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ON THE CLASSIFICATION OF CONTACT RIEMANNIAN MANIFOLDS SATISFYING THE CONDITION (C)

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Abstract. Given a contact form η, there is a one-to-one correspondence between the Riemannian structures (η, g) and the CR-structures (η, L). It is interesting to study the interaction between the two associated structures. We approach the geometry of contact Riemannian manifolds in connection with their associated CR-structures. In this context, for a contact Riemannian manifold $(M; \eta, g)$ we consider the Jacobi-type operator $R_{\dot\gamma} = R(\cdot, \dot\gamma)\dot\gamma$ along a self-parallel curve γ with respect to the (generalized) Tanaka connection $\hat\nabla$.

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1. Introduction. The contact structure η is a global differentiable one-form on a smooth manifold M^{2n+1} such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. It is well-known that there exists an associated Riemannian structure (metric) g and a (1,1)-type tensor ϕ, where g and ϕ are canonically related. We call the pair (η, g) a contact Riemannian structure and $M = (M; \eta, g)$ a contact Riemannian manifold. S. Sasaki and Y. Hatakeyama [**22**] defined the normality of the contact Riemannian structure (see section 2). A normal contact Riemannian manifold is said to be a Sasakian manifold. In [**19**] it was proved that a Sasakian manifold which is locally symmetric ($\nabla R = 0$) must have constant curvature $+1$, where ∇ is the Levi-Civita connection. This fact means that local symmetry is a very strong condition for a Sasakian manifold. For this reason, T. Takahashi ([**24**]) introduced the notion of Sasakian locally ϕ-symmetric spaces which may be considered as the analogues of locally Hermitian symmetric spaces. A contact Riemannian locally ϕ-symmetric space is defined as a generalization of a Sasakian locally ϕ-symmetric space and investigated in [**5**].

One the other hand, the associated CR-structure of a given contact Riemannian manifold $M = (M; \eta, g)$ is given by the holomorphic subbundle

$$\mathscr{H} = \{X - i\bar\phi X : X \in D\}$$

of the complexification $TM^{\mathbb{C}}$ of the tangent bundle TM, where D is the subbundle of TM defined by the kernel of η and $\bar\phi = \phi|D$, the restriction of ϕ to D. Then we see that

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each fibre $\mathscr{H}_x$ $(x \in M)$ is of complex dimension n and $\mathscr{H} \cap \bar{\mathscr{H}} = \{0\}$. Furthermore, we have $\mathbb{C}D = \mathscr{H} \oplus \bar{\mathscr{H}}$. We say that *the associated* CR-*structure is integrable* if $[\mathscr{H}, \mathscr{H}] \subset \mathscr{H}$. For $\mathscr{H}$ we define the Levi form by

$$L : D \times D \to \mathscr{F}(M), \quad L(X, Y) = -d\eta(X, \phi Y)$$

where $\mathscr{F}(M)$ denotes the algebra of differential functions on M. Then we see that the Levi form is Hermitian and positive definite, that is, the CR-structure is *strongly pseudo-convex, pseudo-Hermitian CR-structure*. In fact, for a contact manifold $(M; \eta)$, there is a correspondence between the contact Riemannian structure (η, g) and strongly pseudo-convex, pseudo-Hermitian CR-structure (η, L) by the relation $g = L + \eta \otimes \eta$, where we denote by the same letter L the natural extension of the Levi form to a (0,2)-tensor field on M. N. Tanaka [**25**] defined the canonical affine connection on a nondegenerate integrable CR-manifold. In [**27**] S. Tanno defined the generalized Tanaka connection $\hat{\nabla}$ on a contact Riemannian manifold and further, he proved that for a given contact Riemannian manifold M the associated CR-structure is strongly pseudo-convex integrable if and only if M satisfies the integrability condition $Q = 0$ (see section 2), in which case the connection $\hat{\nabla}$ coincides with the Tanaka connection. Here, we note that the normality of a contact Riemannian structure implies the integrability of the associated CR-structure, but the converse does not always hold. The associated CR-structures of 3-dimensional contact Riemannian manifolds are always integrable (see [**27**]). Also, we see that their associated CR-structures are integrable for (contact Riemannian) (k, μ)-spaces (see [**2**], [**6**] or [**12**]).

It is interesting to study the geometry of a given contact Riemannian manifold $(M; \eta, g)$ in connection with the associated CR-structure, particularly with the generalized Tanaka connection. In this context, we define the Jacobi-type operator $R_{\dot{\gamma}} = R(\cdot, \dot{\gamma})\dot{\gamma}$ along a unit $\hat{\nabla}$-geodesic γ. Here, we observe that the geodesics of the Levi-Civita connection and the generalized Tanaka connection do not coincide in general. In the preceding paper [12] the first author has introduced a new class of contact Riemannian manifolds satisfying the condition (C), i.e., the Jacobi-type operator field $R_{\dot{\gamma}}$ is diagonalizable by a $\hat{\nabla}$-parallel orthonormal frame field along γ and its eigenvalues are constant along γ, or equivalently,

$$(\hat{\nabla}_{\dot{\gamma}} R)(\cdot, \dot{\gamma})\dot{\gamma} = 0 \tag{C}$$

for any unit $\hat{\nabla}$-geodesic γ, where $\hat{\nabla}$ is the generalized Tanaka connection. Further, in [**12**] it has been shown that $(k, 2)$-spaces $(k \neq 1)$, including the standard contact Riemannian structure of the unit tangent sphere bundle T_1M of M with constant curvature -1, are examples that are neither Sasakian nor locally symmetric but satisfy the condition (C) for any $\hat{\nabla}$-geodesic γ. Also, it is remarkable that a (k, μ)-space with $k = \mu = 0$ of dimension ≥ 5, which is a product of $(n+1)$-dimensional flat manifold and n-dimensional space of constant curvature 4, is locally symmetric but M fails to satisfy the condition (C) for any $\hat{\nabla}$-geodesic γ. Continuing the preceding work, in this paper we develop further the results in [**12**]. More precisely, in section 3 we prove

THEOREM A. *Let M be a (k, μ)-space. Then M satisfies the condition* (C) *for any $\hat{\nabla}$-geodesic γ if and only if* (1) $k = 1$ *and M is Sasakian locally ϕ-symmetric or* (2) $\mu = 0$ *in which case M is* 3*-dimensional, or* (3) $\mu = 2$.

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In [12] it was also proved that the standard contact Riemannian structure of the unit tangent sphere bundle T_1M of a 2-dimensional Riemannian manifold M satisfies condition (C) for any $\hat{\nabla}$-geodesic γ if and only if M has constant Gauss curvature 1, 0 or -1. In section 4, we prove this result for arbitrary dimension. Namely, we prove

THEOREM B. *Let M be a $(n+1)$-dimensional Riemannian manifold. Then the standard contact Riemannian structure of the unit tangent sphere bundle T_1M satisfies the condition* (C) *for any $\hat{\nabla}$-geodesic γ if and only if the base manifold M is of constant curvature $c = 1$, $n = 1$ and $c = 0$, or $c = -1$.*

REMARK 1. Recently, it was proved in [4] ([20] and [21], respectively) that the base manifold is of constant curvature $c = -1$, $c = 1$ ($c = 0$, $c = 1$, respectively) if and only if the standard contact Riemannian structure on the unit tangent sphere bundle is a critical point of some functional on the set of associated Riemannian metrics $\mathcal{M}(\eta)$ of a given contact form η.

