CORRESPONDENCE.

ON THE VALUATION OF POLICIES SUBJECT TO CONTINGENT DEBTS.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—In Mr. Sunderland's valuable paper in the *Journal* (vol. xxix, p. 419) on the subject of the issue of policies subject to a contingent debt in lieu of a rating-up in the ordinary manner, there

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does not appear to be any reference to the question as to the proper method of valuing such policies at an investigation period or for purposes of surrender. I have therefore thought it might be interesting to examine this point, with the view of determining whether or not it would be sufficient for all practical purposes to value policies of this description, as if they were in force for their full face-value at the actual age of the assured. Mr. Sunderland deals only with the case where the contingent debt is a *decreasing* one, but I propose to consider also the case in which the debt is fixed at a *constant* amount for a given number of years.

The latter of these cases being the simpler, I shall take it first in order. The debt in this case is chargeable against the sum assured only in the event of death happening within a certain number of years, which may be arbitrarily fixed, but which is generally conveniently taken as the time in which the premiums payable will amount to the sum assured at the rate of interest assumed in the company's periodical valuations. This term is sometimes spoken of as the "probationary term."

Then if P'_x=the office premium at the *actual* age, κ_{x+n}=the loading per unit contained in the office premium for the rated-up age, t=the number of years composing the probationary term, and X=the constant contingent debt for a policy of 100 on a life of actual age x taken at an advance of n years; and if we further assume the life in question to be for all purposes exactly equivalent to one really aged $x+n$, we shall have the value of the benefit without any loading whatever for expenses, contingencies, or profits

$$=\frac{(100-\mathrm{X})(\mathrm{M}_{x+n}-\mathrm{M}_{x+n+t})+100\mathrm{M}_{x+n+t}}{\mathrm{D}_{x+n}} \quad . \quad . \quad . \quad (1)$$

$$=\frac{100\mathrm{M}_{x+n}-\mathrm{X}(\mathrm{M}_{x+n}-\mathrm{M}_{x+n+t})}{\mathrm{D}_{x+n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$=\frac{(100-\mathrm{X})\mathrm{M}_{x+n}+\mathrm{X}\mathrm{M}_{x+n+t}}{\mathrm{D}_{x+n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Multiplying (3) by $1+\kappa_{x+n}$, the value of the loaded benefit will be

$$\frac{(1+\kappa_{x+n})\{(100-\mathrm{X})\mathrm{M}_{x+n}+\mathrm{X}\mathrm{M}_{x+n+t}\}}{\mathrm{D}_{x+n}},$$

and the value of the annual premiums receivable being

$$\frac{100\mathrm{P}'_x\,\mathrm{N}_{x+n-1}}{\mathrm{D}_{x+n}},$$

we have $(1+\kappa_{x+n})\{(100-\mathrm{X})\mathrm{M}_{x+n}+\mathrm{X}\mathrm{M}_{x+n+t}\}=100\mathrm{P}'_x\mathrm{N}_{x+n-1}$,

from which $$\mathrm{X}=\frac{100\left(\mathrm{M}_{x+n}-\dfrac{\mathrm{P}'_x}{1+\kappa_{x+n}}\mathrm{N}_{x+n-1}\right)}{\mathrm{M}_{x+n}-\mathrm{M}_{x+n+t}} \quad . \quad . \quad . \quad (4)$$

If net premiums only were taken account of, the formula for the debt would become

$$\mathrm{X}=\frac{100(\mathrm{M}_{x+n}-\mathrm{P}_x\mathrm{N}_{x+n-1})}{\mathrm{M}_{x+n}-\mathrm{M}_{x+n+t}} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

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P_x here representing the *net* premium at age x. Further, if we suppose the office premiums to be loaded with an equal percentage at all ages, the amount of the debt will still be that shown by formula (5).

Now, in order to illustrate the practical effect of this method in actual working I shall take an example derived from the published rates of premium of one of the Australian offices, these rates *not* being formed by the addition of an equal percentage of the net premiums at all ages. Take the case of a life aged 30 which has been accepted at the rate for age 35, and where it is proposed to pay the premium for the lower age with a debt upon the policy which is to remain constant during a term equal to the number of years in which the premium actually payable will amount at 4 per-cent interest to the sum assured. This probationary period will be, in the case supposed, equal to 25 years, the rates of premium being at age 30 £2. 8*s*. 2*d*., and at age 35 £2. 15*s*. 4*d*. per £100. The rate of loading at age 35, the net premium being taken by the H^M 4 per-cent table, is ·405 per unit. Therefore $(1+\kappa_{35})$ in formula (4) becomes 1·405, and the resulting value of $X=23{\cdot}614$. In actual practice this would probably be taken to the nearest integer, but in what follows I shall assume that the exact value is adhered to.

