Canad. Math. Bull. Vol. **31** (2), 1988

REDUCED SOBOLEV INEQUALITIES

BY

R. A. ADAMS

ABSTRACT. The Sobolev inequality of order m asserts that if $p \geqq 1$, $mp < n$ and $1/q = 1/p - m/n$, then the L^q-norm of a smooth function with compact support in $\mathbf{R}^n$ is bounded by a constant times the sum of the L^p-norms of the partial derivatives of order m of that function. In this paper we show that that sum may be reduced to include only the completely mixed partial derivatives or order m, and in some circumstances even fewer partial derivatives.

1. **Introduction.** Sobolev's inequality of order m, namely

$$(1)\quad \|u\|_q \leqq K \sum_{|\alpha|=m} \|D^\alpha u\|_p, \quad \text{where} \quad q = \begin{cases} \dfrac{np}{n-mp} & \text{if} \quad mp < n \\ \infty & \text{if} \quad p = 1, m = n \end{cases}$$

holds, with fixed constant K, for all functions $u \in C_0^\infty(\mathbf{R}^n)$, the space of infinitely differentiable functions with compact support in $\mathbf{R}^n$, or, more generally, for all sufficiently smooth functions u which decay sufficiently rapidly at infinity. Here, of course, $\|\cdot\|_p$ denotes the norm in the space $L^p(\mathbf{R}^n)$, $p \geqq 1$, and $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}$, where $D_j = \partial/\partial x_j$, $1 \leqq j \leqq n$, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-dimensional multi-index of nonnegative integers of order $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

The purpose of this paper is to show that the sum on the right side of Sobolev's inequality (1) can, if $m \geqq 2$, be replaced by a reduced sum taken over only those partial derivatives of order m which are "completely mixed" in the sense that all m differentiations are taken with respect to different variables. Denoting

$$\mathscr{M} = \mathscr{M}(n, m) = \{\alpha : |\alpha| = m, \alpha_j = 0 \text{ or } 1 \text{ for } 1 \leqq j \leqq n\},$$

we shall show (Theorem 3.3 below) that all $u \in C_0^\infty(\mathbf{R}^n)$ satisfy a *reduced Sobolev inequality* of the form

Received by the editors June 16, 1986.

Research partially supported by the Centre for Mathematical Analysis of the Australian National University, and by the Natural Sciences and Engineering Research Council of Canada under Operating Grant A8869.

AMS Subject Classification (1980): 46E35.

https://doi.org/10.4153/CMB-1988-024-1 Published online by Cambridge University Press

$$(2) \qquad \|u\|_q \leqq K \sum_{\alpha \in \mathscr{M}} \|D^\alpha u\|_p,$$

where q has the same value as in (1). The special case $p = 1$ of (2) was remarked by Stein [3, p. 160]. As an example, if $n = 3$, $m = 2$ and $1 \leqq p < 3/2$, we can find a constant K such that for all $u \in C_0^\infty(\mathbf{R}^3)$,

$$\|u\|_{3p/(3-2p)} \leqq K(\|D_1D_2u\|_p + \|D_1D_3u\|_p + \|D_2D_3u\|)_p,$$

the sum on the right involves only three of the six partial derivatives of u of order 2. Observe also that for $m = n$ and $p = 1$ the set $\mathscr{M}$ has only one element, $\alpha = (1, 1, \ldots, 1)$, and so (2) says, in this case,

$$\|u\|_\infty \leqq K\|D_1D_2 \ldots D_nu\|_1$$

which follows at once (with $K = 1$) from the representation

$$u(x) = \int_{-\infty}^{x_1} dy_1 \int_{-\infty}^{x_2} dy_2 \ldots \int_{-\infty}^{x_n} D_1D_2 \ldots D_nu(y)dy_n.$$

(We shall see later that K can be taken to be $1/2^n$.)