Finally, in section 5, we give a local and a global classification of 3-dimensional contact Riemannian manifolds satisfying the condition (C) for any $\hat{\nabla}$-geodesic γ. More precisely, we prove

THEOREM C (local classification). *Let M be a 3-dimensional contact Riemannian manifold. Then M satisfies the condition* (C) *for any $\hat{\nabla}$-geodesic γ if and only if M is locally isometric to one of the following spaces:*

(1) *a Sasakian ϕ-symmetric space;*

(2) $SU(2)$ (*or* $SO(3)$), $SL(2,\mathbb{R})$ (*or* $O(1,2)$) *with a special left-invariant contact metric which is not Sasakian, respectively;*

(3) *a flat manifold.*

In [7] the authors gave a classification of Sasakian ϕ-symmetric spaces (complete and simply connected Sasakian locally ϕ-symmetric spaces). Together with this classification we have

THEOREM D (global classification). *Let M be a complete and simply connected 3-dimensional contact Riemannian manifold. Then M satisfies the condition* (C) *for any $\hat{\nabla}$-geodesic γ if and only if M is isometric to one of the following spaces:*

(1) *the standard unit sphere S^3; $SU(2)$, $\widetilde{SL}(2,\mathbb{R})$ (the universal covering of $SL(2,\mathbb{R})$) or the Heisenberg group H with a left-invariant Sasakian metric, respectively;*

(2) $SU(2)$, $\widetilde{SL}(2,\mathbb{R})$ *with a special left-invariant contact metric which is not Sasakian, respectively;*

(3) $\mathbb{R}^3$.

2. Preliminaries. We start by collecting some fundamental material about contact Riemannian geometry and refer to [2] for further details. All manifolds in the present paper are assumed to be connected and of class C^∞.

A $(2n+1)$-dimensional manifold M^{2n+1} is said to be a contact manifold if it admits a global one-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η, there exists a unique vector field ξ, called the characteristic vector field, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X. It is well-known that there also exists a Riemannian metric g and a $(1, 1)$-tensor field ϕ such that

$$\eta(X) = g(X,\xi), \quad d\eta(X, Y) = g(X,\phi Y), \quad \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

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where X and Y are vector fields on M. From (2.1), it follows that

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

A Riemannian manifold M equipped with structure tensors (η, g) satisfying (2.1) is said to be a *contact Riemannian manifold* or *contact metric manifold* and it is denoted by $M = (M; \eta, g)$. Given a contact Riemannian manifold M, we define a $(1, 1)$-tensor field h by $h = \frac{1}{2}L_\xi\phi$, where L denotes Lie differentiation. Then we may observe that h is symmetric and satisfies

$$h\xi = 0, \quad h\phi = -\phi h, \tag{2.3}$$

$$\nabla_X\xi = -\phi X - \phi hX, \tag{2.4}$$

where ∇ is Levi-Civita connection. From (2.3) and (2.4), we see that each trajectory of ξ is a geodesic. We denote by R the Riemannian curvature tensor defined by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

for all vector fields X, Y, Z. Along a trajectory of ξ, the Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ is a symmetric $(1, 1)$-tensor field. We have

$$(\text{trace } R_\xi) = g(S\xi, \xi) = 2n - (\text{trace } h^2), \tag{2.5}$$

$$R_\xi = \phi R_\xi \phi - 2(h^2 + \phi^2), \tag{2.6}$$

$$\nabla_\xi h = \phi - \phi R_\xi - \phi h^2, \tag{2.7}$$

$$g(R(X, Y)\xi, Z) = g((\nabla_Y\phi)X - (\nabla_X\phi)Y, Z) + g((\nabla_Y\phi h)X - (\nabla_X\phi h)Y, Z) \tag{2.8}$$

for all vector fields X, Y, Z on M, where S is the Ricci $(1, 1)$-tensor on M. A contact Riemannian manifold for which ξ is a Killing vector field is called a K-contact manifold. It is easy to see that a contact Riemannian manifold is K-contact if and only if $h = 0$. For a contact Riemannian manifold M, one may define naturally an almost complex structure J on $M \times \mathbb{R}$ by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

where X is a vector field tangent to M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable, M is said to be normal or Sasakian. It is known that M is normal if and only if M satisfies

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ. A Sasakian manifold is also characterized by the condition

$$(\nabla_X\phi)Y = g(X, Y)\xi - \eta(Y)X \tag{2.9}$$

for all vector fields X and Y on the manifold and this is equivalent to

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.10}$$

for all vector fields X and Y.

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For a contact Riemannian manifold M, the tangent space T_pM of M at each point $p \in M$ is decomposed as $T_pM = D_p \oplus \{\xi\}_p$(direct sum), where we denote $D_p = \{v \in T_pM \mid \eta(v) = 0\}$. Then $D : p \to D_p$ defines a distribution orthogonal to ξ. The $2n$-dimensional distribution D is called the *contact distribution*. We see that the restriction $\bar{\phi} = \phi|D$ of ϕ to D defines an almost complex structure on D, and furthermore that the associated Levi form, which is defined by $L(X, Y) = -d\eta(X, \bar{\phi}Y)$, $X, Y \in D$, is positive definite and Hermitian. We call the pair $(\eta, \bar{\phi})$ a *strongly pseudo-convex, pseudo-hermitian structure* on M. Since $d\eta(\phi X, \phi Y) = d\eta(X, Y)$, we see that $[\bar{\phi}X, \bar{\phi}Y] - [X, Y] \in D$ for $X, Y \in D$. Further if M satisfies the condition

$$[\bar{\phi}, \bar{\phi}](X, Y) = 0$$

for $X, Y \in D$, then the pair $(\eta, \bar{\phi})$ is called a *strongly pseudo-convex integrable CR-structure*, (associated with the contact Riemannian structure (η, g)). Taking account of (2.9) we see that for a Sasakian manifold the associated CR-structure is strongly pseudo-convex integrable (cf. [**16**]).

Now, we review the *generalized Tanaka connection* ([**27**]) on a contact Riemannian manifold $M = (M; \eta, g)$. The generalized Tanaka connection $\hat{\nabla}$ is defined by

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi$$

for all vector fields X and Y on M. Together with (2.4), $\hat{\nabla}$ may be rewritten as

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y + \eta(Y)(\phi X + \phi hX) - g(\phi X + \phi hX, Y)\xi \tag{2.11}$$

and we see that the generalized Tanaka connection $\hat{\nabla}$ has the torsion $\hat{T}(X, Y) = 2g(X, \phi Y)\xi + \eta(Y)\phi hX - \eta(X)\phi hY$. We put

$$A(X, Y) = \eta(X)\phi Y + \eta(Y)(\phi X + \phi hX) - g(\phi X + \phi hX, Y)\xi \tag{2.12}$$

for all vector fields X and Y on M. Then A is a (1,2)-tensor field on M and $\hat{\nabla}_X Y = \nabla_X Y + A(X, Y)$. In particular, for a K-contact Riemannian manifold we get

$$A(X, Y) = \eta(X)\phi Y + \eta(Y)\phi X - g(\phi X, Y)\xi, \tag{2.13}$$

where X and Y are vector fields. For a given contact Riemannian manifold M the associated CR-structure is strongly pseudo-convex integrable if and only if M satisfies the integrability condition $Q = 0$, where Q is a (1,2)-tensor field on M defined by

$$Q(X, Y) = (\nabla_X \phi)Y - g(X + hX, Y)\xi + \eta(Y)(X + hX) \tag{2.14}$$

for all vector fields X, Y on M (see [**27**, Proposition 2.1]). Further, the following result was proved.

