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A Formal Chevalley Restriction Theorem for Kac–Moody Groups

CLAUS MOKLER
Fachbereich Mathematik, Universität Wuppertal, Gaußstraße 20, D-42097 Wuppertal, Germany. e-mail: mokler@math.uni-wuppertal.de

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Abstract. Let G be a symmetrizable Kac–Moody group over a field of characteristic zero, let T be a split maximal torus of G. By using a completion of the algebra of strongly regular functions on G, and its restriction on T, we give a formal Chevalley restriction theorem. Specializing to the affine case, and to the field of complex numbers, we obtain a convergent Chevalley restriction theorem, by choosing the formal functions, which are convergent on the semigroups of trace class elements $G^{\mathrm{tr}} \subset G$ resp. $T^{\mathrm{tr}} \subset T$.

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Introduction

Let G be a reductive linear algebraic group, and equip its coordinate ring $\mathbb{C}[G]$ with the G-action, induced by the conjugation action of G on itself. Let T be a maximal torus of G with corresponding Weyl group $\mathcal{W}$, and equip the coordinate ring $\mathbb{C}[T]$ of T with the $\mathcal{W}$-action, induced by the conjugation action of $\mathcal{W} = N_G(T)/T$ on T. Let $\mathcal{X}(T)^+$ be the monoid of dominant characters of T with respect to some Borel subgroup containing T. Denote by Tr_Λ the character of the rational irreducible G-representation $(L(\Lambda), \pi_\Lambda)$, belonging to the highest weight $\Lambda \in \mathcal{X}(T)^+$.

The Chevalley restriction theorem says that the restriction map $r: \mathbb{C}[G] \to \mathbb{C}[T]$ induces an isomorphism of the invariant algebras $\mathbb{C}[G]^G$ and $\mathbb{C}[T]^{\mathcal{W}}$. Furthermore, the characters Tr_Λ, $\Lambda \in \mathcal{X}(T)^+$, form a $\mathbb{C}$-base of $\mathbb{C}[G]^G$. If G is in addition semisimple, simply connected, then $\mathbb{C}[G]^G$ is a polynomial algebra in the characters Tr_{N_i} belonging to the fundamental dominant weights N_i, $i = 1, \ldots, n$.

V. Kac and D. Peterson constructed in [K, P 1], to a Kac–Moody algebra $\mathbf{g}$ over a field $\mathbb{F}$ of characteristic 0, a group analogue of a semisimple, simply connected algebraic group, the Kac–Moody group G. In the symmetrizable case, they defined and investigated in [K, P 2] the algebra of strongly regular functions $\mathbb{F}[G]$ on G. They showed that it admits a Peter and Weyl theorem, i.e., $\mathbb{F}[G] \cong \bigoplus_{\Lambda \in P^+} L^*(\Lambda) \otimes L(\Lambda)$

as a $G \times G$-module, (P^+ the set of dominant weights). Due to this result, it seems reasonable to interpret the algebra of strongly regular functions as the analogue of the coordinate ring of a semisimple, simply connected algebraic group. (For a precise investigation of this question, see [M 1].)

However, in the nonclassical case, there is no direct analogue of the Chevalley restriction theorem. If the generalized Cartan matrix has no component of finite type, then, with the exception of one-dimensional representations, the irreducible highest weight representations $L(\Lambda)$ are infinite-dimensional, and their traces cannot be realized as functions on G, resp. T. The invariant algebras $\mathbb{F}[G]^G$, resp. $\mathbb{F}[T]^{\mathcal{W}}$ are spanned only by the traces of these one-dimensional representations on G, resp. T. (Here $\mathbb{F}[T]$ denotes the restriction of the algebra of strongly regular functions on T.)

To obtain nevertheless a formal analogue of the Chevalley restriction theorem, we complete the algebras $\mathbb{F}[G]$, $\mathbb{F}[T]$ in a natural way, obtaining a G-algebra $\widehat{\mathbb{F}[G]}$, and a $\mathcal{W}$-algebra $\widehat{\mathbb{F}[T]}$. The restriction map extends, inducing an isomorphism of the invariant algebras $\widehat{\mathbb{F}[G]}^G$ and $\widehat{\mathbb{F}[T]}^{\mathcal{W}}$. These invariant algebras are, in a certain sense, spanned by the formal G resp. T-characters of the modules $L(\Lambda)$, $\Lambda \in P^+$. We also obtain an algebraic description of these algebras.

Specializing to the field of complex numbers, every irreducible highest weight module $L(\Lambda)$, $\Lambda \in P^+$, carries a contravariant positive definite Hermitian form, unique up to a nonzero positive scalar factor. Denote by G^{tr} the semigroup of elements $g \in G$, such that for all $\Lambda \in P^+$ the linear map $\pi_\Lambda(g)$ extends to a trace class operator on the Hilbert space completion of $L(\Lambda)$. Denote by T^{tr} the intersection of G^{tr} with T.

To obtain a convergent Chevalley restriction theorem in the affine case, we realize certain subalgebras of $\widehat{\mathbb{C}[G]}$, $\widehat{\mathbb{C}[T]}$, as algebras of functions $\mathbb{C}[G^{\mathrm{tr}}]$, $\mathbb{C}[T^{\mathrm{tr}}]$ on the semigroups G^{tr}, T^{tr}. In particular, the formal characters are now realized as functions on G^{tr}, respectively T^{tr}. These algebras carry a G-, respectively a $\mathcal{W}$-action, and the restriction map induces an injective homomorphism of $\mathbb{C}[G^{\mathrm{tr}}]^G$ into $\mathbb{C}[T^{\mathrm{tr}}]^{\mathcal{W}}$.

We restrict to the affine case because, for an indefinite Kac–Moody group, G^{tr} is not invariant under G-conjugation. Even more worse, every element of T^{tr} is G-conjugate to some element not contained in G^{tr}, compare [M 2]. It remains open whether there is a similar result, now using a subalgebra of $\widehat{\mathbb{C}[G]}$, which can be realized as an algebra of functions on $\bigcup_{g\in G} gG^{\mathrm{tr}}g^{-1}$.

There are the following relations to the work of other people: E. Looijenga developed in [Loo], section 4, an invariant theory of exponential type associated to certain root data of generalized Cartan matrices. Using the fans $\Lambda - Q_0^+$, $\Lambda \in P^+$, he built a $\mathbb{Z}$-algebra A. He defined and investigated the algebra of $\mathcal{W}$-invariants $A^{\mathcal{W}}$ and $\mathcal{W}$-anti-invariants $A^{-\mathcal{W}}$. In particular he obtained an algebraic description of $A^{\mathcal{W}}$ in the affine case. The invariant algebra $\widehat{\mathbb{F}[T]}^{\mathcal{W}}$ used for the formal Chevalley restriction theorem is constructed in a similar way as $A^{\mathcal{W}}$. Starting with $\mathbb{F}[T]$, we build the algebra $\widehat{\mathbb{F}[T]}$, and then the invariant algebra $\widehat{\mathbb{F}[T]}^{\mathcal{W}}$. But to define $\widehat{\mathbb{F}[T]}$, we use the set of weights of $L(\Lambda)$ instead of the fans $\Lambda - Q_0^+$, $\Lambda \in P^+$. Therefore, our space $\widehat{\mathbb{F}[T]}$ is much smaller than the corresponding space A adopted to the field $\mathbb{F}$. Our

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invariant space $\widehat{\mathbb{F}[T]}^{\mathcal{W}}$ is for a generalized Cartan matrix, which has a component of nonindefinite type, smaller than $A^{\mathcal{W}}$ adopted to $\mathbb{F}$. This modification is required because, in the case of a generalized Cartan matrix of finite type, we want $\widehat{\mathbb{F}[T]}$ to equal the classical coordinate ring of the torus, and $\widehat{\mathbb{F}[T]}^{\mathcal{W}}$ to equal its invariant algebra.

We obtain an algebraic description of the invariant algebra $\widehat{\mathbb{F}[T]}^{\mathcal{W}}$ for an arbitrary generalized Cartan matrix, generalizing the corresponding description of $A^{\mathcal{W}}$ by Looijenga in the affine case. Our main aid to prove this description is a finite covering of $P(\Lambda) \cap P^+$, $(\Lambda \in P^+)$, by certain set of weights, which are related to cones build of imaginary roots. This covering can be considered as a generalization of the covering by imaginary root strings in the affine case.

Let $\mathbb{F} = \mathbb{C}$. For $\Lambda \in P^+$, the formal T-character

$$\chi_\Lambda = \sum_{\lambda \in P(\Lambda)} m_\lambda e_\lambda, \quad m_\lambda := \dim(L(\Lambda)_\lambda),$$

determines a function on T^{tr} by $\chi_\Lambda(t) := \mathrm{Tr}(\pi_\Lambda(t))$, $t \in T^{\mathrm{tr}}$. These functions, as well as the domain of convergence T^{tr}, have been studied by several people starting with Moody and Lepowsky [L, Mo], Meurman [Meu] in the rank two hyperbolic case, Slodowy [Sl 2], Kac and Peterson, whose results can be found in [K], sections 10.6 and 11.10.