It is well known that Sobolev's inequality (1), (and therefore also (2)), is invariant under dilation of u. Indeed, if $u \in C_0^\infty(\mathbf{R}^n)$ is fixed and $u_\lambda(x) = u(\lambda x)$ then $u_\lambda \in C_0^\infty(\mathbf{R}^n)$ for any $\lambda > 0$ and

$$\|u_\lambda\|_q = \lambda^{-n/q}\|u\|_q,$$
$$\|D^\alpha u_\lambda\|_p = \lambda^{m-n/p}\|D^\alpha u\|_p \quad \text{for} \quad |\alpha| = m.$$

Hence (1) or (2) imply that

$$\lambda^{-n/q-m+n/p} \leqq \frac{K \sum \|D^\alpha u\|_p}{\|u\|_q},$$

which cannot hold for all $\lambda > 0$ unless

$$\frac{n}{q} = \frac{n}{p} - m,$$

that is, unless q is given as in (1). In Section 4 of this paper we will consider the possibility of further reducing (2) so that the sum on the right side extends over a subset of $\mathscr{M}$. The above argument shows that no such reduction can lead to a different value for q.

2. **Mixed norms.** Our proof of the reduced Sobolev inequality (2) is based on mixed norm estimates in a manner similar to their use in Fournier [2] and Adams [1]. We give a brief summary here of the elementary facts about mixed norms that we shall need. See [1] or [2] for more details.

If $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, where $0 < p_j \leqq \infty$ for each j, we construct the number $\|u\|_{\mathbf{p}}$ by first taking the L^{p_1} norm of u with respect to x_1, then the L^{p_2} norm of the result with respect to x_2, and so on, finishing with the L^{p_n} norm with respect to x_n. (Of course these are not actually norms unless each $p_j \geqq 1$.)

$$\|u\|_{\mathbf{p}} = \| \ldots \| \|u\|_{L^{p_1}(dx_1)}\|_{L^{p_2}(dx_2)} \cdots \|_{L^{p_n}(dx_n)}.$$

Evidently $\|u\|_{(p,p,\ldots,p)} = \|u\|_p$. We require the mixed norm *Hölder inequality*

$$\left\|\prod_{j=1}^{k} u_j\right\|_{\mathbf{q}} \leqq \prod_{j=1}^{k} \|u_j\|_{\mathbf{p}_j}$$

where $1/\mathbf{q} = \sum_{j=1}^{k} (1/\mathbf{p}_j)$, that is, where $\mathbf{q} = (q_1, \ldots, q_n)$ has components given by

$$\frac{1}{q_i} = \sum_{j=1}^{k} \frac{1}{(p_j)_i} \quad \text{for} \quad i = 1, \ldots, n.$$

The definition of $\|\cdot\|_{\mathbf{p}}$ requires that the individual L^{p_j} norms be evaluated in component order. This order can be altered by means of a permutation σ of $\{1, 2, \ldots, n\}$. If $\sigma\mathbf{p} = (p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(n)})$, $\sigma x = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$, and $\sigma u(\sigma x) = u(x)$, then $\|\sigma u\|_{\sigma\mathbf{p}}$ is called a permuted mixed norm of u; it involves the same L^{p_j} norms with respect to the same variables as does $\|u\|_{\mathbf{p}}$, but taken in a different order. In general the value of $\|\sigma u\|_{\sigma\mathbf{p}}$ varies with σ; the *permutation inequality* states that the largest value for $\|\sigma u\|_{\sigma\mathbf{p}}$ occurs for any σ for which the components of $\sigma\mathbf{p}$ are in non-increasing order:

$$p_{\sigma(1)} \geqq p_{\sigma(2)} \geqq \ldots \geqq p_{\sigma(n)}.$$

In general the value of a mixed norm is increased if the order of the two adjacent L^{p_j} norms is transposed resulting in the larger L^{p_j} norm being evaluated earlier.