PROPOSITION 2.1 ([**27**]). *The generalized Tanaka connection $\hat{\nabla}$ on a contact Riemannian manifold $M = (M; \eta, g)$ is the unique linear connection satisfying the following conditions:*

(i) $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$;
(ii) $\hat{\nabla}g = 0$;
(iii(a)) $\hat{T}(X, Y) = 2d\eta(X, Y)\xi$, $X, Y \in D$;
(iii(b)) $\hat{T}(\xi, \phi Y) = -\phi\hat{T}(\xi, Y)$, $Y \in D$;
(iv) $(\hat{\nabla}_X \phi)Y = Q(X, Y)$, $X, Y \in TM$.

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The Tanaka connection ([**25**]) on a nondegenerate integrable CR-manifold is defined as the unique linear connection satisfying (i), (ii), (iii(a)), (iii(b)) and $\hat{\nabla}\phi = 0$. The metric affine connection $\hat{\nabla}$ is a natural and proper generalization of the Tanaka connection. In fact, in [**1**] the authors deal with the use of $\hat{\nabla}$ in the non-integrable case.

Let γ be a $\hat{\nabla}$-geodesic parametrized by the arc-length parameter s, where a $\hat{\nabla}$-geodesic means a geodesic with respect to $\hat{\nabla}$. From (2.11) we see that a $\hat{\nabla}$-geodesic does not coincide with a ∇-geodesic. Define the Jacobi operator $R_{\dot{\gamma}}$ by $R_{\dot{\gamma}} = R(\cdot, \dot{\gamma})\dot{\gamma}$ along γ, where $\dot{\gamma}$ is the unit tangent vector field of γ. Then $R_{\dot{\gamma}}$ is a symmetric (1, 1)-tensor field along γ. Moreover, from (i) of Proposition 2.1 we observe that $\eta(\dot{\gamma})$ is constant along γ, and thus a $\hat{\nabla}$-geodesic whose tangent initially belongs to D remains in D. We call such a $\hat{\nabla}$-geodesic which is tangent to D a *horizontal* $\hat{\nabla}$*-geodesic*.

We recall the definition of a Sasakian locally ϕ-symmetric space ([**24**]).

DEFINITION 2.2. A Sasakian manifold $M = (M; \eta, g)$ is said to be *locally* ϕ*-symmetric* if $\phi^2(\nabla_V R)(X, Y)Z = 0$ for all vector fields $V, X, Y, Z \in D$.

As a generalization of the above Sasakian one, a contact Riemannian locally ϕ-symmetric space is defined in [**5**] by the same condition and is called ([**8**]) a *locally* ϕ*-symmetric space in the weak sense*. In [**12**] we have the following characterization of a Sasakian locally ϕ-symmetric space.

THEOREM 2.3. *A Sasakian manifold M is locally ϕ-symmetric if and only if M satisfies the condition* (C) *for any horizontal $\hat{\nabla}$-geodesic γ, or if and only if M satisfies the condition* (C) *for any $\hat{\nabla}$-geodesic γ.*

3. A contact Riemannian (k, μ)**-space.** In [**6**], the (k, μ)-nullity distribution of a contact Riemannian manifold M, for the pair $(k, \mu) \in \mathbb{R}^2$, is defined by

$$N(k, \mu) : p \to N_p(k, \mu) = \{z \in T_pM \,|\, R(x, y)z = k(g(y, z)x - g(x, z)y) \\ + \mu(g(y, z)hx - g(x, z)hy) \text{ for any } x, y \in T_pM\}.$$

A (k, μ)*-space* is a contact Riemannian manifold with ξ belonging to the (k, μ)-nullity distribution, that is,

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY). \tag{3.1}$$

It was shown in [**6**] that the (k, μ)-spaces are invariant under a D-homothetic deformation. As mentioned in the introduction, the associated CR-structures of the (k, μ)-spaces are integrable, that is, $Q = 0$. This class contains Sasakian manifolds ($k = 1$ and $h = 0$). The unit tangent sphere bundle is a (k, μ)-space if and only if the base manifold is of constant curvature c with $k = c(2 - c)$ and $\mu = -2c$ ([**6**]). (By virtue of the result of Y. Tashiro [**29**], we know that for $c \neq 1$, the unit tangent sphere bundle is non-Sasakian.) Very recently, E. Boeckx [**9**] presented explicit examples for all possible dimensions and all possible (k, μ).

THEOREM 3.1 ([**12**]). *Let M be a (k, μ)-space. If M satisfies the condition* (C) *for any $\hat{\nabla}$-geodesic γ, then we have:*

(i) $k = 1$ *and M is a Sasakian locally ϕ-symmetric space;*
(ii) $\mu = 0$ *and M is a 3-dimensional locally ϕ-symmetric space in the weak sense;*
(iii) $\mu = 2$ *and M is a locally ϕ-symmetric space in the weak sense.*

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In **[8]** it has been proved that all (k, μ)-spaces are locally ϕ-symmetric in the strong sense, i.e., the characteristic reflections are local isometries, and hence also in the weak sense. Thus, we have

PROPOSITION 3.2. *A (k, μ)-space $(k < 1)$ is locally ϕ-symmetric in the weak sense.*

Therefore, together with Proposition 3.2 and the proof of Theorem 3.1 (see **[12]**), we have Theorem A.

REMARK 2. An example of a contact flat Riemannian structure on $\mathbb{R}^3(x^1, x^2, x^3)$ is given by $\eta = \frac{1}{2}(\cos x^3 dx^1 + \sin x^3 dx^2)$ and $g_{ij} = \frac{1}{4}\delta_{ij}$. For dimension at least 5 a contact manifold cannot admit a contact Riemannian structure of vanishing curvature (cf. **[2]**). Also, it was proved that a contact Riemannian manifold M^{2n+1} which satisfy $R(X, Y)\xi = 0$ for all vector fields X and Y (i.e., ξ belonging to the (0,0)-nullity distribution) is locally a product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive constant sectional curvature equal to 4. Hence, we see that a contact Riemannian manifold M^{2n+1} $(n \geq 2)$ satisfying $R(X, Y)\xi = 0$ is locally symmetric but it does not satisfy the condition (C) for any $\hat{\nabla}$-geodesic.