In the affine case G. Brüchert determined in [B] a conjugation invariant subsemigroup $G^{>1}$ of G^{tr}, and conjectured equality. He showed, that the functions Tr_Λ on G^{tr} defined by $\mathrm{Tr}_\Lambda(g) := \mathrm{Tr}(\pi_\Lambda(g))$, $g \in G^{\mathrm{tr}}$, are conjugation invariant on $G^{>1}$, $\Lambda \in P^+$.

We shall prove the conjecture of G. Brüchert. The functions χ_Λ, Tr_Λ fit into our framework. The conjugation invariance of Tr_Λ will be deduced from a more general theorem for the functions of $\mathbb{C}[G^{\mathrm{tr}}]$.

For an affine Kac–Moody group $\tilde{G}$ of holomorphic loops (which contains the Kac–Moody group G of Laurent polynomial loops as a subgroup), P. Etingov, I. Frenkel, and A. Kirillov defined and investigated in [E, F, K] spherical functions. These are functions on certain $\tilde{G}'$-conjugation invariant parts $\tilde{G}_q$ of $\tilde{G}$, $0 < q < 1$, with certain holomorphy properties, and certain properties with respect to the $\tilde{G}'$-conjugation action. The union of these parts $\tilde{G}_q$, $0 < q < 1$, is disjoint. It gives a subsemigroup of $\tilde{G}$, which extends the subsemigroup $G^{>1}$ of G.

In particular, Etingov, Frenkel, and Kirillov showed, that the characters of the integrable modules at level k form a basis in the space of $\tilde{G}'$-conjugacy invariant functions on $\tilde{G}_q$ of degree k, $k > 0$.

1. Preliminaries

In this section we recall some basic facts about Kac–Moody algebras, Kac–Moody groups, and the algebra of strongly regular functions, which are used later, merely to introduce our notation.

The Kac–Moody group given in [K, P 1], [K, P 3] corresponds to the derived Kac–Moody algebra. We work with a slightly enlarged group, corresponding to the full Kac–Moody algebra, as in [Ti], [Mo, Pi].

All the material stated in this subsection about Kac–Moody algebras can be found in the books [K] (most results also valid for a field of characteristic zero with the same proofs), [Mo, Pi], about Kac–Moody groups in [K, P 1], [K, P 3], [Mo, Pi], about the algebra of strongly regular functions in [K, P 2], and about the faces of the Tits cone in [Loo], [Sl 1].

Furthermore, we will prove some properties of the set of imaginary roots, and of the set of weights of irreducible admissible highest weight modules, which will be important in the following sections at several places.

We denote by $\mathbb{N} = \mathbb{Z}^+$, $\mathbb{Q}^+$, resp. $\mathbb{R}^+$ the sets of strictly positive numbers of $\mathbb{Z}$, $\mathbb{Q}$, resp. $\mathbb{R}$. The sets $\mathbb{N}_0 = \mathbb{Z}_0^+$, $\mathbb{Q}_0^+$, $\mathbb{R}_0^+$ contain, in addition, the zero. In the whole paper, $\mathbb{F}$ is a field of characteristic 0, and $\mathbb{F}^\times$ its group of units.

1.1. GENERALIZED CARTAN MATRICES

Starting point for the construction of a Kac–Moody algebra, and its associated simply connected Kac–Moody group is a *generalized Cartan matrix*, which is a matrix $A = (a_{ij}) \in M_n(\mathbb{Z})$ with $a_{ii} = 2$, $a_{ij} \leqslant 0$ for all $i \neq j$, and $a_{ij} = 0$ if and only if $a_{ji} = 0$. Denote by l the rank of A, and set $I := \{1, 2, \ldots, n\}$.

For the properties of the generalized Cartan matrices, in particular their classification, we refer to the book [K]. In this paper we assume A to be symmetrizable.

A nonempty subset $J \subseteq I$ is said to have a property, if the corresponding submatrix of A, which is a generalized Cartan matrix, has this property.

1.2. REALIZATIONS

A *simply connected minimal free realization* of A consists of dual free $\mathbb{Z}$-modules H, P of rank $2n - l$, and linear independent sets $\Pi^\vee = \{h_1, \ldots, h_n\} \subseteq H$, $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subseteq P$ such that $\alpha_i(h_j) = a_{ji}$, $i, j = 1, \ldots, n$. Furthermore, there exist (in general nonuniquely determined) weights $N_1, \ldots, N_n \in P$ such that $N_i(h_j) = \delta_{ij}$, $i, j = 1, \ldots, n$.

P is called the *weight lattice* and $Q := \mathbb{Z}\text{-span}\{\alpha_i \mid i \in I\}$ the *root lattice*. Set

$$Q_0^\pm := \mathbb{Z}_0^\pm\text{-span}\{\alpha_i \mid i \in I\}, \qquad Q^\pm := Q_0^\pm \setminus \{0\}.$$

We fix $N_1, \ldots, N_n \in P$ as above, and set $P_I := \mathbb{Z}\text{-span}\{N_1, \ldots, N_n\}$. Extending $h_1, \ldots, h_n \in H$, $N_1, \ldots, N_n \in P$ to a pair of dual bases $h_1, \ldots, h_{2n-l} \in H$, $N_1, \ldots, N_{2n-l} \in P$ gives a *system of fundamental dominant weights* $N_1, \ldots, N_{2n-l}$.

1.3. THE WEYL GROUP, THE TITS CONE, AND ITS FACES

Define the following vector spaces over $\mathbb{F}$:

$$\mathbf{h} := \mathbf{h}_\mathbb{F} := H \otimes_\mathbb{Z} \mathbb{F}, \qquad \mathbf{h}^* := \mathbf{h}_\mathbb{F}^* := P \otimes_\mathbb{Z} \mathbb{F}.$$

H and P are identified with $H \otimes 1$, $P \otimes 1$, and $\mathbf{h}^*$ is interpreted as the dual of $\mathbf{h}$. Order the elements of $\mathbf{h}^*$ by $\lambda \leqslant \lambda'$ if and only if $\lambda' - \lambda \in Q_0^+$.

Because A is symmetrizable, we can choose a symmetric matrix $B \in M_n(\mathbb{Q})$, and a diagonal matrix $D = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_n)$, $\epsilon_1, \ldots, \epsilon_n \in \mathbb{Q}^+$, such that $A = DB$. Define a nondegenerate symmetric bilinear form on $\mathbf{h}$ by

$$(h_i|h) = (h|h_i) = \alpha_i(h)\epsilon_i \quad (i \in I,\ h \in \mathbf{h}),$$
$$(h_i|h_j) = 0 \quad (i, j = n+1, \ldots, 2n-l).$$

The induced nondegenerate symmetric form on $\mathbf{h}^*$ is also denoted by $(\,|\,)$.

The *Weyl group* $\mathcal{W} = \mathcal{W}(A)$ is the Coxeter group with generators σ_i, $i \in I$, and relations:

$$\sigma_i^2 = 1 \quad (i \in I), \qquad (\sigma_i\sigma_j)^{m_{ij}} = 1 \quad (i, j \in I,\ i \neq j).$$

The m_{ij} are given by

$a_{ij}a_{ji}$	0	1	2	3	$\geqslant 4$
m_{ij}	2	3	4	6	no relation between σ_i and σ_j.

The Weyl group $\mathcal{W}$ acts faithfully and contragrediently by

$$\sigma_i h := h - \alpha_i(h)h_i, \qquad i \in I, \quad h \in \mathbf{h},$$
$$\sigma_i\lambda := \lambda - \lambda(h_i)\alpha_i, \qquad i \in I, \quad \lambda \in \mathbf{h}^*,$$

on $\mathbf{h}$ and $\mathbf{h}^*$, leaving the lattices H, Q, P and the forms invariant. $\Delta_{\mathrm{re}} := \mathcal{W}\{\alpha_i \mid i \in I\} \subseteq Q$ is called the set of *real roots*.