3. **Mixed-norm and reduced Sobolev inequalities.** Our proof of the reduced Sobolev inequality (2) relies on the following mixed-norm version of the first order Sobolev inequality.

3.1 THEOREM. *Let $n \geqq 2$ and $1 \leqq p \leqq q$. Let r satisfy*

$$\frac{n}{r} = \frac{1}{p} + \frac{n-1}{q} - 1 > 0. \tag{3}$$

For $j = 1, \ldots, n$ let $\mathbf{v}_j(p, q) = (q, q, \ldots, p, \ldots, q)$ have all components equal to q except the j'th component which is p. There exists a constant K such that for all $u \in C_0^\infty(\mathbf{R}^n)$,

$$\|u\|_r \leqq K \sum_{j=1}^{n} \|D_j u\|_{\mathbf{v}_j(p,q)}. \tag{4}$$

https://doi.org/10.4153/CMB-1988-024-1 Published online by Cambridge University Press

PROOF. Let $s \geqq 1$. Starting with the identity

$$|u(x)|^s = \int_{-\infty}^{x_j} D_j|u(x)|^s dx_j$$

we obtain the inequality

$$\sup_{x_j} |u(x)|^s \leqq s \int_{-\infty}^{\infty} |u(x)|^{s-1}|D_j u(x)|dx_j. \tag{5}$$

Let $\lambda \geqq 1$ be given by

$$\frac{1}{\lambda} = \frac{1}{q} + \frac{1}{p'} = \frac{1}{q} + 1 - \frac{1}{p}.$$

(Here p' is the exponent conjugate to p.) Taking the L^λ norm of both sides of (5) we obtain

$$\|\sigma|u|^s\|_{\sigma\mathbf{v}_j(\infty,\lambda)} \leqq s\|\sigma|u|^{s-1}D_j u\|_{\sigma\mathbf{v}_j(1,\lambda)},$$

where σ is any permutation of $\{1, 2, \ldots, n)$ for which $\sigma(1) = j$. An application of Hölder's inequality sandwiched between two applications of the permutation inequality for mixed norms gives us

$$\begin{aligned}\|u\|^s_{\mathbf{v}_j(\infty,s\lambda)} &= \|\,|u|^s\|_{\mathbf{v}_j(\infty,\lambda)} \leqq \|\sigma|u|^s\|_{\sigma\mathbf{v}_j(\infty,\lambda)}\\ &\leqq s\|\sigma|u|^{s-1}D_j u\|_{\sigma\mathbf{v}_j(1,\lambda)}\\ &\leqq s\|\sigma|u|^{s-1}\|_{\sigma\mathbf{v}_j(p',p')}\|\sigma D_j u\|_{\sigma\mathbf{v}_j(p,q)}\\ &\leqq s\|u\|^{s-1}_{(s-1)p'}\|D_j u\|_{\mathbf{v}_j(p,q)}.\end{aligned}$$

Note that $p \leqq q$ is needed to justify the last inequality above. We now have

$$\|u\|_{\mathbf{v}_j(\infty,s\lambda)} \leqq K\|u\|^{1-1/s}_{(s-1)p'}\|D_j u\|^{1/s}_{\mathbf{v}_j(p,q)}.$$

(Throughout this and subsequent proofs K represents various constants independent of $u \in C_0^\infty(\mathbf{R}^n)$, and may change from line to line.) Let t satisfy $1/\mathbf{t} = \sum_{j=1}^n (1/\mathbf{v}_j(\infty, s\lambda))$. Evidently $\mathbf{t} = (t, t, \ldots, t)$ where $t = s\lambda/(n-1)$. Using Hölder's inequality again we obtain

$$\begin{aligned}\|u\|^n_{nt} &= \left\|\prod_{j=1}^n u\right\|_{\mathbf{t}} \leqq \prod_{j=1}^n \|u\|_{\mathbf{v}_j(\infty,s\lambda)}\\ &\leqq K\|u\|^{n-n/s}_{(s-1)p'} \prod_{j=1}^n \|D_j u\|^{1/s}_{\mathbf{v}_j(p,q)}.\end{aligned} \tag{6}$$