4. The unit tangent sphere bundles. The basic facts and fundamental formulas about tangent bundles are well-known (cf. **[13]**, **[17]**, **[32]**). We briefly review of notations and their definitions. Let $M = (M, G)$ be a $(n+1)$-dimensional Riemannian manifold and TM denote its tangent bundle with the projection $\pi : TM \to M$, $\pi(x, u) = x$. For a vector $X \in T_xM$, we denote by X^H and X^V, the *horizontal lift* and the *vertical lift*, respectively. Then we can define a Riemannian metric $\tilde{g}$, *Sasaki metric*, on TM in a natural way. That is,

$$\tilde{g}(X^H, Y^H) = \tilde{g}(X^V, Y^V) = G(X, Y) \circ \pi, \quad \tilde{g}(X^H, Y^V) = 0$$

for all vector fields X and Y on M. Also, a natural almost complex structure tensor J of TM is defined by $JX^H = X^V$ and $JX^V = -X^H$. Then we easily see that $(TM; \tilde{g}, J)$ is an almost Hermitian manifold. We note that J is integrable if and only if (M, G) is locally flat (**[17]**). Now we consider the unit tangent sphere bundle (T_1M, g'), which is isometrically embedded hypersurface in $(TM, \tilde{g})$ with unit normal vector field $N = u^V$. For $X \in T_xM$, we define the *tangential lift* of X to $(x, u) \in T_1M$ by

$$X^T_{(x,u)} = X^V_{(x,u)} - G(X, u)N_{(x,u)}.$$

Clearly, the tangent space $T_{(x,u)}T_1M$ spanned by vectors of the form X^H and X^T where $X \in T_xM$. We put

$$\xi' = -JN, \quad \phi' = J - \eta' \otimes N.$$

Then we find $g'(X, \phi' Y) = 2d\eta'(X, Y)$. By taking $\xi = 2\xi'$, $\eta = \frac{1}{2}\eta'$, $\phi = \phi'$, and $g = \frac{1}{4}g'$, we get the standard contact Riemannian structure (ϕ, ξ, η, g). Indeed, we easily check these tensors satisfy (2.1). Here, we notice that ξ determines the geodesic flow. The tensors ξ and ϕ are explicitly given by

$$\begin{aligned} \xi &= 2u^H, \\ \phi X^T &= -X^H + 1/2G(X, u)\xi, \\ \phi X^H &= X^T \end{aligned} \tag{4.1}$$

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where X and Y are vector fields on M. From now, we consider $T_1M = (T_1M; \eta, g)$ with the standard contact Riemannian structure. We list fundamental formulas, which are needed for the proof of our Theorem, without proofs (cf. **[2]**, **[3]**, **[10]**, **[28]**, **[29]**). We denote by ∇ and R, the Levi-Civita connection and the Riemannian curvature tensor associated with g, respectively.

$$\begin{aligned} \nabla_{X^T} Y^T &= -G(Y, u)X^T, \\ \nabla_{X^T} Y^H &= 1/2(K(u, X)Y)^H, \\ \nabla_{X^H} Y^T &= (\nabla_X Y)^T + 1/2(K(u, Y)X)^H, \\ \nabla_{X^H} Y^H &= (\nabla_X Y)^H - 1/2(K(X, Y)u)^T, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} R(X^T, Y^H)Z^H &= -1/2\{K(Y, Z)(X - G(X, u)u)\}^T \\ &\quad + 1/4\{R(Y, K(u, X)Z)u\}^T - 1/2\{(D_Y K)(u, X)Z\}^H, \\ R(X^H, Y^H)Z^H &= (K(X, Y)Z)^H + 1/2\{K(u, K(X, Y)u)Z\}^H \\ &\quad - 1/4\{K(u, K(Y, Z)u)X - K(u, K(Y, Z)u)Y\}^H \\ &\quad + 1/2\{(D_Z K)(X, Y)u\}^T \end{aligned} \tag{4.3}$$

for all vector fields X, Y and Z on M, where we denote by D and K, the Levi-Civita connection and the Riemannian curvature tensor associated with G, respectively. From (4.1) and (4.2), we have

$$\nabla_{X^T}\xi = -2\phi X^T - (K_u X)^H, \ \nabla_{X^H}\xi = -(K_u X)^H, \tag{4.4}$$

where $K_u = K(\cdot, u)u$ is the Jacobi operator associated with the unit vector u. From (2.4) and (4.4), it follows that

$$\begin{aligned} hX^T &= X^T - (K_u X)^T, \\ hX^H &= -X^H + 1/2G(X, u)\xi + (K_u X)^H. \end{aligned} \tag{4.5}$$

Using the formula (4.3), we get

$$\begin{aligned} R_\xi X^T &= \left(K_u^2 X\right)^T + 2(K_u' X)^H, \\ R_\xi X^H &= 4(K_u X)^H - 3\left(K_u^2 X\right)^H + 2(K_u' X)^T, \end{aligned} \tag{4.6}$$

where $K' = (D_u K)(\cdot, u)u$ and $K^2 = K(K(\cdot, u)u, u)u$. By using (2.7), (4.1) and (4.3) we obtain

$$\begin{aligned} (\nabla_\xi h)X^T &= -2(K_u X)^H + 2\left(K_u^2 X\right)^H - 2(K_u' X)^T, \\ (\nabla_\xi h)X^H &= -2(K_u X)^T + 2\left(K_u^2 X\right)^T + 2(K_u' X)^H. \end{aligned} \tag{4.7}$$

The above formulae (4.4)–(4.7) are also found in **[10]**. Finally, from (4.2) and (4.6) we compute

$$\begin{aligned} R'_\xi X^T &= 4(K_u' K_u X + K_u K_u' X)^T + 4\left(K_u'' X + K_u^2 X - K_u^3 X\right)^H, \\ R'_\xi X^H &= 8(K_u' X - K_u' K_u X - K_u K_u' X)^H + 4\left(K_u'' X + K_u^2 X - K_u^3 X\right)^T. \end{aligned} \tag{4.8}$$

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Now, we prove Theorem B. Suppose that $T_1M = (T_1M; \eta, g)$ with the standard contact Riemannian structure satisfies the condition (C) for any $\hat{\nabla}$-geodesic γ. Since the geodesic flow vector field ξ also determine $\hat{\nabla}$-geodesic flow, we consider the condition

$$(\hat{\nabla}_\xi R)(\cdot, \xi)\xi = 0. \tag{4.9}$$

Then from (2.10) and (4.9), we see that (4.9) is equivalent to the condition

$$R'_\xi = R_\xi \phi - \phi R_\xi. \tag{4.10}$$

From (4.10), together with the (4.1), (4.6) and (4.8), we have

$$\begin{gathered}(K'_u K_u X + K_u K'_u X)^T + (K'_u X)^T = 0,\\ \left(K_u^3 X - K_u X - K''_u X\right)^T = 0,\end{gathered} \tag{4.11}$$

and

$$\begin{gathered}2(K'_u K_u X + K_u K'_u X)^H - (K'_u X)^H = 0,\\ \left(K_u^3 X - K_u X - K''_u X\right)^H = 0.\end{gathered} \tag{4.12}$$

On the other hand, differentiating (2.6) covariantly with respect to ξ, then taking account of (4.10) we have

$$(\nabla_\xi h)h + h(\nabla_\xi h) = 0. \tag{4.13}$$

From (4.5), (4.7) and (4.13), we obtain

$$\begin{gathered}(K'_u K_u X + K_u K'_u X)^T - 2(K'_u X)^T = 0\\ \text{and } \ (K'_u K_u X + K_u K'_u X)^H - 2(K'_u X)^H = 0,\end{gathered}$$

hence it follows that

$$K'_u K_u X + K_u K'_u X - 2K'_u X = 0. \tag{4.14}$$

Thus, from (4.11) and (4.12), together with (4.14) we have

$$K'_u X = 0 \tag{4.15}$$

and

$$K_u^3 X - K_u X = 0 \tag{4.16}$$

for all vector field X on M. Here, we note that M satisfies the condition (4.15) if and only if M is locally symmetric (see [**15**], [**31**]). Further, from (4.16) we see that the eigenvalues of K_u are constant and -1, 0 or 1, that is, M is a globally Osserman space (i.e., the eigenvalues of K_u depend neither on the point p nor on the choice of u). But, we know that a locally symmetric globally Osserman space is locally flat or locally isometric to rank one symmetric space (cf. [**14**]). Thus, we see that the base manifold M is a space of constant curvature $c = 1$, 0 or -1. As we have mentioned before, the unit tangent sphere bundle is a (k, μ)-space if and only if the base manifold is of constant curvature c with $k = c(2 - c)$ and $\mu = -2c$. Therefore, together with Theorem A, we have proved Theorem B.