To illustrate the action of $\mathcal{W}$ on $\mathbf{h}^*_{\mathbb{R}}$ geometrically, for $J \subseteq I$ set

$$F_J := \{\lambda \in \mathbf{h}^*_{\mathbb{R}} \mid \lambda(h_i) = 0 \text{ for } i \in J, \quad \lambda(h_i) > 0 \text{ for } i \in I \setminus J\},$$
$$\bar{F}_J := \{\lambda \in \mathbf{h}^*_{\mathbb{R}} \mid \lambda(h_i) = 0 \text{ for } i \in J, \quad \lambda(h_i) \geqslant 0 \text{ for } i \in I \setminus J\}.$$

Call $\bar{C} := \bar{F}_\emptyset = \{\lambda \in \mathbf{h}^*_{\mathbb{R}} \mid \lambda(h_i) \geqslant 0 \text{ for } i \in I\}$ the *fundamental chamber*. The *Tits cone* $X := \mathcal{W}\bar{C}$ is a convex $\mathcal{W}$-invariant cone with edge $c := F_I = \bar{F}_I = \{\lambda \in \mathbf{h}^*_{\mathbb{R}} \mid \lambda(h_i) = 0 \text{ for } i \in I\}$. A $\mathcal{W}$-invariant partition into facets is given by $\{\sigma F_J \mid \sigma \in \mathcal{W},\ J \subseteq I\}$. The chamber $\bar{C} = \dot{\bigcup}_{J \subseteq I} F_J$ is a fundamental region of X, and the parabolic subgroup $\mathcal{W}_J$ of $\mathcal{W}$ is the stabilizer of every element $\lambda \in F_J$.

Every face of the convex cone X is exposed, and $\mathcal{W}$-conjugate to exactly one of the faces

$$R(\Theta) := X \cap \{\lambda \in \mathbf{h}^*_{\mathbb{R}} \mid \lambda(h_i) = 0 \text{ for all } i \in \Theta\} = \mathcal{W}_{\Theta^\perp}\bar{F}_\Theta,$$

where $\Theta \subseteq I$ is special, which means either $\Theta = \emptyset$ or else all connected components of Θ are of nonfinite type, and $\Theta^\perp := \{i \in I \mid a_{ij} = 0 \text{ for all } j \in \Theta\}$.

1.4. THE KAC–MOODY ALGEBRA

The Kac–Moody algebra $\mathbf{g} = \mathbf{g}(A)$ is the Lie algebra over $\mathbb{F}$ generated by the Abelian Lie algebra $\mathbf{h}$ and $2n$ elements e_i, f_i $(i \in I)$, with the following relations, which hold for any $i, j \in I$, $h \in \mathbf{h}$:

$$[e_i, f_j] = \delta_{ij} h_i, \quad [h, e_i] = \alpha_i(h) e_i, \quad [h, f_i] = -\alpha_i(h) f_i,$$
$$(ad\, e_i)^{1-a_{ij}} e_j = (ad\, f_i)^{1-a_{ij}} f_j = 0 \quad (i \neq j).$$

The *Chevalley involution* $*$ of $\mathbf{g}$ is the involutive anti-automorphism determined by $e_i^* = f_i$, $f_i^* = e_i$, $h^* = h$, $(i \in I,\ h \in H)$.

The space $\mathbf{h}$ and the elements e_i, f_i, $(i \in I)$, can be identified with their images in $\mathbf{g}$. The nondegenerate symmetric bilinear form $(\,|\,)$ on $\mathbf{h}$ can be uniquely extended to a nondegenerate symmetric invariant bilinear form $(\,|\,)$ on $\mathbf{g}$. We have the *root space decomposition*:

$$\mathbf{g} = \bigoplus_{\alpha \in \mathbf{h}^*} \mathbf{g}_\alpha \quad \text{where } \mathbf{g}_\alpha := \{x \in \mathbf{g} \mid [h, x] = \alpha(h) x \text{ for all } h \in \mathbf{h}\}.$$

In particular $\mathbf{g}_0 = \mathbf{h}$, $\mathbf{g}_{\alpha_i} = \mathbb{F} e_i$, $\mathbf{g}_{-\alpha_i} = \mathbb{F} f_i$, $(i \in I)$.

The set of roots $\Delta := \{\alpha \in \mathbf{h}^* \setminus \{0\} \mid \mathbf{g}_\alpha \neq 0\}$ is invariant under the Weyl group and spans the root lattice Q. We have $\Delta_{\mathrm{re}} \subseteq \Delta$, and $\Delta_{\mathrm{im}} := \Delta \backslash \Delta_{\mathrm{re}}$ is called the set of *imaginary roots*.

Δ, Δ_{re} and Δ_{im} decompose into the disjoint union of the sets of *positive* and *negative* roots $\Delta^\pm = \Delta \cap Q^\pm$, $\Delta^\pm_{\mathrm{re}} := \Delta_{\mathrm{re}} \cap Q^\pm$, $\Delta^\pm_{\mathrm{im}} := \Delta_{\mathrm{im}} \cap Q^\pm$, and we have $\Delta^\pm = -\Delta^\mp$, $\Delta^\pm_{\mathrm{re}} = -\Delta^\mp_{\mathrm{re}}$, $\Delta^\pm_{\mathrm{im}} = -\Delta^\mp_{\mathrm{im}}$.

The roots belonging to the cone $X \cup (-X)$ are exactly the imaginary roots, moreover $\Delta^-_{\mathrm{im}} = \Delta \cap X$. Therefore, to describe the negative imaginary roots, it is sufficient to describe their intersection with the fundamental chamber: For $q = \sum_{i \in I} k_i \alpha_i \in Q$ set $\mathrm{supp}(q) := \{i \in I | k_i \neq 0\}$. In [K], Theorem 5.4, it is shown, that

$$\Delta^-_{\mathrm{im}} \cap \bar{C} = \{\gamma \in (Q_0^- \cap \bar{C}) \backslash \{0\} \mid \mathrm{supp}(\gamma) \text{ is connected}\}.$$

We need an easy conclusion of this description. For a special set $\Theta \subseteq I$, we get a nonempty subsemigroup of Q by

$$K(\Theta) := \{\gamma \in Q_0^- \cap \bar{C} \mid \mathrm{supp}(\gamma) = \Theta\}.$$

Note that $K(\emptyset) = \{0\}$. It is also easy to see, that for $\Theta \neq \emptyset$, the subsemigroup $K(\Theta)$ is the intersection of Q with a pointed, finitely generated, convex, Q-rational cone in $Q \otimes_{\mathbb{Z}} \mathbb{R}$.

PROPOSITION 1.1. (1) *If Θ is nonempty, special, with connected components $\Theta_1, \ldots, \Theta_m$, then $K(\Theta) = K(\Theta_1) + \cdots + K(\Theta_m)$.*
(2) *We have*

$$\Delta^-_{\mathrm{im}} \cap \bar{C} = \dot{\bigcup_{\Theta \text{ special, connected, } \neq \emptyset}} K(\Theta), \quad Q_0^- \cap \bar{C} = \dot{\bigcup_{\Theta \text{ special}}} K(\Theta).$$

Proof. In (1), we only have to show the inclusion '$\subseteq$'. Let $\gamma = \sum_{i \in \Theta} n_i \alpha_i \in K(\Theta)$. Then γ can be written as a sum $\gamma = \gamma_1 + \cdots + \gamma_m$, with $\gamma_p := \sum_{i \in \Theta_p} n_i \alpha_i \in Q_0^-$, $p = 1, \ldots, m$. We have $\gamma_p \in \bar{C}$, because of

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$$\gamma_p(h_j) = \gamma(h_j), \quad \text{for } j \in \Theta_p,$$
$$\gamma_p(h_j) = \sum_{i \in \Theta_p} \underbrace{a_{ji}}_{\leqslant 0} \underbrace{n_i}_{\leqslant 0} \geqslant 0, \quad \text{for } j \in I \setminus \Theta_p.$$

Therefore $\gamma_p \in K(\Theta_p)$, $p = 1, \dots, m$.

In (2), we only have to show the equation for $Q_0^- \cap \bar{C}$. Then the equation for $\Delta_{\mathrm{im}}^- \cap \bar{C}$ follows from the description of [K], Theorem 5.4, stated above. It is sufficient to show, that for any element $\gamma \in Q_0^- \cap \bar{C}$ its support $\Theta := \mathrm{supp}(\gamma)$ is special.

We have $\gamma = 0$ if and only if $\Theta = \emptyset$, and this set is special. Let $\gamma = \sum_{i \in \Theta} n_i \alpha_i \neq 0$, and let $\Theta_1, \dots, \Theta_m$ be the connected components of Θ. Then for $p \in \{1, \dots, m\}$ we have

$$0 \leqslant \gamma(h_j) = \sum_{i \in \Theta_p} a_{ji} \underbrace{n_i}_{<0} \quad \text{for all } j \in \Theta_p.$$

Due to [K], Corollary 4.3, Θ_p is not of finite type. □

Corresponding to the decomposition into positive and negative roots there is a *triangular decomposition* $\mathbf{g} = \mathbf{n}^- \oplus \mathbf{h} \oplus \mathbf{n}^+$, where $\mathbf{n}^{\pm} := \oplus_{\alpha \in \Delta^{\pm}} \mathbf{g}_\alpha$.

For a real root α, the subalgebra $\mathbf{g}_\alpha \oplus [\mathbf{g}_\alpha, \mathbf{g}_{-\alpha}] \oplus \mathbf{g}_{-\alpha}$ of $\mathbf{g}$ is isomorphic to $sl(2, \mathbb{F})$.