Clearly we want to choose s so that

$$(s-1)p' = nt = \frac{ns\lambda}{n-1} = \frac{ns}{n-1}\frac{qp'}{q+p'} \tag{7}$$

https://doi.org/10.4153/CMB-1988-024-1 Published online by Cambridge University Press

Solution of (7) for s leads to the common value $(s - 1)p' = nt = r$, where r is given by (3). Cancellation of the common factor in (6) then gives us

$$\|u\|_r \leqq K\left(\prod_{j=1}^{n} \|D_j u\|_{\mathbf{v}_j(p,q)}\right)^{1/n}$$

$$\leqq K \prod_{j=1}^{n} \|D_j u\|_{\mathbf{v}_j(p,q)},$$

as required. □

3.2 REMARK. Inequality (4) is also invariant under dilation and cannot hold for all $u \in C_0^\infty(\mathbf{R}^n)$ unless r satisfies (3). Therefore we can avoid the algebra to solve (7) – it must lead to the correct value for r.

3.3 THEOREM. *Let $p \geqq 1$, $m \geqq 1$, $mp < n$, and let r satisfy*

$$\frac{n}{r} = \frac{n}{p} - m.$$

Then there exists a constant K such that for all $u \in C_0^\infty(\mathbf{R}^n)$,

$$\|u\|_r \leqq K \sum_{\alpha \in \mathscr{M}} \|D^\alpha u\|_p.$$

PROOF. We proceed by induction on m. The case $m = 1$ is the usual first-order version of Sobolev's inequality, and it is also the special case $q = p$ of Theorem 3.1. Suppose, therefore, that the case $m - 1$ has been proved. We consider the case m. By Theorem 3.1 we have

$$\|u\|_r \leqq K \prod_{j=1}^{n} \|D_j u\|_{\mathbf{v}_j(p,q)}$$

where $p \leqq q$ and r satisfies

$$\frac{n}{r} = \frac{1}{p} + \frac{n-1}{q} - 1.$$

Now apply the induction hypothesis to $D_j u$, considered as a function of the $n - 1$ variables excluding x_j:

$$\|D_j u\|_{L^q(\mathbf{R}^{n-1})} \leqq K \sum_{\substack{\beta \in \mathscr{M}(n,m-1) \\ \beta_j = 0}} \|D^\beta D_j u\|_{L^p(\mathbf{R}^{n-1})} \tag{8}$$

where

$$\frac{n-1}{q} = \frac{n-1}{p} - (m - 1). \tag{9}$$

https://doi.org/10.4153/CMB-1988-024-1 Published online by Cambridge University Press

Observe that q, as determined by (9), is indeed larger than p. We take the L^p norm of (8) with respect to the remaining variable x_j. Since $p \leqq q$ we can transpose that L^p norm into its correct (j'th) position and hence obtain

$$\|D_j u\|_{v_j(p,q)} \leqq K \sum_{\substack{\beta \in \mathcal{M}(n,m-1) \\ \beta_j = 0}} \|D^\beta D_j u\|_p.$$

Thus

$$\|u\|_r \leqq K \sum_{j=1}^{n} \sum_{\substack{\beta \in \mathcal{M}(n,m-1) \\ \beta_j = 0}} \|D^\beta D_j u\|_p \leqq K \sum_{\beta \in \mathcal{M}(n,m)} \|D^\alpha u\|_p,$$

where

$$\frac{n}{r} = \frac{1}{p} + \frac{n-1}{q} - 1 = \frac{1}{p} + \frac{n-1}{p} - (m-1) - 1 = \frac{n}{p} - m,$$

and the induction is complete. □

4. **Further reductions.** Is it possible to replace $\mathcal{M}$ in (2) with a proper subset of $\mathcal{M}$? For some values of m, n and p the answer is yes. However, the techniques we are using in this paper are well suited to address this question only for the special case $p = 1$. Only partial results are accessible if $p > 1$.