5. Three dimensional contact Riemannian manifolds. In this section we prove Theorem C and Theorem D. It was proved in **[27]** that a 3-dimensional contact Riemannian manifold always satisfies the condition $Q = 0$, i.e.

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX). \tag{5.1}$$

From (2.8) and (5.1) we have

$$R(X, Y)\xi = \eta(Y)(X + hX) - \eta(X)(Y + hY) + \phi((\nabla_Y h)X - (\nabla_X h)Y) \tag{5.2}$$

for all vector fields X and Y. We have already noted that a K-contact manifold is characterized by the condition $h = 0$ and it is easily seen and well-known that a 3-dimensional K-contact manifold is Sasakian. Hence, we have

LEMMA 5.1. *A* 3*-dimensional contact Riemannian manifold is Sasakian if and only if* $h = 0$.

Let $(M^3; \eta, g)$ be a 3-dimensional contact Riemannian manifold satisfying the condition (C) for any $\hat{\nabla}$-geodesic γ. It is well-known that the curvature tensor R of a 3-dimensional Riemannian manifold is expressed by

$$\begin{aligned} R(Y, X)Z = \rho(X, Z)Y - \rho(Y, Z)X + g(X, Z)SY - g(Y, Z)SX \\ - \tfrac{1}{2}\tau\{g(X, Z)Y - g(Y, Z)X\} \end{aligned} \tag{5.3}$$

for all vector fields X, Y, Z, where $\rho(Y, X) = g(SY, X)$ and τ is the scalar curvature of the manifold. From (5.3) and the assumption we have

$$\begin{aligned} 0 &= (\hat{\nabla}_x R)(y, x)x \\ &= (\hat{\nabla}_x \rho)(x, x)y - (\hat{\nabla}_x \rho)(y, x)x + g(x, x)(\hat{\nabla}_x S)y - g(y, x)(\hat{\nabla}_x S)x \\ &\quad - \tfrac{1}{2}(x\tau)\{g(x, x)y - g(y, x)x\}, \end{aligned} \tag{5.4}$$

for any $x, y \in T_pM$ and any $p \in M$. For any unit v orthogonal to ξ, let $\{v, \phi v, \xi\}$ be an adapted orthonormal basis of $T_pM (p \in M)$. Then from (4.4) we get $g((\hat{\nabla}_x R)(v, x)x, v) = 0$, $g((\hat{\nabla}_x R)(\phi v, x)x, \phi v) = 0$ and $g((\hat{\nabla}_x R)(\xi, x)x, \xi) = 0$, and summing up these three equalities, we have

$$(\hat{\nabla}_x \rho)(x, x) = 0. \tag{5.5}$$

Also, from (5.4) we get $(\hat{\nabla}_v R)(\phi v, v)v = 0$, $(\hat{\nabla}_v R)(\xi, v)v = 0$ and thus we have

$$(\hat{\nabla}_v \rho)(\phi v, \phi v) = (\hat{\nabla}_v \rho)(\xi, \xi). \tag{5.6}$$

From (2.12), we have

$$A(x, y) = \eta(x)\phi y + \eta(y)(\phi x + \phi hx) - g(\phi x + \phi hx, y)\xi \tag{5.7}$$

for $x, y \in T_pM$ and $p \in M$. From (2.11) and (5.7) we have the formulas (5.8) and (5.9)

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which are equivalent to (5.5) and (5.6), respectively:

$$(\nabla_x\rho)(x,x) = 4\eta(x)\rho(\phi x, x) + 2\{\eta(x)\rho(\phi hx, x) - \eta(Sx)g(\phi hx, x)\}, \tag{5.8}$$

$$(\nabla_v\rho)(\xi,\xi) - (\nabla_v\rho)(\phi v, \phi v) = 2\{(2 + g(hv, v))\rho(\xi, \phi v) + \rho(\phi hv, \xi)\}, \tag{5.9}$$

for any unit $x \in T_pM$ and unit vector v orthogonal to ξ.

Let W be the subset of M on which the number of distinct eigenvalues of h is constant. Then W is an open and dense subset of M. We fix any point q in W. Then, from (2.3), there exists a C^∞ function λ and a local orthonormal frame field $\{e_1, e_2 = \phi e_1, e_3 = \xi\}$ on a neighborhood $N(q)(\subset W)$ containing q such that $he_1 = \lambda e_1$, $he_2 = -\lambda e_2$, $h\xi = 0$. We denote $\Gamma_{ijk} = g(\nabla_{e_i}e_j, e_k)$, $\rho_{ij} = \rho(e_i, e_j)$, $\nabla_i\rho_{jk} = (\nabla_{e_i}\rho)(e_j, e_k)$ and $\nabla_h R_{ijkl} = g((\nabla_h R)(e_i, e_j)e_k, e_l)$ for $h, i, j, k, l = 1, 2, 3$. Then we get at once

$$\Gamma_{ijj} = 0.$$

Further from (2.4) we get

$$\Gamma_{132} = -\Gamma_{123} = -(1+\lambda), \quad \Gamma_{231} = -\Gamma_{213} = 1-\lambda \tag{5.10}$$

and

$$\Gamma_{113} = \Gamma_{223} = \Gamma_{331} = \Gamma_{332} = 0. \tag{5.11}$$

Also, from (2.7) and taking account of (2.5) and (5.2), we have

$$\xi\lambda = \rho_{12} \tag{5.12}$$

and

$$4\lambda\Gamma_{312} = \rho_{22} - \rho_{11}. \tag{5.13}$$

Moreover, from (5.8) we get

$$\nabla_1\rho_{11} = 0, \quad \nabla_2\rho_{22} = 0 \tag{5.14}$$

and

$$\nabla_3\rho_{33} = 0. \tag{5.15}$$

Differentiating (2.5) covariantly in the direction ξ and taking account of (5.15) we obtain that $\xi\lambda = 0$. Thus, from (5.12) we have

$$\rho_{12} = 0. \tag{5.16}$$

If we substitute $x = \frac{1}{\sqrt{2}}(e_1 + e_2)$ and $x = \frac{1}{\sqrt{2}}(e_1 - e_2)$, respectively in (5.8) and take account of (5.14), we have

$$2\nabla_1\rho_{12} + 2\nabla_2\rho_{12} + \nabla_1\rho_{22} + \nabla_2\rho_{11} = -4\lambda(\rho_{31} + \rho_{32})$$

and

$$-2\nabla_1\rho_{12} + 2\nabla_2\rho_{12} + \nabla_1\rho_{22} - \nabla_2\rho_{11} = 4\lambda(\rho_{31} - \rho_{32}).$$

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By summing these two equalities, we have

$$\nabla_1\rho_{22} + 2\nabla_2\rho_{12} = -4\lambda\rho_{23} \tag{5.17}$$

and subtracting from the preceding one, we have

$$\nabla_2\rho_{11} + 2\nabla_1\rho_{12} = -4\lambda\rho_{13}. \tag{5.18}$$

If we substitute $x = \frac{1}{\sqrt{2}}(e_1 + e_3)$ and $x = \frac{1}{\sqrt{2}}(e_1 - e_3)$, respectively in (5.8) and take account of (5.15) and (5.16), we have

$$2\nabla_1\rho_{13} + 2\nabla_3\rho_{31} + \nabla_1\rho_{33} + \nabla_3\rho_{11} = 2(\lambda + 2)\rho_{23}$$

and

$$-2\nabla_1\rho_{13} + 2\nabla_3\rho_{31} + \nabla_1\rho_{33} - \nabla_3\rho_{11} = 2(\lambda + 2)\rho_{23}.$$