1.5. THE KAC–MOODY GROUP

To construct the Kac–Moody group, call a representation (V, π) of $\mathbf{g}$ *admissible* if:

(1) V is $\mathbf{h}$-diagonalizable with set of weights $P(V) \subseteq P$.
(2) $\pi(x)$ is locally nilpotent for all $x \in \mathbf{g}_\alpha$, $\alpha \in \Delta_{\mathrm{re}}$.

Examples are the adjoint representation $(\mathbf{g}, ad)$, and the irreducible highest weight representation $(L(\Lambda), \pi_\Lambda)$, $\Lambda \in P^+ := P \cap \bar{C}$.

The *Kac–Moody group* $G = G(A)$ can be characterized in the following way:

- The group G acts on every admissible representation. Two elements $g, g' \in G$ are equal if and only if for all admissible modules V, and all $v \in V$ we have $gv = g'v$.
- (1) For every $h \in H$, $s \in \mathbb{F}^\times$ there exists a unique element $t_h(s) \in G$, such that for any admissible representation (V, π) we have

  $$t_h(s) v_\lambda = s^{\lambda(h)} v_\lambda, \quad v_\lambda \in V_\lambda, \quad \lambda \in P(V).$$

  (2) For every $x \in \mathbf{g}_\alpha$, $\alpha \in \Delta_{\mathrm{re}}$, there exists a unique element $\exp(x) \in G$, such that for any admissible representation (V, π) we have

  $$\exp(x) v = \exp(\pi(x)) v, \quad v \in V.$$

  G is generated by the elements of (1) and (2).

The *Chevalley involution* $*\colon G \to G$ is the involutive anti-isomorphism determined by

$$\exp(x_\alpha)^* := \exp(x_\alpha^*), \qquad t_h(s)^* := t_h(s) \qquad (x_\alpha \in \mathbf{g}_\alpha, \alpha \in \Delta_{\mathrm{re}},\ h \in H,\ s \in \mathbb{F}^\times).$$

The Kac–Moody group has the following important structural properties: (a) The elements of (1) induce an embedding of the torus $H \otimes_{\mathbb{Z}} \mathbb{F}^\times$ into G. Its image is denoted by T.

For $\alpha \in \Delta_{\mathrm{re}}$ the elements of (2) induce an embedding of $(\mathbf{g}_\alpha, +)$ into G. Its image U_α is called the root group belonging to α.

Let $\alpha \in \Delta_{\mathrm{re}}^+$. Let $x_\alpha \in \mathbf{g}_\alpha$, $x_{-\alpha} \in \mathbf{g}_{-\alpha}$, such that $[x_\alpha, x_{-\alpha}] = h_\alpha$. There exists an injective homomorphism of groups ϕ_α: $\mathrm{SL}(2, \mathbb{F}) \to G$ with

$$\phi_\alpha\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} := \exp(sx_\alpha), \qquad \phi_\alpha\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} := \exp(sx_{-\alpha}) \quad (s \in \mathbb{F}^\times).$$

(b) Denote by N the subgroup generated by T and $n_\alpha := \phi_\alpha\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\alpha \in \Delta_{\mathrm{re}}$. Then N/T can be identified with the Weyl group $\mathcal{W}$, the isomorphism κ: $N/T \to \mathcal{W}$ given by $\kappa(n_\alpha T) := \sigma_\alpha$, $\alpha \in \Delta_{\mathrm{re}}$. We denote an arbitrary element $n \in N$ with $\kappa(nT) = \sigma \in \mathcal{W}$ by n_σ. The set of weights $P(V)$ of an admissible $\mathbf{g}$-module is $\mathcal{W}$-invariant, and $n_\sigma V_\lambda = V_{\sigma\lambda}$, $\lambda \in P(V)$.

(c) Let $B^\pm$ be the subgroups generated by T and U_α, $\alpha \in \Delta_{\mathrm{re}}^\pm$. Let $U^\pm$ be the subgroups generated by U_α, $\alpha \in \Delta_{\mathrm{re}}^\pm$.

Then $(G, (U_\alpha)_{\alpha\in\Delta_{\mathrm{re}}}, T)$ is a *root groups data system*, leading to the *twinned BN-pairs* $(B^\pm, N)$, which have the property $B^+ \cap B^- = B^+ \cap N = B^- \cap N = T$. We have the *Bruhat* and *Birkhoff decompositions*:

$$G = \dot{\bigcup_{\sigma\in\mathcal{W}}} B^\epsilon \sigma B^\delta \qquad (\epsilon, \delta) = \underbrace{(+,+), (-,-)}_{\text{Bruhat}}, \underbrace{(+,-), (-,+)}_{\text{Birkhoff}}.$$

The derived group G' is identical with the Kac–Moody group as defined in [K, P 1]. It is generated by the root groups U_α, $\alpha \in \Delta_{\mathrm{re}}$, and we have $G = G' \rtimes T_{\mathrm{rest}}$, where T_{rest} is the subtorus of T generated by the elements $t_{h_i}(s)$, $i = n+1, \ldots, 2n-l$, $s \in \mathbb{F}^\times$.

1.6. PROPERTIES OF THE ADMISSIBLE IRREDUCIBLE HIGHEST WEIGHT REPRESENTATIONS, AND THEIR SET OF WEIGHTS

For $\Lambda \in P^+ := \bar{C} \cap P$ there exists a nondegenerate symmetric bilinear form $\langle\langle\ |\ \rangle\rangle$: $L(\Lambda) \times L(\Lambda) \to \mathbb{F}$, which is contravariant, i.e., $\langle\langle v \mid xw \rangle\rangle = \langle\langle x^*v \mid w \rangle\rangle$ for all $v, w \in L(\Lambda)$, $x \in g$, resp., $x \in G$. This form is uniquely determined up to a nonzero multiplicative scalar.

For the properties of the set of weights $P(\Lambda)$ of $L(\Lambda)$ we refer to the book [K], sections 11.1, 11.2 and 11.3. We prove some more properties, which will be important later.

THEOREM 1.2. *Let $\Lambda_1, \Lambda_2 \in P^+$. We have:*

(a) $L(\Lambda_1) \otimes L(\Lambda_2) \cong \oplus_{\Lambda\in P(\Lambda_1+\Lambda_2)\cap P^+} n_\Lambda L(\Lambda)$, *with* $n_\Lambda \in \mathbb{N}_0$, $n_{\Lambda_1+\Lambda_2} = 1$.
(b) *If* $\Lambda_1 \in P(\Lambda_2)$, *then* $P(\Lambda_1) \subseteq P(\Lambda_2)$.

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(c) $P(\Lambda_1) + P(\Lambda_2) = P(\Lambda_1 + \Lambda_2)$.

Proof. To (*a*). Due to [K], Corollary 10.7 b),

$$L(\Lambda_1) \otimes L(\Lambda_2) \cong \bigoplus_{\Lambda \in P^+,\ \Lambda \leqslant \Lambda_1+\Lambda_2} n_\Lambda L(\Lambda)$$

with $n_\Lambda \in \mathbb{N}_0$, $n_{\Lambda_1+\Lambda_2} = 1$. Also $n_\Lambda \neq 0$ is only possible for $\Lambda \in P(L(\Lambda_1) \otimes L(\Lambda_2))$. Therefore it is sufficient to show $P(L(\Lambda_1) \otimes L(\Lambda_2)) \cap P^+ \subseteq P(\Lambda_1 + \Lambda_2) \cap P^+$.

Recall from [K], section 11.2, that an element $\lambda \in P$ is called *nondegenerate* with respect to $\Lambda \in P^+$, if either $\lambda = \Lambda$, or else $\lambda < \Lambda$, and for every connected component S of $\operatorname{supp}(\Lambda - \lambda)$ we have $S \cap \{i \in I \mid \Lambda(h_i) \neq 0\} \neq \emptyset$. Due to [K], Proposition 11.2(a), the set of weights $P(\Lambda_1 + \Lambda_2) \cap P^+$ consists of the elements of $\lambda \in P^+$, which are nondegenerate with respect to $\Lambda_1 + \Lambda_2$. Therefore it is sufficient to show, that every weight in $P(L(\Lambda_1) \otimes L(\Lambda_2)) = P(\Lambda_1) + P(\Lambda_2)$ is nondegenerate with respect to $\Lambda_1 + \Lambda_2$.