Let $\mathcal{S}$ be a subset of $\mathcal{M}(n, m)$ satisfying the condition

$$(10) \qquad \sum_{\alpha \in \mathcal{S}} \alpha_j = k \geqq 1, \qquad (j = 1, 2, \ldots, n),$$

where k is independent of j. If c is the number of elements in $\mathcal{S}$ then

$$(11) \qquad nk = \sum_{j=1}^{n} \sum_{\alpha \in \mathcal{S}} \alpha_j = \sum_{\alpha \in \mathcal{S}} \sum_{j=1}^{n} \alpha_j = mc.$$

We shall show that, at least for $p = 1$, the set $\mathcal{M}$ in (2) can be replaced with $\mathcal{S}$. For $\mathcal{S} = \mathcal{M}$ we have

$$c = \binom{n}{m} = \frac{n!}{m!(n-m)!}$$

while

$$k = \binom{n-1}{m-1} = \frac{(n-1)!}{(m-1)!(n-m)!}.$$

If $n = 4$ and $m = 2$ there are several possibilities for the choice of $\mathcal{S}$, among them the sets

https://doi.org/10.4153/CMB-1988-024-1 Published online by Cambridge University Press

$$\mathscr{S}_1 = \{(1, 1, 0, 0), (0, 0, 1, 1)\},$$

$$\mathscr{S}_2 = \{(1, 1, 0, 0), (0, 1, 1, 0), (0, 0, 1, 1), (1, 0, 0, 1)\}.$$

For $\mathscr{S}_1$ we have $k = 1$, $c = 2$; for $\mathscr{S}_2$, $k = 2$, $c = 4$. Both $\mathscr{S}_1$ and $\mathscr{S}_2$ are proper subsets of $\mathscr{M}(4, 2)$, which has six elements.

4.1 THEOREM. *Let $m < n$ and let $\mathscr{S}$ be a subset of $\mathscr{M}(n, m)$ satisfying* (10) *and having c elements. If $q = n/(n - m)$ then the reduced Sobolev inequality*

$$\|u\|_q \leqq \frac{1}{2^m c} \sum_{\alpha \in \mathscr{S}} \|D^\alpha u\|_1, \tag{12}$$

holds for all $u \in C_0^\infty(\mathbf{R}^n)$.

PROOF. Since

$$u(x) = \int_{-\infty}^{x_1} D_1 u(\xi, x_2, \ldots, x_n) d\xi = -\int_{x_1}^{\infty} D_1 u(\xi, x_2, \ldots, x_n) d\xi,$$

therefore

$$\sup_{x_1} |u(x)| \leqq \frac{1}{2} \int_{-\infty}^{\infty} |D_1 u(x)| dx_1.$$

Iterating this inequality to take successive suprema with respect to $x_2, \ldots, x_m$ we obtain

$$\sup_{x_1, x_2, \ldots, x_m} |u(x)| \leqq \frac{1}{2^m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |D_1 D_2 \ldots D_m u| dx_1 \ldots dx_m.$$

Integrating the remaining variables leads to

$$\|u\|_{(\infty, \ldots, \infty, 1, \ldots, 1)} \leqq \frac{1}{2^m} \|D_1 D_2 \ldots D_m u\|_1.$$

Similarly, for any $\alpha \in \mathscr{M}(n, m)$ we have, by the permutation inequality,

$$\|u\|_{\mathbf{w}_\alpha} \leqq \frac{1}{2^m} \|D^\alpha u\|_1$$

where $\mathbf{w}_\alpha$ has j'th component given by

$$(\mathbf{w}_\alpha)_j = \begin{cases} \infty & \text{if } \alpha_j = 1 \\ 1 & \text{if } \alpha_j = 0. \end{cases}$$

