Thus ι_L is the permutation of L which corresponds to the permutation of L^* induced by $-w_L$. In particular ι_L is of order less than or equal to 2. It follows from the definitions that ι_L is the identity permutation if and only if L satisfies the (-1)-condition.

1.14. *Let $L \subset S$ be such that (W_L, L) is finite and irreducible. Then a necessary and sufficient condition that ι_L is not the identity permutation of L is that the Coxeter graph of L is of one of the following types: A_n $(n > 1)$, D_{2n+1}, E_6, $I_2(2p+1)$.*

Proof. This follows from 1.11 and 1.12.

For L of types A_n $(n > 1)$, D_{2n+1}, and E_6, a description of ι_L is given in [2, p. 251, p. 257 and p. 261]. If L is of type $I_2(2p+1)$, then ι_L interchanges the two vertices of the diagram.

2. Proof of Theorem A

2.1. Proof of *(a)*. Let $c \in W$ be an involution. By 1.3 there exists a subset K of S such that W_K is finite and c is conjugate to an element of W_K. Hence we may assume that W is finite and that B is positive definite on E. We will identify W with $\sigma(W)$, which is a finite subgroup of $O(E)$ generated by reflections. Let $H = \{w.H_s \mid w \in W \text{ and } s \in S\}$. If $\varphi \in \Phi$, let s_φ denote the reflection

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in the hyperplane H_φ orthogonal to φ . Clearly $s_\varphi \in W$ and $H = \{H_\varphi \mid \varphi \in \Phi\}$.

Let E_- (respectively E_+) denote the -1 eigenspace (respectively $+1$ eigenspace) of c on E ; thus E is the orthogonal direct sum of E_- and E_+ . Let $\Phi_c = \Phi \cap E_-$. By a standard result on reflection groups [2, Chapter V, §3.3, Proposition 2], c may be written as a product of reflections s_φ , $\varphi \in \Phi_c$. It follows easily from this that Φ_c spans E_- and that $E_+ = \bigcap_{\varphi \in \Phi_c} H_\varphi$.

We consider chambers and facets of E with respect to the family of hyperplanes H (we follow the terminology of [2, Chapter V]). Since E_+ is the intersection of hyperplanes in the family H , E_+ is a union of facets of E . Let F be a facet of E which is contained in E_+ and is a relatively open subset of E_+ . Let

$$C = \{x \in E \mid B(x, e_s) > 0 \text{ for every } s \in S\} .$$

By 1.4, C is a chamber of E . If J is a subset of S , let

$$C_J = \{x \in E \mid B(x, e_s) = 0 \text{ for } s \in J \text{ and } B(x, e_s) > 0 \text{ for } s \in (S-J)\} .$$

Let $\overline{C}$ denote the closure of C . Then each C_J is a facet contained in $\overline{C}$ and $\overline{C}$ is the disjoint union of the C_J's , $J \subset S$. Since $\overline{C}$ is a fundamental domain for the action of W on E , there exists $w \in W$ and $J \subset S$ such that $w(F) = C_J$. Now C_J is, by definition, an open subset of $E_J^\perp$. Thus w maps E_+ onto $E_J^\perp$. Hence w maps E_- onto E_J . It follows immediately that J satisfies the (-1)-condition and that $c_J = wcw^{-1}$. This proves *(a)*.

2.2. Proof of *(b)*. For this part of the proof we can no longer assume that W is finite and that B is positive definite. Let $J \in \mathcal{J}$, let $d_J = |J|$ and let $A_J = \bigcap_{s \in J} H_s$. Now c_J is the identity map on A_J and acts by multiplication by -1 on E_J . Hence $A_J \cap E_J = \{0\}$. Since $\dim E_J = d_J$ and since A_J is of codimension at most d_J in E , we see

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that E is the direct sum of A_J and E_J. Consequently E_J (respectively A_J) is the -1 eigenspace (respectively $+1$ eigenspace) of c_J on E.