Summing these two equalities we have

$$\nabla_1\rho_{33} + 2\nabla_3\rho_{13} = 2(\lambda + 2)\rho_{23}. \tag{5.19}$$

A similar calculation for $x = \frac{1}{\sqrt{2}}(e_2 + e_3)$ and $x = \frac{1}{\sqrt{2}}(e_2 - e_3)$ gives

$$\nabla_2\rho_{33} + 2\nabla_3\rho_{23} = 2(\lambda - 2)\rho_{13}. \tag{5.20}$$

On the one hand, by applying the second Bianchi identity in (5.3), then taking account of (5.14) we have

$$2\nabla_2\rho_{12} + 2\nabla_3\rho_{13} - \nabla_1\rho_{22} - \nabla_1\rho_{33} = 0. \tag{5.21}$$

$$2\nabla_1\rho_{21} + 2\nabla_3\rho_{23} - \nabla_2\rho_{11} - \nabla_2\rho_{33} = 0. \tag{5.22}$$

From (5.17), (5.19) and (5.21) (resp. (5.18), (5.21) and (5.24)), we have (5.23) (resp. (5.24)):

$$\nabla_1\rho_{22} + \nabla_1\rho_{33} = -(\lambda - 2)\rho_{23}, \tag{5.23}$$

$$\nabla_2\rho_{11} + \nabla_2\rho_{33} = -(\lambda + 2)\rho_{13}. \tag{5.24}$$

On the other hand, from (5.9) we have

$$\nabla_1\rho_{33} - \nabla_1\rho_{22} = 4(\lambda + 1)\rho_{23} \tag{5.25}$$

and

$$\nabla_2\rho_{33} - \nabla_2\rho_{11} = 4(\lambda - 1)\rho_{13}. \tag{5.26}$$

Thus, from (5.23)–(5.26) we have

$$\nabla_1\rho_{33} = \frac{3}{2}(\lambda + 2)\rho_{23}, \quad \nabla_2\rho_{33} = \frac{3}{2}(\lambda - 2)\rho_{13} \tag{5.27}$$

and

$$\nabla_1\rho_{22} = -\frac{1}{2}(5\lambda + 2)\rho_{23}, \quad \nabla_2\rho_{11} = -\frac{1}{2}(5\lambda - 2)\rho_{13}. \tag{5.28}$$

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Also, from (5.17), (5.18) and (5.28), we have

$$\nabla_1\rho_{12} = -\frac{1}{4}(3\lambda + 2)\rho_{13} \quad \text{and} \quad \nabla_2\rho_{21} = -\frac{1}{4}(3\lambda - 2)\rho_{23}. \tag{5.29}$$

Differentiating (2.5) covariantly in the directions e_1 and e_2 and taking account of (5.10), (5.11) and (5.27), we have

$$(\lambda - 2)\rho_{23} = 8\lambda(e_1\lambda) \tag{5.30}$$

and

$$(\lambda + 2)\rho_{13} = 8\lambda(e_2\lambda), \tag{5.31}$$

respectively.

If we also differentiate (5.16) covariantly in the direction ξ, then we have

$$\nabla_3\rho_{12} = \Gamma_{312}(\rho_{11} - \rho_{22}). \tag{5.32}$$

Substituting $x = \xi$ in (5.4), we get $\hat{\nabla}_3\rho_{12} = 0$, and from (5.7) we get $\hat{\nabla}_3\rho_{12} = \nabla_3\rho_{12} + \rho_{11} - \rho_{22}$. Thus, we obtain

$$\nabla_3\rho_{12} = \rho_{22} - \rho_{11}. \tag{5.33}$$

We prove

LEMMA 5.2. *λ is locally constant.*

Proof. We set $N(q) = N^0(q) \cup N^1(q)$, where $N^0 = \{p \in N(q) | \rho_{11}(p) \neq \rho_{22}(p)\}$ and $N^1 = \{p \in N(q) | \rho_{11}(p) = \rho_{22}(p)\}$. We divide our arguments into three cases: (i) $N = N^0$, (ii) $N = N^1$ or (iii) N^0 and N^1 are both non-empty. (i) $N = N^0$. Then $\rho_{11} \neq \rho_{22}$ on N, and from (5.33) we get $\Gamma_{312} = -1$ on N. Thus (5.13) becomes $4\lambda = \rho_{11} - \rho_{22}$ on N. Differentiating this covariantly in the directions e_1 and e_2 and taking account of (5.10) and (5.11), we have $\nabla_1\rho_{22} = -2(\lambda + 1)\rho_{23} - 4(e_1\lambda)$ and $\nabla_2\rho_{11} = -2(\lambda - 1)\rho_{13} + 4(e_2\lambda)$. Thus taking account of (5.28), we have

$$\begin{aligned} (\lambda - 2)\rho_{23} &= 8(e_1\lambda), \\ -(\lambda + 2)\rho_{13} &= 8(e_2\lambda) \end{aligned} \tag{5.34}$$

on N. So, from (5.30), (5.31) and (5.34) we have

$$\begin{aligned} (\lambda - 1)(e_1\lambda) &= 0, \\ (\lambda + 1)(e_2\lambda) &= 0. \end{aligned} \tag{5.35}$$

From (5.35) we see that λ is locally constant on N. We consider the case (ii) $N = N^1$. Then $\rho_{11} = \rho_{22}$ on N. Differentiating $\rho_{12} = 0$ covariantly in the direction e_1 and e_2, then from the assumption, (5.10) and (5.11), we have $\nabla_1\rho_{12} = -(1 + \lambda)\rho_{13}$ and $\nabla_2\rho_{21} = (1 - \lambda)\rho_{23}$, respectively. Thus from (5.29) taking account of (5.30) and (5.31), we have

$$\begin{aligned} (\lambda + 2)\rho_{13} &= 0, \\ (\lambda - 2)\rho_{23} &= 0. \end{aligned} \tag{5.36}$$

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So, from (5.30), (5.31) and (5.36) we have

$$\lambda(e_1\lambda) = 0,$$
$$\lambda(e_2\lambda) = 0. \tag{5.37}$$

From (5.37) we see that λ is locally constant on N. In case (iii), that is, N^0 and N^1 are non-empty, then in view of the above cases (i) and (ii) and by using a continuity argument, we see that λ is constant. □

Thus, from (5.30) and (5.31) we have

$$(\lambda - 2)\rho_{23} = 0 \text{ and } (\lambda + 2)\rho_{13} = 0. \tag{5.38}$$

If $\lambda = 0$, then by Lemma 4.1 M is a Sasakian manifold, and further by Theorem 2.3 we see that M is locally ϕ-symmetric. From now, we suppose that $\lambda \neq 0$ and we argue by three cases: (I) $\lambda \neq \pm 2$, (II) $\lambda = 2$ and (III) $\lambda = -2$.