Now $\sum_{\alpha \in \mathscr{S}} (1/\mathbf{w}_\alpha) = 1/\mathbf{r}$, where, by (11),

$$\frac{1}{r_j} = c - k = k\frac{n - m}{m} = \frac{1}{r} \text{ (independent of } j).$$

https://doi.org/10.4153/CMB-1988-024-1 Published online by Cambridge University Press

Also, $q = n/(n - m) = cr$, so by Hölder's inequality

$$\|u\|_q^c = \| \, |u|^c \|_r = \left\| \prod_{\alpha \in \mathcal{S}} u \right\|_{\mathbf{r}} \leqq \prod_{\alpha \in \mathcal{S}} \|u\|_{\mathbf{w}_\alpha} \leqq \prod_{\alpha \in \mathcal{S}} \frac{1}{2^m} \|D^\alpha u\|_1.$$

The desired inequality (12) now follows by virtue of the inequality between geometric and arithmetic means. □

It seems reasonable to conjecture that if $p > 1$ and $mp < n$ then

$$\|u\|_q \leqq K \sum_{\alpha \in \mathcal{S}} \|D^\alpha u\|_p \tag{13}$$

holds for all $u \in C_0^\infty(\mathbf{R}^n)$ provided $q = np/(n - mp)$ and $\mathcal{S}$ satisfies (10). The author does not know how to prove this in general; the mixed-norm techniques used here are not adequate. Some special cases, however, can be confirmed. For instance (13) holds provided $m = 2$ and provided the number k in (10) satisfies $k \geqq n/2$. To see this, pick j and let $S_j = \{i \neq j : \alpha_i = \alpha_j = 1 \text{ for some } \alpha \in \mathcal{S}\}$. Evidently S_j has k elements and since $2p < n \leqq 2k$ we can apply the ordinary first order Sobolev inequality to $D_j u$ considered as a function of the k variables $\{x_i : i \in S_j\}$ to obtain

$$\|D_j u\|_{L^r(\mathbf{R}^k)} \leqq K \sum_{i \in S_j} \|D_{ij} u\|_{L^p(\mathbf{R}^k)},$$

where $k/r = (k/p) - 1$. Taking L^p norms with respect to the remaining variables leads to

$$\|D_j u\|_{\mathbf{w}_j(p,r,S_j)} \leqq K \sum_{i \in S_j} \|D_{ij} u\|_p,$$

where $\mathbf{w}_j(p, r\ S_j)$ has i'th component equal to r if $i \in S_j$ and equal to p otherwise. Now the proof of Theorem 3.1 can be easily modified to show that

$$\|u\|_q \leqq K \sum_{j=1}^{n} \|D_j u\|_{\mathbf{w}_j(p,r,S_j)}$$

provided

$$\frac{n}{q} = \frac{n-k}{p} + \frac{k}{r} - 1 > 0.$$

Thus we have

$$\|u\|_q \leqq K \sum_{j=1}^{n} \sum_{i \in S_j} \|D_{ij} u\|_p \leqq K \sum_{\alpha \in \mathcal{S}} \|D^\alpha u\|_p$$

provided

https://doi.org/10.4153/CMB-1988-024-1 Published online by Cambridge University Press

$$\frac{n}{q} = \frac{n-k}{p} + \frac{k}{p} - 1 - 1 = \frac{n}{p} - 2.$$

References

1. R. A. Adams, *Anisotropic Sobolev inequalities*, Research Report CMA-R05-86, Centre for Mathematical Analysis, The Australian National University.

2. John J. F. Fournier, *Mixed norms and rearrangements: Sobolev's inequality and Littlewood's inequality*, to appear Ann. Mat. Pura Appl.

3. Elias M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, 1970.

Department of Mathematics,
The University of British Columbia,
Vancouver, B. C., Canada V6T 1Y4

https://doi.org/10.4153/CMB-1988-024-1 Published online by Cambridge University Press