Now let $J, K \in \mathcal{J}$. Then by 1.2 and 1.9, the restriction of B to $E_J \times E_J$ (respectively to $E_K \times E_K$) is positive definite and the restriction of W_J (respectively W_K) to E_J (respectively E_K) is a finite reflection group. If there exists $w \in W$ such that $w(J^*) = K^*$, then it is clear that $wc_Jw^{-1} = c_K$. Assume conversely that there exists $w \in W$ such that $wc_Jw^{-1} = c_K$. Since E_J (respectively E_K) is the -1 eigenspace of c_J (respectively c_K), it is clear that $w(E_J) = E_K$. It follows from 1.10 that $w(\Phi_J) = \Phi_K$. Let $H_J = \{H_\varphi \cap E_J \mid \varphi \in \Phi_J\}$ and let $H_K = \{H_\varphi \cap E_K \mid \varphi \in \Phi_K\}$. Then H_J (respectively H_K) is a family of hyperplanes in E_J (respectively E_K) and $w(H_J) = H_K$. Let

$$D_J = \{x \in E_J \mid B(x, e_s) > 0 \text{ for every } s \in J\}$$

and let D_K be defined similarly. Then by 1.4 (as applied to the Coxeter group (W_J, J)), D_J is a chamber of E_J with respect to the family of hyperplanes H_J and the set of walls of D_J is $\{H_s \cap E_J \mid s \in J\}$. Similarly for D_K. Since $w(H_J) = H_K$, $w(D_J)$ is a chamber of E_K. Since W_K acts transitively on the set of chambers of E_K (by [2, Chapter V, §3.3, Theorem 1]), there exists $w' \in W_K$ such that $w'w(D_J) = D_K$. Let $s \in J$. Since H_s is a wall of D_J, we see that $w'w(H_s)$ is a wall of $w'w(D_J) = D_K$. Hence there exists $s' \in K$ such that $w'w(e_s) = \pm e_{s'}$. If $x \in D_J$, then $B(x, e_s) > 0$ and hence $B(w'w(x), w'w(e_s)) > 0$. Since $w'w(x) \in D_K$ we see that $w'w(e_s) = e_{s'}$. Thus $w'w(J^*) \subset K^*$. Since $|J| = |K|$, we have $w'w(J^*) = K^*$. This proves *(b)* and completes the proof of Theorem A.

3. An algorithm for determining W-equivalent subsets

In this section we will give an elementary algorithm, which is essentially due to Howlett [5] and Deodhar [4], for determining when two subsets J and K of S are W-equivalent.

If $J \subset S$ and $s \in S$, we let $L(s, J)$ denote the connected component of $J \cup \{s\}$ which contains s. We let $A(J)$ be the set of $s \in (S-J)$ such that, letting $L = L(s, J)$, the Coxeter group (W_L, L) is a finite Coxeter group which does not satisfy the (-1)-condition. It follows from 1.11 and 1.12 that, for $s \in (S-J)$, we have $s \in A(J)$ if and only if the Coxeter graph of $L(s, J)$ is of one of the following types: A_n $(n > 1)$, D_{2n+1}, E_6 or $I_2(2p+1)$.

Let $J \subset S$, let $s \in A(J)$, let $L = L(s, J)$ and let $s' = \iota_L(s)$. We define a subset $K(s, J)$ of S by

$$K(s, J) = (J \cup \{s\}) - \{s'\} .$$

We say that J and $K = K(s, J)$ are related by an *elementary equivalence* and we denote this by $J \vdash K$, or, if reference to s is wanted, by $J \vdash_s K$. Since ι_L is of order 2 it is clear that $J \vdash K$ implies that $K \vdash J$.

If L is a subset of S such that (W_L, L) is finite and irreducible, if $s \in L$ and if $M = (L-\{s\})$, we define an element $v(s, L) \in W_L$ by $v(s, L) = w_L w_M$, where w_L and w_M are as defined in 1.13.

LEMMA 3.1. *Let* $J \subset S$, *let* $s \in A(J)$, *let* $L = L(s, J)$ *and let* $v = v(s, L)$. *Then* $v(J^*) = K(s, J)^*$.

Proof. Let $M = (L-\{s\})$ and let $J' = \{t \in J \mid t \notin L\}$. Then J is the disjoint union of M and J' and $K = K(s, J)$ is the disjoint union of J' and $(L-\{\iota_L(s)\}) = \iota_L(M)$. It is clear that if $t \in J'$, then $v(e_t) = e_t$. We have $w_M(M^*) = -M^*$ and hence

$$v(M^*) = -w_L(M^*) = \iota_L(M)^* .$$

Thus $v(J^*) = K^*$.

We see from Lemma 3.1 that if J and K are related by an elementary equivalence, then they are W-equivalent. The following proposition shows that every W-equivalence can be obtained by a finite sequence of elementary equivalences.