An element $\lambda_i \in P(\Lambda_i)$ is of the form $\lambda_i = \Lambda_i - q_i$ with $q_i = \sum_j k_j^{(i)} \alpha_j \in Q_0^+$. Due to [K], Lemma 11.2, λ_i is nondegenerate with respect to Λ_i. $(i = 1, 2)$. Clearly $\lambda_1 + \lambda_2 \leqslant \Lambda_1 + \Lambda_2$. Let $\lambda_1 + \lambda_2 \neq \Lambda_1 + \Lambda_2$, let S be a connected component of $\operatorname{supp}(\Lambda_1 + \Lambda_2 - (\lambda_1 + \lambda_2))$. Choose an element $i_0 \in S$. We have $k_{i_0}^{(1)} \neq 0$ or $k_{i_0}^{(2)} \neq 0$, and we may assume $k_{i_0}^{(1)} \neq 0$. Let S' be a connected component of $\operatorname{supp}(\Lambda_1 - \lambda_1)$ with $i_0 \in S'$. Due to $S' \subseteq \{i \in I \mid k_i^{(1)} + k_i^{(2)} \neq 0\}$ we get $S' \subseteq S$. Because λ_1 is nondegenerate with respect to Λ_1, we find

$$\emptyset \neq S' \cap \left\{i \in I \mid \Lambda_1(h_i) \neq 0\right\} \subseteq S \cap \left\{i \in I \mid (\Lambda_1 + \Lambda_2)(h_i) \neq 0\right\}.$$

To (*b*). Denote by ‘co’ the convex hull in $\mathbf{h}_{\mathbb{R}}^*$. Due to the $\mathcal{W}$-invariance of $P(\Lambda_2)$, and due to [K], Proposition 11.3(a), we have $\mathcal{W}\Lambda_1 \subseteq P(\Lambda_2) \subseteq \operatorname{co}(\mathcal{W}\Lambda_2)$. Therefore $\operatorname{co}(\mathcal{W}\Lambda_1) \subseteq \operatorname{co}(\mathcal{W}\Lambda_2)$. Because of $\Lambda_1 \leqslant \Lambda_2$, we also have $\Lambda_1 - Q_0^+ \subseteq \Lambda_2 - Q_0^+$. Using [K], Proposition 11.3(a), once more, we find $P(\Lambda_1) \subseteq P(\Lambda_2)$.

To (*c*). For $n \in \mathbb{N}$ we have $P(nL(\Lambda)) = P(\Lambda)$. Using (a), we find

$$P(\Lambda_1) + P(\Lambda_2) = P(L(\Lambda_1) \otimes L(\Lambda_2)) = \bigcup_{\Lambda \in P(\Lambda_1+\Lambda_2) \cap P^+,\ n_\Lambda \neq 0} P(\Lambda).$$

Due to (b), the union on the right is a subset of $P(\Lambda_1 + \Lambda_2)$. We have equality because of $n_{\Lambda_1+\Lambda_2} \neq 0$. □

Recall that c denotes the edge of the Tits cone.

PROPOSITION 1.3. *Let A have no component of finite type. For $\Lambda \in P^+$, we have $P(\Lambda) \cap c \neq \emptyset$ if and only if $\Lambda \in c$.*

Proof. We only have to show the direction ‘⇒’. Every element of $P(\Lambda)$ can be written in the form $\Lambda - \sum_{i \in I} n_i \alpha_i$, where $\sum_{i \in I} n_i \alpha_i \in Q_0^+$. Let $\Lambda - \sum_{i \in I} n_i \alpha_i \in c$. Then by applying this element to h_j, $j \in I$, we find

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$$0 \leqslant \Lambda(h_j) = \sum_{i\in I} a_{ji} \underbrace{n_i}_{\geqslant 0}.$$

Due to [K], Theorem 4.3, we get $\Lambda(h_j) = \sum_{i\in I} a_{ji}n_i = 0$ for all $j \in I$. $\square$

For an affine Kac–Moody algebra, the set of weights $P(\Lambda) \cap P^+$, $(\Lambda \in P^+)$, can be covered by finitely many imaginary root strings, i.e., sets of the form $\lambda - \mathbb{N}_0\delta$, where δ denotes the minimal positive imaginary root. Compare [K], Proposition 12.6. Furthermore, $-\mathbb{N}_0\delta = Q_0^- \cap \bar{C}$. Now the only special sets are $\emptyset$ and I, and we can write $-\mathbb{N}_0\delta$ as the union $-\mathbb{N}_0\delta = K(\emptyset) \cup K(I)$, where $K(\emptyset) = \{0\}$, $K(I) = -\mathbb{N}\delta$. Therefore every root string $\lambda - \mathbb{N}_0\delta$ is the union of $\lambda + K(\emptyset) = \{\lambda\}$ and $\lambda + K(I) = \lambda - \mathbb{N}\delta$.

Next we give a generalization for an arbitrary Kac–Moody algebra. We find a finite covering of $P(\Lambda) \cap P^+$, $(\Lambda \in P^+)$, by sets of the form $\lambda + K(\Theta)$, Θ special:

Call $\Xi \subseteq I$ relevant for $\Lambda \in P^+$, if either $\Xi = \emptyset$, or else every connected component of Ξ intersects $\{i \in I \mid \Lambda(h_i) \neq 0\}$ nontrivially. Proposition 11.2(a) of [K] can be written in the form

$$P(\Lambda) \cap P^+ = \dot{\bigcup_{\Xi \text{ rel. for } \Lambda}} \Lambda + S(\Xi), \tag{1}$$

where $S(\Xi) := \{q \in Q_0^- \mid \operatorname{supp}(q) = \Xi, \Lambda + q \in P^+\}$. Note that $S(\emptyset) = \{0\}$.

THEOREM 1.4. *Let $\Xi \subseteq I$ be relevant for $\Lambda \in P^+$. Denote by Ξ^0, resp. Ξ^∞, the union of all connected components of Ξ of finite, resp. non-finite, type. Then Ξ^0, Ξ^∞ are also relevant for Λ, and we have*

$$S(\Xi) = S(\Xi^0) + S(\Xi^\infty). \tag{2}$$

Furthermore, $S(\Xi^0)$ is finite, and there exists a finite set $M(\Xi^\infty) \subseteq S(\Xi^\infty)$ such that

$$S(\Xi^\infty) = K(\Xi^\infty) \cup \bigcup_{\Theta \text{ special},\ \Theta \subseteq \Xi^\infty} (M(\Xi^\infty) + K(\Theta)). \tag{3}$$

Proof. The sets Ξ^0, Ξ^∞ are relevant, because they are unions of connected components of Ξ, or empty. To show the inclusion '$\subseteq$' of (2), decompose $q = \sum_{i\in\Xi} n_i\alpha_i \in S(\Xi)$ in the form $q = q^0 + q^\infty$ with $q^0 := \sum_{i\in\Xi^0} n_i\alpha_i$ and $q^\infty := \sum_{i\in\Xi^\infty} n_i\alpha_i$, a sum over the empty set to be equal to zero. We have $\Lambda + q^0 \in P^+$ due to

$$(\Lambda + q^0)(h_i) = (\Lambda + q)(h_i) \geqslant 0, \qquad \text{for } i \in \Xi^0,$$
$$(\Lambda + q^0)(h_i) \geqslant \Lambda(h_i) \geqslant 0, \qquad \text{for } i \notin \Xi^0.$$

Therefore $q^0 \in S(\Xi^0)$. Similar we find $q^\infty \in S(\Xi^\infty)$. To show the reverse inclusion, let $q^0 \in S(\Xi^0)$, $q^\infty \in S(\Xi^\infty)$. Then

$$(\Lambda + q^0 + q^\infty)(h_i) \geqslant \Lambda(h_i), \qquad \text{for } i \notin \Xi^0 \cup \Xi^\infty,$$
$$(\Lambda + q^0 + q^\infty)(h_i) = (\Lambda + q^0)(h_i) \geqslant 0, \qquad \text{for } i \in \Xi^0,$$

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$$(\Lambda + q^0 + q^\infty)(h_i) = (\Lambda + q^\infty)(h_i) \geqslant 0, \qquad \text{for } i \in \Xi^\infty.$$

Therefore $q^0 + q^\infty \in S(\Xi)$.

Let $\mathbf{g}_{\Xi^0}$ be the Lie algebra generated by $\mathbf{g}_{\pm\alpha_i}$, $i \in \Xi^0$. Let $\mathbf{n}^-_{\Xi^0}$ be its subalgebra generated by $\mathbf{g}_{-\alpha_i}$, $i \in \Xi^0$. Then $U(\mathbf{n}^-_{\Xi^0})L(\Lambda)_\Lambda$ is a finite-dimensional irreducible highest weight module of the finite-dimensional split reductive Lie algebra $\mathbf{g}_{\Xi^0} + \mathbf{h}$. Due to $\Lambda + S(\Xi^0) \subseteq P(U(\mathbf{n}^-_{\Xi^0})L(\Lambda)_\Lambda)$, the set $S(\Xi^0)$ is finite.

We only have to show (3) in the nontrivial case $\Xi^\infty \neq \emptyset$. We only have to find a finite set $M(\Xi^\infty) \subseteq S(\Xi^\infty)$, and show the inclusion '$\subseteq$' of (3). Then the reverse inclusion is also satisfied. This follows immediately from the definition of $S(\Xi^\infty)$, using $M(\Xi^\infty) \subseteq S(\Xi^\infty)$ and $K(\Xi^\infty), K(\Theta) \subseteq Q_0^- \cap \bar{C}$.