The question now arises how such policies should be valued at the periodical investigations. Their value, on the common assumption that a rated-up life is always to be considered as equal to a select (or average) life so many years older, will be found as follows. In the first place it will be necessary to determine the net premium for the benefit actually granted, that is, in the case just supposed, an assurance of 76·386 for the first 25 years, together with a deferred assurance of 100 after the expiry of that term. Thus we have

$$\frac{76{\cdot}386(M_{35}-M_{60})+100M_{60}}{N_{34}}$$

$$=\frac{100\,M_{35}-23{\cdot}614(M_{35}-M_{60})}{N_{34}}=1{\cdot}714=\pi' \text{ say.}$$

This is necessarily equal to $\frac{100\ P'_{30}}{1+\kappa_{35}}=\frac{2{\cdot}408}{1{\cdot}405}=1{\cdot}714$, for the transaction is equivalent to insuring the life at a rate of premium which, though charged as for age 30, contains the rate of loading involved in the rate for age 35. It follows that the value of the policy after m years $(m<t)$ will be

$$\frac{76{\cdot}386(M_{35+m}-M_{60})+100\,M_{60}}{D_{35+m}}-(1+a_{35+m})\times 1{\cdot}714.$$

Of course, when the probationary period has just expired this will become

$$\frac{100\,M_{60}}{D_{60}}-(1+a_{60})\times 1{\cdot}714,$$

and similarly for subsequent years.

The numerical values derived from the above formula are given in col. (3) of the subjoined table, where they are also compared with the values of an ordinary policy for 100 on a life aged 30 at entry, and further with those of the same policy valued at the rated-up age, 35. It will be seen from this table that to value such policies as being of their full ultimate amount and at the real age of the assured will considerably understate their true values, while to value them as at the assumed age gives values which at first are slightly too great and afterwards somewhat too small. In practice such a valuation as at the rated-up ages would no doubt be sufficiently accurate, except perhaps for very long durations or where the rating-up is heavy. It will be noticed that even after the expiry of the probationary term the value of the policy still remains greater than that of an ordinary policy on a life of similar rated age. This must necessarily be so, seeing that the net premium for the benefit is *less* than in the case of the ordinary policy. In fact the true value then exceeds that of such a policy by an amount equal to $(\pi-\pi')(1+a_{x+n+m})$, m being the duration of the policy. For instance, in the above example we have $(1{\cdot}969-1{\cdot}714)(1+a_{60})=2{\cdot}667$, the difference between the values in col. 2 and col. 3 for duration 25 years.

To come now to the case where the debt instead of being constant is a decreasing one, this being the case considered by Mr. Sunderland in his paper above referred to. Here if X′ be put for the initial debt we shall have the equation

$$(1+\kappa_{x+n})\left\{(100-\mathrm{X}')\mathrm{M}_{x+n}+\frac{\mathrm{X}'}{t}(\mathrm{R}_{x+n+1}-\mathrm{R}_{x+n+t+1})\right\}$$
$$=100\mathrm{P}'_x.\mathrm{N}_{x+n-1},$$

whence

$$\mathrm{X}'=\frac{100\left(\mathrm{M}_{x+n}-\dfrac{\mathrm{P}'_x}{1+\kappa_{x+n}}\mathrm{N}_{x+n-1}\right)}{\mathrm{M}_{x+n}-\dfrac{1}{t}(\mathrm{R}_{x+n+1}-\mathrm{R}_{x+n+t+1})} \quad . \quad . \quad . \quad . \quad . \quad (6),$$

the debt diminishing by equal annual decrements for t years, after which time the policy becomes one for its full face-value. In this case t may conveniently be taken as equal to the *expectation of life* for the actual age at entry. If it were taken as equal to the number of years in which the premiums at 4 per-cent interest would amount to the sum assured, the value of t would be *less* than the expectation, and the initial debt consequently *greater* than if the deduction were spread over the longer term. This would, no doubt, act as a deterrent to possible assurers under this scheme, especially if the number of years added to the age was at all considerable.

To take, as before, an actual example: Let the rates of premium be as above stated in the former illustration, and let t now be taken as the integer nearest to the expectation of life at the actual age, *i.e.*, in this case, 35. Making the necessary substitutions in formula (6), we shall have $\mathrm{X}'=32{\cdot}406$ and the annual decrease $=\mathrm{X}'\div 35={\cdot}926$.

To find, now, the value of a policy effected on this scale, it will be necessary, as in the former case, to calculate the true net premium for the benefit, which in this instance will be

$$\frac{100M_{35}-32{\cdot}406\left\{M_{35}-\frac{1}{35}(R_{36}-R_{71})\right\}}{N_{34}}=1{\cdot}714,$$

being exactly the same as was formerly arrived at in the case of the *constant* debt. This must necessarily be the case, seeing that the two benefits are at the inception of the policies precisely equal in present value. This equality, however, will not hold as regards the policy-values, even if the same term were taken for both the constant and the variable debt, owing to the different distribution of the debt over the subsequent years of life; but these values will, of course, be identical at any time after the expiry of the probationary term. In the case of the decreasing debt, the formula for the policy-value after m years ($m<t$) will be

$$\frac{M_{35+m}\left[100-32{\cdot}406\left(1-\frac{m}{35}\right)\right]+\frac{32{\cdot}406}{35}(R_{36+m}-R_{71})}{D_{35+m}}$$
$$-(1+a_{35+m})\times 1{\cdot}714.$$

At the end of the 35 years probationary term this will become

$$\frac{100M_{70}}{D_{70}}-(1+a_{70})\times 1{\cdot}714.$$

These values are shown in col. 5 of the appended table, and may be compared as before with the values of an ordinary policy valued according to either the true or the rated-up age at entry.