PROPOSITION 3.2 (Howlett, Deodhar). *Let J and K be subsets of S. Then J and K are W-equivalent if and only if they can be connected by a finite sequence of elementary equivalences:*

$$J = J_0 \vdash J_1 \vdash \dots \vdash J_n = K .$$

Proof. If J and K are connected by a finite sequence of elementary equivalences, then they are W-equivalent by Lemma 3.1. Let $w \in W$ be such that $w(J^*) = K^*$. We need to prove that J and K are connected by a finite sequence of elementary equivalences. It is shown in [4, §5] that there exists a sequence of elements $s_0, \dots, s_{n-1}$ and a sequence $J_0, \dots, J_n$ of subsets of S such that the following conditions hold:

(i) $J_0 = J$ and $J_n = K$;

(ii) if $L_j = L(s_j, J_j)$, then W_{L_j} is a finite group and $v(s_j, L_j)(J_j^*) = J_{j+1}^*$; and

(iii) $w = v(s_{n-1}, L_{n-1}) \dots v(s_0, L_0)$.

For each index j there are two possible cases:

(a) $s_j \notin A(J_j)$;

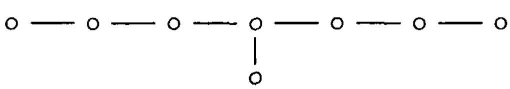
(b) $s_j \in A(J_j)$.

In case (a) we have $J_{j+1} = J_j$ and in case (b), $J_j \vdash_{s_j} J_{j+1}$. Thus we see that $J_0 = J$ and $J_n = K$ are connected by a finite sequence of elementary equivalences. This proves Proposition 3.2.

3.3 REMARKS. (a) Let $J \subset S$, let $s \in A(J)$ and let $v = v(s, L(s, J))$. Then $v(J^*) = K(s, J)^*$. The bijective mapping $J^* \to K(s, J)^*$ given by v may be a bit complicated. However, in applying Proposition 3.2 to determine W-equivalence classes of subsets of S, one does not need any knowledge of this bijection. One only needs to describe the *set* $K(s, J)$, and this is easy to do by inspection of the Coxeter diagram of S .

(b) If $|S|$ is not too large (say $|S| \le 10$), it is quite easy to apply the algorithm of Proposition 3.2 to get a complete classification of W-equivalence classes of elements of S . As a test of the algorithm, we did this for the affine Weyl groups associated to root systems of types E_6, E_7 , and E_8 and the computations were quite easy to carry out by hand. A few particular cases for type E_7 are discussed below.

3.4. AN EXAMPLE. In order to illustrate the algorithm for W-equivalence classes given by Proposition 3.2, we discuss a few cases involving a non-trivial example. Let (W, S) be the affine Weyl group corresponding to a root system of type E_7 . The Coxeter diagram of S is



(see [2, p. 265]). We label the vertices of the Coxeter diagram as follows:

$$\begin{matrix} s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ & & & s_7 & & & \end{matrix} .$$

Let J_3 denote the subset $\{s_7, s_4, s_6\}$. The Coxeter graph of J_3 is of type $3A_1 = A_1 + A_1 + A_1$. We have $A(J_3) = \{s_3, s_5\}$. If $s \in A(J_3)$, it is easy to see that $J_3 \vdash_s J_3$. Hence J_3 is not W-equivalent to any other subset of S . A similar argument shows that $J_3' = \{s_0, s_2, s_7\}$ is not W-equivalent to any other subset of S . On the other hand the subsets $K_3 = \{s_0, s_2, s_4\}$ and $K_3' = \{s_2, s_4, s_7\}$ are connected by the following sequence of elementary equivalences:

$$\{s_2, s_4, s_7\} \vdash_{s_5} \{s_2, s_5, s_7\} \vdash_{s_1} \{s_1, s_5, s_7\} \vdash_{s_3} \{s_1, s_3, s_5\}$$
$$\vdash_{s_0} \{s_0, s_3, s_5\} \vdash_{s_2} \{s_0, s_2, s_5\} \vdash_{s_4} \{s_0, s_2, s_4\} .$$

A few more arguments show that there are exactly 3 W-equivalence classes of subsets J of S of type $3A_1$. As representatives one can take J_3, J_3' and K_3. Similar arguments allow one to determine all W-equivalence classes of subsets of S. For example, there are exactly 20 W-equivalence classes of subsets of S which satisfy the (-1)-condition. Thus, by Theorem A, there are 20 conjugacy classes of involutions in W.