Let $q^\infty = \sum_i n_i\alpha_i \in S(\Xi^\infty)$. Ξ^∞ has no component of finite type. Due to the characterizations of [K], Corollary 4.3, there exists at least one element $j \in \Xi^\infty$, such that $\sum_{i\in\Xi^\infty} a_{ji}n_i \geqslant 0$. Therefore one of the following finitely many cases holds:

(a) The case $\sum_{\Xi^\infty} a_{ji}n_i \geqslant 0$ for all $j \in \Xi^\infty$. Because of $\sum_{\Xi^\infty} a_{ji}n_i \geqslant 0$ for all $j \notin \Xi^\infty$, we have $q^\infty \in P^+$. Together with $q^\infty \in Q_0^-, \mathrm{supp}(q^\infty) = \Xi^\infty$, we get $q^\infty \in K(\Xi^\infty)$.

(b) There exists a partition $\Xi^\infty = \Xi_1 \dot{\cup} \Xi_2$, $\Xi_1 \neq \emptyset$, $\Xi_2 \neq \emptyset$, and there exists a tuple of integers $m_- := (m_j)_{j\in\Xi_1}$ with $0 > m_j \geqslant -\Lambda(h_j)$, such that

$$\sum_{i\in\Xi^\infty} a_{ji}n_i = m_j \quad \text{for } j \in \Xi_1, \qquad \sum_{i\in\Xi^\infty} a_{ji}n_i \geqslant 0, \quad \text{for } j \in \Xi_2.$$

By decomposing the generalized Cartan submatrix belonging to Ξ^∞ into blocks with respect to Ξ_1, Ξ_2, we can write these equations in the form

$$m_- = A_{\Xi_1}n_- + Bn_+, \qquad 0 \leqslant Cn_- + A_{\Xi_2}n_+.$$

Note that $B,\ C$ are matrices with nonpositive entries. Every component of Ξ_1 is of finite type, due to $A_{\Xi_1}(-n_-) = -m_- + (-B)(-n_+) > 0$, compare [K], Corollary 4.3.

The tuple $(-n_+, Cn_- + A_{\Xi_2}n_+)$ belongs to the set

$$\left\{(-\tilde{n}_+, C\tilde{n}_- + A_{\Xi_2}\tilde{n}_+) \in \mathbb{N}^{|\Xi_2|} \times \mathbb{N}_0^{|\Xi_2|} \mid \tilde{n}_- = A_{\Xi_1}^{-1}m_- - A_{\Xi_1}^{-1}B\tilde{n}_+ < 0\right\}. \tag{4}$$

Equip $\mathbb{N}_0^{2|\Xi_2|}$ with the product order of the natural order of $\mathbb{N}_0$. It is well known, that a nonempty subset of $\mathbb{N}_0^{2|\Xi_2|}$ contains only finitely many minimal elements, and every element of this subset lies over some minimal element. Let $(-n_+^{\min}, Cn_-^{\min} + A_{\Xi_2}n_+^{\min})$ be a minimal element of the set (4), which is smaller than, or equal to $(-n_+, Cn_- + A_{\Xi_2}n_+)$. Then we have

$$n_+ - n_+^{\min} \leqslant 0, \tag{5}$$

$$C(n_- - n_-^{\min}) + A_{\Xi_2}(n_+ - n_+^{\min}) \geqslant 0, \tag{6}$$

$$A_{\Xi_1}(n_- - n_-^{\min}) + B(n_+ - n_+^{\min}) = 0. \tag{7}$$

B has only nonpositive entries. With (5) follows $B(n_+ - n_+^{\min}) \geqslant 0$. With (7) follows $A_{\Xi_1}(n_- - n_-^{\min}) \leqslant 0$. Because Ξ_1 has only components of finite type, we get from [K], Theorem 4.3,

$$n_- - n_-^{\min} \leqslant 0. \tag{8}$$

Write q^∞ as the sum $q^\infty = q^{\min} + \beta$, with

$$q^{\min} := \sum_{i\in\Xi^\infty} n_i^{\min}\alpha_i, \quad \text{and} \quad \beta := \sum_{i\in\Xi^\infty}(n_i - n_i^{\min})\alpha_i.$$

Due to (5) and (8), we have $\beta \in Q_0^-$. Due to (7) and (6), we have $\beta \in \bar{C}$. Because of $\beta \in Q_0^- \cap \bar{C} = \bigcup_{\Theta\,\text{special}} K(\Theta)$, we find $\operatorname{supp}(\beta)$ is special. Due to the definition of β, we have $\operatorname{supp}(\beta) \subseteq \Xi^\infty$.

Due to the conditions defining the set (4), we have $n_-^{\min} < 0$, $n_+^{\min} < 0$. Therefore $\operatorname{supp}(q^{\min}) = \Xi^\infty$. Also due the conditions defining the set (4), we find

$$A_{\Xi_1}n_-^{\min} + Bn_+^{\min} - m_- = 0 \quad \text{and} \quad Cn_-^{\min} + A_{\Xi_2}n_+^{\min} \geqslant 0.$$

From this follows $(\Lambda + q^{\min})(h_i) \geqslant 0$ for all $i \in \Xi^\infty$. We have also $\Lambda(h_i)+(q^{\min})(h_i) \geqslant 0$ for all $i \notin \Xi^\infty$. Therefore $q^{\min} \in S(\Xi^\infty)$.

Take as $M(\Xi^\infty)$ the finite set of these elements $q^{\min}$, i.e., the occurring minimal elements $q^{\min}$, for all occurring partitions $\Xi^\infty = \Xi_1 \dot\cup \Xi_2$, $\Xi_1 \neq \emptyset$, $\Xi_2 \neq \emptyset$, and occurring tuple of integers $m_- := (m_j)_{j\in\Xi_1}$ with $0 > m_j \geqslant -\Lambda(h_j)$. □

1.7. THE ALGEBRA OF STRONGLY REGULAR FUNCTIONS

For $\Lambda \in P^+$, $v, w \in L(\Lambda)$, and $\langle\langle\ |\ \rangle\rangle$ a nondegenerate symmetric contravariant bilinear form on $L(\Lambda)$, call the function $f_{vw}\colon G \to \mathbb{F}$, given by $f_{vw}(g) := \langle\langle v \,|\, gw\rangle\rangle$, $g \in G$, a *matrix coefficient* of G. The algebra $\mathbb{F}[G]$ generated by all such matrix coefficients is called the *algebra of strongly regular functions* on G. $\mathbb{F}[G]$ is an integrally closed domain. It admits a *Peter–Weyl theorem*: Define an action π of $G \times G$ on $\mathbb{F}[G]$, and an involutive automorphism $*$ of $\mathbb{F}[G]$, which we also call *Chevalley involution*, by

$$(\pi(g, h)f)(x) := f(g^{-1}xh), \qquad f^*(x) := f(x^*), \quad g, x, h \in G,\ f \in \mathbb{F}[G].$$

For every $\Lambda \in P^+$, fix on $L(\Lambda)$ a nondegenerate symmetric contravariant bilinear form. Define an action π on $L(\Lambda) \otimes L(\Lambda)$ by

$$\pi(g, h)(v \otimes w) := (g^{-1})^* v \otimes hw, \quad g, h \in G, \quad v, w \in L(\Lambda).$$

Then the map $\oplus_{\Lambda\in P^+} L(\Lambda) \otimes L(\Lambda) \to \mathbb{F}[G]$, induced by $v \otimes w \mapsto f_{vw}$, is an isomorphism of $G \times G$-modules. It identifies the direct sum of the switch maps of the factors with the Chevalley involution.

An embedding of the linear space $\mathbb{F}[G]$ into the dual of the universal enveloping algebra $U(\mathbf{g})$ is induced by

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$$f_{vw}(x) := \langle\langle v \mid xw \rangle\rangle, \qquad v, w \in L(\Lambda), \quad \Lambda \in P^+, \; x \in U(\mathbf{g}).$$

Restricting the functions of $\mathbb{F}[G]$ onto G', resp. T_{rest} gives the algebras $\mathbb{F}[G']$, resp. $\mathbb{F}[T_{\text{rest}}]$, the first identical with the algebra of strongly regular functions as defined in [K, P 2], the second the classical coordinate ring of the torus T_{rest}. $\mathbb{F}[G]$ is isomorphic to $\mathbb{F}[G'] \otimes \mathbb{F}[T_{\text{rest}}]$ by the comorphism dual to the multiplication map $G' \times T_{\text{rest}} \to G$.

1.8. THE FIELD OF COMPLEX NUMBERS $\mathbb{F} = \mathbb{C}$

Some of the above constructions can be modified for the field of complex numbers. For the sake of simplicity we shall use the same notations.