Had the *decreasing* debt been spread over the same period as the *constant* debt—in this case 25 years—the necessary initial debt would have been 44·266 and the annual decrease 1·771; the policy-values being shown in col. 4 of the table already referred to. In this case the values in question become identical after the expiry of the 25 years whether the policy were originally issued subject to a constant or to a decreasing debt, although during the currency of that term the policy, subject to the decreasing debt, has a somewhat larger value.

Further, if we base the formula for the debt upon *net* premiums, as in formula (5), or if we assume that the premiums at all ages are loaded with a uniform percentage (which, as already stated, produces exactly the same result) we shall necessarily have the net premium for the benefit equal to the ordinary net premium for the actual age. In the case of a constant debt where a life aged 30 is rated-up 5 years the debt on this assumption would be 27·784 for a period of 25 years (as against 23·614 when the loaded premiums were used), and the premium for the benefit $1{\cdot}669=\pi_{30}$. The policy-values are shown in col. 6 of the table.

If the *decreasing* debt be calculated on the basis of net premiums and spread over the expectation of life, then the debts will be found identical with those quoted in Mr. Sunderland's paper, that for age 30 rated-up five years, starting at 38·090 (annual decrease 1·088), as against 32·406 when the premiums are loaded as stated above. The values of such a policy are given in col. 8.

Finally, if we spread the decreasing debt over the same period as the constant one, namely, 25 years, the necessary initial amount will be 52·029 and the annual decrease 2·081, the corresponding policy-values being given in col. 7.

The only other point to which I wish to advert in the present communication is that as to the proper method of allotting profits to policies under the contingent debt plan. In considering this question Mr. Sunderland divides the loading on the premium into the portion required to provide for expenses and that required to provide for profits, but it does not seem to me that it is always possible thus to split up the loading into its component parts. Most if not all of the Australian offices now employ in the division of surplus what may be termed the "modified contribution method", by which each policy is first credited with the surplus arising from interest earned on the reserve held for it at the previous investigation, and the remaining surplus is then allotted in proportion to the loading on the premiums paid during the period in question. To apply this method strictly to the class of policies we are now considering it will be only necessary to determine as has been done above the true net premium in each case, treating as loading the balance of the actual premium receivable, and to value the policies in the manner I have indicated, or, if this were found too troublesome, by some convenient approximation. In the case of a policy for 100 on a life aged 30 and rated-up 5 years with a constant debt of 23·614 for 25 years, the loading would be 2·408—1·714=·694 as against a loading of ·739 (=2·408—1·669) in the case of an ordinary policy on a select life. The application of the method of distribution I have just described would therefore result in the allocation of smaller "bonuses from loading" in the case of contingent debt policies than those allotted to ordinary policies on select lives, although in the later years of duration this would be partly counterbalanced by the larger "interest bonuses" arising from the higher value of the policy.

Before concluding I would only repeat that in the whole of this investigation I have *assumed* that policies on rated-up lives may properly be treated for all purposes as if the higher age were the real age of the life, although I am of course aware that some theoretical objections have been urged against such a course. Still I believe that this method if it errs does so on the side of safety and therefore may generally with advantage be adopted in practice.

I am,

Yours &c.,

Sydney, N.S.W. D. CARMENT.
10 *July* 1893.

[Some slight alteration has been made in the notation used by Mr. Carment in this letter, in order to make it accord with that employed by Mr. Sunderland in his paper above referred to.—Ed. *J.I.A.*]

TABLE OF POLICY-VALUES—H^M 4 *per-cent.*

Duration $=m$	${}_mV_{30}$	${}_mV_{35}$	Using Loaded Premiums in the Calculation of the Debts			Using Net Premiums in the Calculation of the Debts		
			Constant Debt 23 614 for 25 Years	Decreasing Debt beginning at 44·266 for 25 Years	Decreasing Debt beginning at 32·406 for 35 Years	*Constant Debt* 27 784 for 25 Years	Decreasing Debt beginning at 52·029 for 25 Years	Decreasing Debt beginning at 38 090 for 35 Years
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
2	1·970	2·386	2·275	2·643	2·429	2·256	2·685	2·436
5	5·152	6·176	5·936	6·840	6·230	5·895	6·955	6·326
10	11·010	13·353	12·990	14·654	13·601	12·927	14·881	13·646
15	17·817	21·288	21·136	23·261	21·752	21·108	23·601	21·236
20	25·343	29·972	30 632	32·506	30·749	30·747	32·944	30·891
25	33·580	39·181	41·848	41·848	40·389	42·314	42·314	40·601
30	42·314	48·419	50·681	50·681	50·115	51·077	51·077	50·412
35	51·076	57·591	59·452	59·452	59·452	59·776	59·776	59·776