(I) $\lambda \neq \pm 2$. Then from (5.38) we get

$$\rho_{13} = \rho_{23} = 0,$$

which yield $R(e_1, e_2)\xi = 0$. Hence by using (5.2), we have

$$\lambda(\Gamma_{212}e_1 - \Gamma_{121}e_2) = 0,$$

which gives

$$\Gamma_{212} = \Gamma_{221} = \Gamma_{121} = \Gamma_{112} = 0. \tag{5.39}$$

Since λ is constant, we see that $\rho_{11} = \rho_{22}$ at each point on N or $\rho_{11} - \rho_{22} = 4\lambda$ at each point on N. We first suppose that $\rho_{11} = \rho_{22}$ on N. Then from (5.13) we get $\Gamma_{312} = 0$. Thus, together with (5.10), (5.11) and (5.39), we have

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = (1 - \lambda)e_1, \quad [e_3, e_1] = (1 + \lambda)e_2. \tag{5.40}$$

By virtue of a well-known result of the theory of Lie groups (see [**30**, Proposition 1.9]) and with the help of J. Milnor's classification for 3-dimensional unimodular Lie groups ([**18**]), we see that M is locally isometric to one of the following spaces:

(i) $SU(2)$ (or $SO(3)$) with a left-invariant metric when $0 < \lambda < 1$;

(ii) $SL(2, \mathbb{R})$ (or $O(1, 2)$) with a left-invariant metric when $\lambda > 1$;

(iii) flat when $\lambda = 1$. In fact, if $\lambda = 1$, then from (5.10), (5.11), (5.39) and (5.40) we see that $R = 0$.

Next, we suppose that $\rho_{11} - \rho_{22} = 4\lambda$ on N. Then from (5.13) we get $\Gamma_{312} = -1$. Thus, together with (5.10), (5.11) and (5.39), we have

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = -\lambda e_1, \quad [e_3, e_1] = \lambda e_2. \tag{5.41}$$

By similar arguments as in the former case, we see that M is locally isometric to $SL(2, \mathbb{R})$ (or $O(1, 2)$) with a special left-invariant metric.

If we differentiate (5.16) covariantly in the directions e_1 and e_2, then we have

$$\nabla_1\rho_{12} = \Gamma_{121}(\rho_{22} - \rho_{11}) - (1 + \lambda)\rho_{13},$$
$$\nabla_2\rho_{12} = \Gamma_{212}(\rho_{11} - \rho_{22}) + (1 - \lambda)\rho_{23}.$$

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Together with (5.29), we have

$$4\Gamma_{112}(\rho_{11} - \rho_{22}) = (\lambda + 2)\rho_{13} \quad \text{and} \quad 4\Gamma_{221}(\rho_{22} - \rho_{11}) = (\lambda - 2)\rho_{23}. \tag{5.42}$$

We now consider the case (II) $\lambda = 2$. Then from (5.38) we get $\rho_{13} = 0$. Differentiating this covariantly in the directions e_2 and e_3, we obtain

$$\nabla_2\rho_{13} = (1 - \lambda)(\rho_{11} - \rho_{33}) \quad \text{and} \quad \nabla_3\rho_{13} = -\Gamma_{312}\rho_{23}. \tag{5.43}$$

From (5.19), (5.27) and (5.43) we have

$$(\Gamma_{312} + 1)\rho_{23} = 0.$$

If there are interior points where $\rho_{23} = 0$, then in the same way as the case (I) we see that M is locally isometric to $SL(2, \mathbb{R})$ (or $O(1, 2)$) with a special left-invariant metric. So, we restrict ourselves to the place $\Gamma_{312} = -1$. Then from (5.13) we get $\rho_{11} - \rho_{22} = 8$. Hence from (5.42) we have $\Gamma_{212} = \Gamma_{221} = \Gamma_{121} = \Gamma_{112} = 0$, and thus we have the same conclusion.

Finally, we consider the case (III) $\lambda = -2$. From (5.28) we get $\rho_{23} = 0$. Differentiating this covariantly along e_1 and e_3, we obtain

$$\nabla_1\rho_{23} = (1 + \lambda)(\rho_{22} - \rho_{33}) \quad \text{and} \quad \nabla_3\rho_{23} = \Gamma_{312}\rho_{13}. \tag{5.44}$$

From (5.20), (5.27) and (5.44) we have

$$(\Gamma_{312} + 1)\rho_{13} = 0.$$

In a similar way as in $\lambda = 2$ we see that M is locally isometric to $SL(2, \mathbb{R})$ (or $O(1, 2)$) with a special left-invariant metric. We note that $\rho_{11} \neq \rho_{22}$ in the case $\lambda = -2$.

Conversely, by Theorem 2.3 we see that a Sasakian locally ϕ-symmetric space satisfies the condition (C) for any $\hat{\nabla}$-geodesic γ. Also, we easily see that a locally flat manifold always satisfies the condition (C) for any $\hat{\nabla}$-geodesic γ. Now, we consider a 3-dimensional Lie group with the Lie algebra structure

$$[e_1, e_2] = c_1 e_3,\ [e_2, e_3] = c_2 e_1,\ [e_3, e_1] = c_3 e_2, \tag{5.45}$$

for some constants $c_1 (\neq 0)$, c_2, c_3. Let $\{\omega_i\}$ be the dual 1-forms of the vector fields $\{e_i\}$. By using (5.45) we get $d\omega_3(e_1, e_2) = -d\omega_3(e_2, e_1) = -\frac{c_1}{2}$ and $d\omega_3(e_i, e_j) = 0$ for $(i, j) \neq (1, 2), (2, 1)$. Further we easily check that $\omega_3 \wedge d\omega_3(e_1, e_2, e_3) = -\frac{c_1}{6} (\neq 0)$, and hence ω_3 is a contact form and e_3 is the characteristic vector field. Define a Riemannian metric g and a (1,1)-tensor field ϕ by

$$g(e_i, e_j) = \delta_{ij}, \quad d\omega_3(e_i, e_j) = g(e_i, \phi e_j)$$

for $i, j = 1, 2, 3$. Then, since (ϕ, ω_3, g) has to be a contact Riemannian structure, we must have $g(\phi e_i, \phi e_j) = g(e_i, e_j) - \omega(e_i)\omega(e_j)$ for $i, j = 1, 2, 3$, and hence we have $c_1 = 2$.

We recall the Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ & + g(Y, [Z, X]) + g(Z, [X, Y]) - g(X, [Y, Z]) \end{aligned}$$

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where X, Y, Z are smooth vector fields on the manifold. Then we have

$$\begin{aligned} &\nabla_{e_1}e_1 = 0, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_3}e_3 = 0, \quad \nabla_{e_3}e_1 = \frac{1}{2}(c_2 + c_3 - 2)e_2, \\ &\nabla_{e_1}e_3 = \frac{1}{2}(c_2 - c_3 - 2)e_2, \quad \nabla_{e_2}e_3 = \frac{1}{2}(2 + c_2 - c_3)e_1. \end{aligned} \tag{5.46}$$

