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A NOTE ON COMPACTIFYING ARTINIAN RINGS

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In this note a number of compactifications are discussed within the class of artinian rings. In [**1**] the following was proved:

THEOREM. *For an artinian ring R the following are equivalent:*
(1) *R is equationally compact.*
(2) *$R^+ \simeq B \oplus P$, where B is a finite group, P is a finite direct sum of Prüfer groups, and $R \cdot P = P \cdot R = \{0\}$.*
(3) *R is a retract of a compact topological ring.*

Here, we extend this result to show that the following are also equivalent to (1) for an artinian ring R:

(4) *R is a subring of a compact topological ring.*
(5) *R is a subring of an equationally compact ring.*
(6) *R is quasi-compactifiable.*

Involved in the present discussion are a number of ideas appearing in the proof of the above theorem. To avoid superfluous discussions we will refer at times to arguments which the reader may find in [**1**]. We refer also to [**1**] for terminology.

PROPOSITION 1. *Let R be a ring satisfying $n \cdot R = \{0\}$. If R is quasi-compactifiable, then so is $\mathbf{Z}_n * R$.*

Proof. By [**2**, Theorem 4] we can choose $S \in c(R) \cap \text{Pos}\,(R)$. Then obviously $n \cdot S = \{0\}$, and we claim that $\mathbf{Z}_n * S \in c(\mathbf{Z}_n * R)$. The proof is now totally analogous to that of [**1**, Proposition 4].

PROPOSITION 2. *A torsion-free artinian ring with more that one element is never quasi-compactifiable.*

Proof. In the proof of [**1**, Lemma 5] we take I larger in cardinality than any ring S which quasi-compactifies R; this yields $R = (0)$.

We next settle quasi-compactifiability in unital artinian rings. To do this we first need information on what happens when passing to homomorphic images:

PROPOSITION 3. *Let R and S be rings such that R is noetherian with identity, and let A be an ideal of S. Then $S \in c(R)$ implies $S/A \in c(R/R \cap A)$.*

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Proof. Set $A' = A \cap R$ and assume that $S \in c(R)$. Let Σ be a set of polynomial equations with constants in R/A' and finitely solvable in R/A'. Without loss of generality Σ can be assumed to be of the form

$$\{\phi_i = 0; i \in I\},$$

where ϕ_i is a polynomial with constants in R/A'. "Lift" each ϕ_i to ϕ_i' by replacing all constants by representatives in R. Since R is noetherian with identity there exist $a_1, \ldots, a_n$, elements of A', such that $A' = Ra_1 + \ldots + Ra_n$. Then the system of equations

$$\Sigma' = \{\phi_i' = z_i; i \in I\} \cup \{z_i = z_{i1}a_1 + \ldots + z_{in}a_n; i \in I\}$$

(where the z_i's and z_{jk}'s are assumed to be variables not occurring in Σ), has constants in R and is finitely solvable in R (just "lift" solutions of members of Σ). Hence Σ' is solvable in S. But the z_i's are forced to take on values in A, and thus any solution of Σ' taken modulo A yields a solution of Σ in S/A.

PROPOSITION 4. *Every quasi-compactifiable artinian ring R with identity is finite.*

Proof. Again we may choose $S \in c(R) \cap \text{Pos}(R)$. We show first that $R/J(R)$ is quasi-compactifiable ($J(R)$ denotes the Jacobson radical). Since R is also noetherian, this would be accomplished by Proposition 3 provided an ideal A of S can be found satisfying $R \cap A = J(R)$. Now $J(R) = Ra_1 + \ldots + Ra_n$ for suitable $a_1, \ldots, a_n$, because R is noetherian with identity. We set

$$A = Sa_1 + \ldots + Sa_n$$

and claim that A does the job. Now the two-sidedness of the left ideal $Ra_1 + \ldots + Ra_n$ is expressed by the positive sentence

$$\Phi = (\forall x_1) \ldots (\forall x_n)(\forall y)(\exists z_1) \ldots (\exists z_n)((x_1a_1 + \ldots + x_na_n) \cdot y = z_1a_1 + \ldots + z_na_n).$$