The *compact involution* $*$ of $\mathbf{g}$ is the involutive anti-linear anti-automorphism determined by $e_i^* = f_i$, $f_i^* = e_i$, $h^* = h$, $(i \in I, \; h \in H)$. It induces the *compact involution* $*$ on G by $\exp(x_\alpha)^* := \exp(x_\alpha^*)$, $t_h(s)^* := t_h(\bar{s})$, $(x_\alpha \in \mathbf{g}_\alpha, \; \alpha \in \Delta_{\text{re}}, \; h \in H, \; s \in \mathbb{C}^\times)$.

Define the exponential map $\exp : \mathbf{h} \to T$ by

$$\exp\left(\sum_{i=1}^{2n-l} c_i h_i\right) := \prod_{i=1}^{2n-l} t_{h_i}(\mathrm{e}^{c_i}), \qquad c_i \in \mathbb{C}^\times.$$

For every $h \in \mathbf{h}$, and every admissible module V, we have

$$\exp(h) v_\lambda = \mathrm{e}^{\lambda(h)} v_\lambda, \quad v_\lambda \in V_\lambda, \; \lambda \in P(V)$$

Define $T^+ := \exp(H \otimes \mathbb{R})$, and the *unitary form* $K := \left\{ g \in G \mid g^* = g^{-1} \right\}$. We have the *Iwasawa decompositions* $G = KT^+U^\pm = U^\pm T^+ K$.

For $\Lambda \in P^+$, the irreducible highest weight representation $(L(\Lambda), \pi_\Lambda)$ carries, with respect to the compact involution, a contravariant positive definite Hermitian form $\langle\langle \mid \rangle\rangle$, unique up to a nonzero positive factor. We assume $\langle\langle \mid \rangle\rangle$ to be anti-linear in the first entry. The algebra of strongly regular functions is also generated by the matrix coefficients built by using these Hermitian forms.

2. A Version of Looijenga's Exponential Invariant Theory

In this section, we present a version of Looijenga's exponential invariant theory adopted to the ground field $\mathbb{F}$. We restrict to the facts relevant for our purpose. Starting point is the algebra of strongly regular functions $\mathbb{F}[G]$ of a Kac–Moody group G, restricted onto the torus T. We first describe this restriction:

The group algebra $\mathbb{F}[P]$ of the lattice P can be identified with the classical coordinate ring on T, identifying the elements of the natural base $(e_\lambda)_{\lambda \in P}$ with the functions given by

$$e_\lambda\left(\prod_{i=1}^{2n-l} t_{h_i}(s_i)\right) := \prod_{i=1}^{2n-l} s_i^{\lambda(h_i)} \qquad (s_i \in \mathbb{F}^\times).$$

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The restriction of the algebra of strongly regular functions on T, which we denote by $\mathbb{F}[T]$, is in the nonclassical case only a subalgebra of the classical coordinate ring of T:

PROPOSITION 2.1. *We have* $\mathbb{F}[T] = \mathbb{F}[X \cap P]$.

A variant of this Proposition has been proved in [M 1]. For the convenience of the reader, we sketch its

Proof. For $v_\lambda \in L(\Lambda)_\lambda$, $w_\mu \in L(\Lambda)_\mu$, $\lambda, \mu \in P(\Lambda)$, $\Lambda \in P^+$, we find by checking on the elements of T: $f_{v_\lambda w_\mu}|_T = f_{v_\lambda w_\mu}(1)e_\mu$. Due to the Peter and Weyl theorem for $\mathbb{F}[G]$, and due to $\bigcup_{\Lambda \in P^+} P(\Lambda) = X \cap P$, we get $\mathbb{F}[T] \subseteq \mathbb{F}[X \cap P]$. Due to the nondegeneracy of $\langle\langle \mid \rangle\rangle$ on the weight spaces, we have even equality. $\square$

Next we construct a formal completion of $\mathbb{F}[T]$. The algebra $\mathbb{F}[T]$ is $(X \cap P)$-graded by $\mathbb{F}[T]_\lambda := \mathbb{F}e_\lambda$, $\lambda \in X \cap P$. For an element $f \in \prod_{\lambda \in X \cap P} \mathbb{F}[T]_\lambda$ denote by f_λ the projection of f onto $\mathbb{F}[T]_\lambda$. Set

$$\mathrm{supp}(f) := \{\lambda \in X \cap P \mid f_\lambda \neq 0\} \subseteq X \cap P.$$

Using Theorem 1.2(c) we easily find:

PROPOSITION 2.2.

$$\widehat{\mathbb{F}[T]} := \left\{ f \in \prod_{\lambda \in X \cap P} \mathbb{F}[T]_\lambda \;\middle|\; \exists\, k \in \mathbb{N}, \exists\, \Lambda_1, \ldots, \Lambda_k \in P^+ : \mathrm{supp}(f) \subseteq \bigcup_{i=1}^{k} P(\Lambda_i) \right\}$$

is a commutative associative algebra with unit, the multiplication given by

$$(f\tilde{f})_\lambda := \sum_{\substack{\lambda_1, \lambda_2 \in X \cap P \\ \lambda = \lambda_1 + \lambda_2}} f_{\lambda_1} \tilde{f}_{\lambda_2}. \tag{9}$$

Remark. We identify $\mathbb{F}[T]$ in the obvious way with a subalgebra of $\widehat{\mathbb{F}[T]}$. If A has only components of finite type, then $\mathbb{F}[T] = \widehat{\mathbb{F}[T]}$.

It is useful, to introduce a natural limit concept for the algebra $\widehat{\mathbb{F}[T]}$. Call a sequence $(f_i)_{i \in \mathbb{N}} \subseteq \widehat{\mathbb{F}[T]}$ convergent to $f \in \widehat{\mathbb{F}[T]}$ if:

- There exist $k \in \mathbb{N}$, $\Lambda_1, \ldots, \Lambda_k \in P^+$, such that for all $i \in \mathbb{N}$ we have $\mathrm{supp}(f_i) \subseteq P(\Lambda_1) \cup \cdots \cup P(\Lambda_k)$.
- For every $\lambda \in X \cap P$, there exists an element $i_0 \in \mathbb{N}$, such that for all $i \geqslant i_0$ we have $(f_i)_\lambda = f_\lambda$.

Note that the limit f is uniquely determined by the sequence $(f_i)_{i \in \mathbb{N}}$.

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Call a map $\widehat{\mathbb{F}[T]} \to \widehat{\mathbb{F}[T]}$ continuous, if it maps convergent sequences into convergent sequences.

With respect to this limit concept, $\mathbb{F}[T]$ is dense in $\widehat{\mathbb{F}[T]}$, i.e., every element $f \in \widehat{\mathbb{F}[T]}$ is the limit of a sequence $(f_i)_{i\in\mathbb{N}} \subseteq \mathbb{F}[T]$. The addition, the multiplication by a scalar, and the multiplication of the algebra $\widehat{\mathbb{F}[T]}$ are continuous.

To investigate $\widehat{\mathbb{F}[T]}$, it is also useful to generalize the notion of a linear base. We call a nonempty family $(f_j)_{j\in J} \subseteq \widehat{\mathbb{F}[T]}$ summable, if there exist $k \in \mathbb{N}$, $\Lambda_1, \ldots, \Lambda_k \in P^+$, such that

$$\bigcup_{j\in J} \mathrm{supp}(f_j) \subseteq P(\Lambda_1) \cup \cdots \cup P(\Lambda_k),$$

and for every $\lambda \in X \cap P$, we have $(f_j)_\lambda \neq 0$ for only finitely many $j \in J$. The sum of such a family is defined as $(\sum_{j\in J} f_j)_\lambda := \sum_{j\in J} (f_j)_\lambda, \ \lambda \in X \cap P$. Note that there are only countably many nonzero f_j's. The sum is equal to the series sum, corresponding to the notion of convergence as above, relative to an arbitrary linear order of these elements.

For a set $\neq B \subset \widehat{\mathbb{F}[T]}$, we call a sum of the form $\sum_{b\in B} c_b b$ with $c_b \in \mathbb{F}$ a s-linear combination. Call B s-linear independent, if $\sum_{b\in B} c_b b = 0$ implies $c_b = 0$ for all $b \in B$. Call B an s-base, if every element of $\widehat{\mathbb{F}[T]}$ can be written in the form $\sum_{b\in B} c_b b$ with uniquely determined $c_b \in \mathbb{F}$.

The action of the Weyl group $\mathcal{W}$ on $\mathbb{F}[T]$ extends uniquely to an action on $\widehat{\mathbb{F}[T]}$, by continuous homomorphisms of algebras:

$$(\sigma f)_\lambda := \sum_\lambda c_\lambda e_{\sigma\lambda} \qquad \text{for } f = \sum_\lambda c_\lambda e_\lambda \in \widehat{\mathbb{F}[T]}, \quad \sigma \in \mathcal{W}.$$

In the rest of this section, we investigate the structure of the invariant algebra $\widehat{\mathbb{F}[T]}^{\mathcal{W}}$.