Further, from (5.46) and by the definition of the curvature tensor R, we can compute $R(e_i, e_j)e_k$ $(i, j, k = 1, 2, 3)$. In particular, we get

$$\begin{aligned} R(e_1, e_2)e_1 &= -\frac{1}{4}\{c_3^2 + c_2^2 - 2c_2c_3 + 4(c_2 + c_3) - 12\}e_2, \\ R(e_1, e_3)e_1 &= \frac{1}{4}\{3c_3^2 - c_2^2 - 2c_2c_3 + 4(c_2 - c_3) - 4\}e_3, \\ R(e_2, e_3)e_2 &= \frac{1}{4}\{3c_2^2 - c_3^2 - 2c_2c_3 + 4(c_3 - c_2) - 4\}e_3, \\ R(e_1, e_3)e_3 &= -\frac{1}{4}\{3c_3^2 - c_2^2 - 2c_2c_3 + 4(c_2 - c_3) - 4\}e_1, \\ R(e_2, e_3)e_3 &= -\frac{1}{4}\{3c_2^2 - c_3^2 - 2c_2c_3 + 4(c_3 - c_2) - 4\}e_2, \\ R(e_i, e_j)e_k &= 0 \text{ for } i \neq j \neq k \neq i, \\ &\text{etc.} \end{aligned} \tag{5.47}$$

From (2.4) and (5.46) we obtain

$$he_1 = \frac{c_3 - c_2}{2}e_1, \quad he_2 = -\frac{c_3 - c_2}{2}e_2. \tag{5.48}$$

Also, from (2.11) and (5.48) we have

$$\hat{\nabla}_{e_3}e_1 = \frac{1}{2}(c_2 + c_3)e_2, \quad \hat{\nabla}_{e_3}e_2 = -\frac{1}{2}(c_2 + c_3)e_1, \quad \text{all other } \hat{\nabla}_{e_i}e_j = 0. \tag{5.49}$$

In view of (5.40) and (5.41), we consider the two possible cases: (i) $c_2 + c_3 = 0$, (ii) $c_2 + c_3 = 2$.

Case (i). From (5.49) it follows that $\hat{\nabla}_{e_i}e_j = 0$ for $i, j = 1, 2, 3$. Thus we see that $\hat{\nabla} R = 0$.

Case (ii). Then from (5.47) we get

$$\begin{aligned} &R(e_1, e_2)e_1 = (1 - (c_2 - 1)^2)e_2, \\ &R(e_1, e_3)e_1 = R(e_2, e_3)e_2 = c_2(c_2 - 2)e_3, \\ &R(e_1, e_3)e_3 = -c_2(c_2 - 2)e_1, \ R(e_2, e_3)e_3 = -c_2(c_2 - 2)e_2, \\ &R(e_i, e_j)e_k = 0 \text{ for } i \neq j \neq k \neq i, \\ &\text{etc.} \end{aligned} \tag{5.50}$$

After some long but straightforward computations, we can check that the manifold satisfies $(\hat{\nabla}_x R)(y, x)x = 0$ for all tangent vector x and y.

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REFERENCES

1. E. Barletta and S. Dragomir, Differential equations on contact Riemannian manifolds, *Ann. Scuola Norm. Pisa, Cl. Sci. (4)* **30** (2001), 63–95.

2. D. E. Blair, Riemannian geometry of contact and symplectic manifolds, *Progress in Mathematics* **203** (Birkhäuser, 2003).

3. D. E. Blair, When is the tangent sphere bundle locally symmetric?, in *Geometry and topology* (ed. G. Stratopoulos and G. M. Rassias), (World Scientific, Singapore, 1989), 15–30.

4. D. E. Blair, Critical associated metrics on contact manifolds III, *J. Austral. Math. Soc. Ser. A* **50** (1991), 189–196.

5. D. E. Blair, T. Koufogiorgos and R. Sharma, A classification of 3-dimensional contact metric manifolds with $Q\phi = \phi Q$, *Kodai Math. J.* **13** (1990), 391–401.

6. D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, Contact metric manifolds satisfying a nullity condition, *Israel J. Math.* **91** (1995), 189–214.

7. D. E. Blair and L. Vanhecke, Symmetries and ϕ-symmetric spaces, *Tôhoku Math. J.* **39** (1987), 373–383.

8. E. Boeckx, A class of locally ϕ-symmetric contact metric spaces, *Arch. Math. (Basel)* **72** (1999), 466–472.

9. E. Boeckx, A full classification of contact metric (k, μ)-spaces, *Illinois J. Math.* **44** (2000), 212–219.

10. E. Boeckx and L. Vanhecke, Characteristic reflections on unit tangent sphere bundles, *Houston J. Math.* **23** (1997), 427–448.

11. E. Boeckx, D. Perrone and L. Vanhecke, Unit tangent sphere bundles and two-point homogeneous spaces, *Period. Math. Hungar.* **36** (1998), 79–95.

12. J. T. Cho, A new class of contact Riemannian manifolds, *Israel J. Math* **109** (1999), 299–318.

13. P. Dombroski, On the geometry of tangent bundle, *J. Reine Angew. Math.* **210** (1962), 73–88.

14. P. Gilkey, A. Swann and L. Vanhecke, Isoparametric geodesic spheres and a conjecture of Osserman concerning the Jacobi operator, *Quart. J. Math. Oxford Ser. (2)* **46** (1995), 299–320.

15. A. Gray, Classification des variétés approximativement kählériennes de courbure sectionelle holomorphe constante, *J. Reine Angew. Math.* **279** (1974), 797–800.

16. S. Ianuş, Sulle varietà di Cauchy-Riemann, *Rend. Accad. Sci. Fis. Mat. Napoli (4)* **39** (1972), 191–195.

17. O. Kowalski, Curvature of the induced Riemannian metric of tangent bundle of a Riemannian manifold, *J. Reine Angew. Math.* **250** (1971), 124–129.

18. J. Milnor, Curvature of left invariant metrics on Lie groups, *Adv. in Math.* **21** (1976).

19. M. Okumura, Some remarks on spaces with a certain contact structure, *Tôhoku Math. J.* **14** (1962), 135–145.

20. D. Perrone, Torsion tensor and critical metrics on contact $(2n + 1)$-manifolds, *Monatsh. Math.* **114** (1992), 245–259.

21. D. Perrone, Tangent sphere bundles satisfying $\nabla_\xi \tau = 0$, *J. Geom.* **49** (1994), 178–188.

22. S. Sasaki and Y. Hatakeyama, On differentiable manifolds with certain structures which are close related to almost contact structure II, *Tôhoku Math. J.* **13** (1961), 281–294.

23. S. Sasaki and Y. Hatakeyama, On differentiable manifolds with contact metric structure, *J. Math. Soc. Japan* **14** (1962), 249–271.

24. T. Takahashi, Sasakian ϕ-symmetric spaces, *Tôhoku Math. J.* **29** (1977), 91–113.

25. N. Tanaka, On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections, *Japan J. Math.* **2** (1976), 131–190.

26. S. Tanno, The topology of contact Riemannian manifolds, *Illinois J. Math.* **12** (1968), 700–717.

27. S. Tanno, Variational problems on contact Riemannian manifolds, *Trans. Amer. Math. Soc.* **314** (1989), 349–379.

28. S. Tanno, The standard CR structure on the unit tangent sphere bundle, *Tôhoku Math. J.* **44** (1992), 349–379.

29. Y. Tashiro, On contact structures of unit tangent sphere bundles, *Tôhoku Math. J.* **21** (1969), 117–143.

30. F. Tricerri and L.Vanhecke, *Homogeneous structures on Riemannian manifolds,* London Math. Soc. Lecture Notes No 83 (Cambridge University Press, 1983).

31. L. Vanhecke and T. J. Willmore, Interactions of tubes and spheres, *Math. Anal.* **263** (1983), 31–42.

32. K. Yano and S. Ishihara, *Tangent and cotangent bundles* (M. Dekker Inc., 1973).