4. Involutions in Weyl groups

Involutions in Weyl groups seem to play a special role in a number of problems involving semisimple Lie groups and Lie algebras. For the convenience of the reader who is interested in semisimple Lie groups and Lie algebras, but who is not familiar with the general theory of Coxeter groups, we shall reformulate Theorem A in terms of root systems. Our basic reference for root systems is [2, Chapter VI]. Let R be a reduced root system in a finite dimensional real vector space E, let $W = W(R)$ be the Weyl group of R and let E be given a W-invariant positive definite inner product. If $\alpha \in R$, then s_α denotes the orthogonal reflection in the hyperplane orthogonal to α. Let B be a base of R. If $J \subset B$, let E_J be the subspace of E spanned by J, let $R_J = R \cap E_J$ and let W_J be the subgroup of W generated by $\{s_\alpha \mid \alpha \in J\}$. Then R_J is a root system in E_J, J is a base of R_J and the restriction map $w \mapsto w|_{E_J}$ is an isomorphism of W_J onto $W(R_J)$. We say that two subsets J and K of B are *W-equivalent* if there exists $w \in W$ such that $w(J) = K$. We say that the root system R satisfies the *(-1)-condition* if $-1 \in W(R)$. A subset J of B satisfies the (-1)-condition if the root system R_J satisfies the (-1)-condition. If $J \subset B$ satisfies the (-1)-condition, then we define an involution $c_J \in W_J$ by: $c_J(x) = -x$ if $x \in E_J$ and $c_J(x) = x$ if $x \in E_J^\perp$. For the case of Weyl groups, Theorem A becomes:

THEOREM A'. *Let* $\mathcal{J}$ *denote the set of all subsets of* B *which satisfy the* (-1)*-condition.*

(a) *Every involution* $c \in W$ *is conjugate to some* c_J, $J \in \mathcal{J}$.

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(b) If $J, K \in \mathcal{J}$, then c_J and c_K are conjugate in W if and only if J and K are W-equivalent.

The proof of Theorem A' is simpler than that of Theorem A. It follows easily from [2, Chapter VI, §1.7, Proposition 24] and [2, Chapter V, §3.3, Proposition 2].

4.1. **Classification of W-equivalence classes.** Assume that the root system R is irreducible. We say that two subsets J and K of B are *isomorphic* if there exists an isometry $\eta : E_J \to E_K$ such that $\eta(R_J) = R_K$. If R has only one root length, then J and K are isomorphic if and only if J and K are of the same type, that is if the Dynkin diagrams corresponding to the root systems R_J and R_K are of the same type. If R has more than one root length, then this is no longer the case.

A complete classification of W-equivalence classes of subsets of B is given in [1, pp. 4-5]. (However, see Remark 4.2 below.) It turns out that in most cases, two subsets of B are W-equivalent if and only if they are isomorphic. The only cases in which two subsets J and K of B satisfying the (-1)-condition are isomorphic, but not W-equivalent, are the following:

(a) R of type E_7. There are two W-equivalence classes of subsets of B of type $3A_1 = A_1 + A_1 + A_1$. As representatives for these W-equivalence classes, we may take $J = \{\alpha_1, \alpha_4, \alpha_6\}$ and $K = \{\alpha_2, \alpha_5, \alpha_7\}$. (The numbering of the roots is as in [2, Planche VI, p. 265].)

(b) R of type D_n $(n \geq 4)$. (i) If n is even, say $n = 2m$, there are three W-equivalence classes of subsets of type mA_1. As representatives for these three equivalence classes, we may take:
$J_m = \{\alpha_1, \alpha_3, \ldots, \alpha_{2m-3}, \alpha_{2m-1}\}$; $K_m = \{\alpha_1, \alpha_3, \ldots, \alpha_{2m-5}, \alpha_{2m-1}, \alpha_{2m}\}$; and $L_m = \{\alpha_1, \alpha_3, \ldots, \alpha_{2m-3}, \alpha_{2m}\}$.

(ii) If p is an integer such that $p \geq 2$ and $2m+1 \leq n$, there are

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two W-equivalence classes of type pA_1. As representatives of these two W-equivalence classes, we may choose: $J_p = \{\alpha_1, \alpha_3, \ldots, \alpha_{2p-1}\}$; and $K_p = \{\alpha_1, \alpha_3, \ldots, \alpha_{2p-5}, \alpha_{n-1}, \alpha_n\}$. (The roots are numbered as in [2, Planche IV, p. 257].)

4.2 REMARK. There is a minor error in [1, Proposition 6.3, p. 4] for type D_n. The authors seem to have overlooked the fact that the subsets J_p and K_p (and the subsets J_m and K_m) above are isomorphic. It is clear that they are not W-equivalent. In [1], K_p (respectively K_m) is apparently considered as a subset of type $(p-2)A_1 + D_2$ (respectively $(m-2)A_1 + D_2$) instead of type pA_1 (respectively mA_1).

4.3 REMARK. Using the algorithm of Section 3, it is a straightforward matter to check the results on W-equivalence classes listed in [1, pp. 4-5].

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