Therefore Φ must be true in S, which implies that the left ideal A is also two-sided. Now obviously $A \cap R$ contains $J(R)$. To show the other inclusion, recall that the artinian ring R has $J(R)$ as its largest nilpotent ideal; suppose $J(R)^m = (0)$. This implies the truth in R of the positive sentence

$$\Psi = (\forall x_{ij})_{i=1,\ldots,n,j=1,\ldots m}\left(\prod_j \left(\sum_i x_{ij}a_i\right) = 0\right).$$

But then Ψ must be true in S, which in turn implies the nilpotency of A; hence $A \cap R$ is a two-sided nilpotent ideal of R and therefore contained in $J(R)$. Thus Proposition 3 guarantees that $R/J(R)$ is also quasi-compactifiable. But $R/J(R)$ is a semisimple artinian ring and therefore finite by [**1**, Proposition 9]. Moreover, since every left ideal of R is finitely generated, [**1**, Lemma 3]

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and induction imply that $R/J(R)^n$ is finite for every n. Thus $R/J(R)^m \simeq R$ is also finite and the proof is complete.

We are ready to prove the equivalence of conditions (1)-(6) stated at the outset. The implications $(3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6)$ are obvious, so it suffices to show $(6) \Rightarrow (2)$. The proof begins analogously to that of [**1**, Theorem 3, (i) $\Rightarrow$ (ii)]: R is the sum of its torsion ideal and some torsion-free ideal T; then T is quasi-compactifiable too, and by Proposition 2, $T = (0)$, i.e., R is torsion. Let $R^+ = B \oplus P$ be a decomposition of R^+ into reduced and divisible parts B and P. By [**1**, Proposition 9], $R \cdot P = P \cdot R = \{0\}$, and P is a finite sum of Prüfer groups.

It remains to show the finiteness of B. Let $\bar{B}$ be the subring of R generated by B. If A is an arbitrary left ideal of $\bar{B}$, then A is also a left ideal of R, because

$$R \cdot A = (P + B) \cdot A = P \cdot A + B \cdot A \subseteq (0) + \bar{B} \cdot A \subseteq A.$$

Thus $\bar{B}$ inherits the descending chain condition from R, i.e., $\bar{B}$ is artinian and, of course, quasi-compactifiable. We claim that $\bar{B}^+$ is a bounded torsion group. Since $\bar{B}$ is artinian, the family of ideals $\{m \cdot \bar{B}; m \in \mathbf{N}\}$ has a smallest element, say $n \cdot \bar{B}$, which is of course additively a divisible subgroup of R^+. Hence $n \cdot \bar{B} \subseteq P$. Since $n \cdot B \subseteq B$ we obtain

$$n \cdot B \subseteq n \cdot \bar{B} \cap B \subseteq P \cap B = \{0\},$$

and thus B is bounded torsion. That the underlying group of $\bar{B}$, the ring generated by B, is also bounded torsion, is then elementary. Thus Proposition 1 applies and $\mathbf{Z}_n * \bar{B}$ is quasi-compactifiable. But $\mathbf{Z}_n * \bar{B}$ is artinian because the $\mathbf{Z}_n * \bar{B}$-modules $\bar{B}$ and $\mathbf{Z}_n * \bar{B}/\bar{B}$ are artinian; thus $\mathbf{Z}_n * \bar{B}$ is finite by Proposition 4 and so, of course, is B. The proof is complete.

References

1. D. K. Haley, *Equationally compact artinian rings*, Can. J. Math. *25* (1973), 273–283.
2. G. H. Wenzel, *On* $(\mathfrak{S}, \mathfrak{A}, \mathfrak{m})$*-atomic compact relational systems*, Math. Ann. *194* (1971), 12–18.

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