First we determine certain s-bases. For every $\Lambda \in P^+$, choose an element $S_\Lambda \in \widehat{\mathbb{F}[T]}^{\mathcal{W}}$ with $\mathrm{supp}(S_\Lambda) \subseteq P(\Lambda)$ and $(S_\Lambda)_\Lambda = e_\Lambda$. In particular, we can take the formal T-character corresponding to $L(\Lambda)$:

$$\chi_\Lambda := \sum_{\lambda\in P(\Lambda)} m_\lambda e_\lambda, \qquad m_\lambda := \dim(L(\Lambda)_\lambda). \tag{10}$$

Generalizing the classical case we have:

THEOREM 2.3. *The family $(S_\Lambda)_{\Lambda\in P^+}$ is an s-base of $\widehat{\mathbb{F}[T]}^{\mathcal{W}}$, and its s-linear combinations are given by the sums*

$$\sum_{\Lambda\in(P(\Lambda_1)\cup\cdots\cup P(\Lambda_k))\cap P^+} c_\Lambda S_\Lambda, \tag{11}$$

with $c_\Lambda \in \mathbb{F}, \ \Lambda_1, \ldots, \Lambda_k \in P^+, \ k \in \mathbb{N}$.

Proof. First we show, that every sum of the form (11) is well defined, i.e., the family $(c_\Lambda S_\Lambda)_{\Lambda\in(P(\Lambda_1)\cup\cdots\cup P(\Lambda_k))\cap P^+}$ is summable. Fix $\lambda \in X \cap P$. Due to Theorem 1.2(b), $(c_\Lambda S_\Lambda)_\lambda \neq 0$ implies $\lambda \in P(\Lambda_1) \cup \cdots \cup P(\Lambda_k)$, and we have $\Lambda \in (P(\Lambda_1) \cup \cdots \cup P(\Lambda_k))$

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$\cap P^+$ with $\Lambda \geqslant \lambda$. This is only possible for finitely many Λ. A sum of the form (11) is an element of $\widehat{\mathbb{F}[T]}^{\mathcal{W}}$, because the elements S_Λ belong to $\widehat{\mathbb{F}[T]}^{\mathcal{W}}$, and $\mathcal{W}$ acts continuously.

Let $\sum_{\Lambda \in P^+} c_\Lambda S_\Lambda$ be a s-linear combination of $(S_\Lambda)_{\Lambda \in P^+}$. Due to the definition of summable, there exist $\Lambda_1, \ldots, \Lambda_k \in P^+$, such that we have $\bigcup_{\Lambda \in P^+, c_\Lambda \neq 0} \operatorname{supp}(S_\Lambda) \subseteq P(\Lambda_1) \cup \cdots \cup P(\Lambda_k)$. Because of $\Lambda \in \operatorname{supp}(S_\Lambda)$, this s-linear combination is of the form (11).

We only sketch the next part of the proof, because the idea can be extracted from the proof of Theorem 4.2 in [Loo]: Let $f \in \widehat{\mathbb{F}[T]}^{\mathcal{W}}$ with supp $(f) \subseteq P(\Lambda_1) \cup \cdots \cup P(\Lambda_k)$. We show that f can be obtained as a s-linear combination of $(S_\Lambda)_{\Lambda \in P^+}$. The family $(F_m)_{m \in \mathbb{N}_0}$ defined by

$$F_0 := (P(\Lambda_1) \cup \cdots \cup P(\Lambda_k)) \cap P^+,$$
$$F_{m+1} := F_m \backslash \max(F_m), \quad m \in \mathbb{N}_0,$$

is a filtration of $(P(\Lambda_1) \cup \cdots \cup P(\Lambda_k)) \cap P^+$, in particular $\bigcap_{m \in \mathbb{N}_0} F_m = \emptyset$. Similarly as in [Loo], define recursively the following sequence: Set $g^{(0)} := 0$. For $m \in \mathbb{N}_0$ set

$$g^{(m+1)} := \sum_{\Lambda \in \max(F_m)} c_\Lambda \, S_\Lambda,$$

where the $c_\Lambda's$ are obtained from the decomposition

$$f - \sum_{i=0}^{m} g^{(i)} = \sum_{\Lambda \in \max(F_m)} c_\Lambda \sum_{\lambda \in \mathcal{W}\Lambda} e_\lambda + r \quad \text{with} \quad \operatorname{supp}(r) \subseteq \mathcal{W}F_{m+1}.$$

It is not difficult to check that this sequence is well defined, summable, and $f = \sum_i g^{(i)} = \sum_{\Lambda \in F_0} c_\Lambda S_\Lambda$.

To show the s-linear independence of $(S_\Lambda)_{\Lambda \in P^+}$, let

$$0 = \sum_{\Lambda \in (P(\Lambda_1) \cup \cdots \cup P(\Lambda_k)) \cap P^+} c_\Lambda S_\Lambda = \sum_{m \in \mathbb{N}_0} \sum_{\Lambda \in \max(F_m)} c_\Lambda S_\Lambda.$$

The elements of $\max(F_0)$ are pairwise incomparable, and they are bigger as, or incomparable with the elements of $\max(F_i)$ for $i > 0$. Due to $\operatorname{supp}(S_\Lambda) \subseteq \Lambda - Q_0^+$, we find $c_\Lambda = 0$ for all $\Lambda \in \max(F_0)$. Repeating the same argument, we find successively $c_\Lambda = 0$ for all $\Lambda \in \max(F_i)$, $i \in \mathbb{N}$. □

To investigate the structure of $\widehat{\mathbb{F}[T]}^{\mathcal{W}}$ as an algebra, we choose a family $(S_\Lambda)_{\Lambda \in P^+}$ as above, with $S_\Lambda S_{\Lambda'} = S_{\Lambda + \Lambda'}$ for $\Lambda, \Lambda' \in P^+$. (Choosing the S_Λ's for a system of fundamental dominant weights determines such a family.)

For an $\mathbb{F}$-algebra B, and a subsemigroup S of the lattice P, denote by $B[S]$ the semigroup-algebra, i.e., the $\mathbb{F}$-algebra of all finite sums in the symbols $\tilde{s}_\Lambda$, $\Lambda \in S$, with coefficients in B. If S is contained in the $\mathbb{N}_0$-span of a base of P, denote by $B[[S]]$ the B-algebra of all formal sums in the symbols s_γ, $\gamma \in S$, with coefficients in B.

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Define $\bar{C}_\emptyset := \bar{C}$. Let $\emptyset \neq \Theta \subseteq I$ be special with connected components $\Theta_1, \ldots, \Theta_m$, and define

$$\bar{C}_\Theta := \bar{C} \setminus (\bar{F}_{\Theta_1} \cup \cdots \cup \bar{F}_{\Theta_m}).$$

Note that $\bar{C}_\Theta \cap P^+$ consists of the elements $\Lambda \in P^+$, for which Θ is relevant. $\bar{C}_\Theta$ is obtained from $\bar{C}$ by removing the facets, which are common with the faces $R(\Theta_1), \ldots, R(\Theta_m)$ of the Tits cone. From this follows $\bar{C}_\Theta + \bar{C}_{\Theta'} \subseteq \bar{C}_{\Theta\cup\Theta'}$. Because we have also $K(\Theta) + K(\Theta') \subseteq K(\Theta \cup \Theta')$, the sum

$$\bigoplus_{\Theta \text{ special}} \mathbb{F}[[K(\Theta)]][\bar{C}_\Theta \cap P^+]$$

is a subalgebra of $\mathbb{F}[[Q_0^- \cap \bar{C}]][P^+]$. We write the elements of $\mathbb{F}[[Q_0^- \cap \bar{C}]][P^+]$ in the form

$$\sum_{\Lambda \in E,\ \gamma \in Q_0^- \cap \bar{C}} c_{\gamma\Lambda} s_\gamma \tilde{s}_\Lambda, \qquad c_{\gamma\Lambda} \in \mathbb{F},\ \ E \subseteq P^+ \text{ finite.}$$

THEOREM 2.4. *We get a surjective homomorphism of algebras*

$$\phi : \bigoplus_{\Theta \text{ special}} \mathbb{F}[[K(\Theta)]][\bar{C}_\Theta \cap P^+] \to \widehat{\mathbb{F}[T]}^{\mathcal{W}}$$

by putting $\phi(\sum_{\Lambda\gamma} c_{\gamma\Lambda} s_\gamma \tilde{s}_\Lambda) := \sum_{\Lambda\gamma} c_{\gamma\Lambda} S_{\gamma+\Lambda}$. *The elements of the kernel are finite sums of elements*

$$\sum_{N \in (\Lambda_1 + K(\Theta_1)) \cap (\Lambda_2 + K(\Theta_2))} c_N \left(s_{N-\Lambda_1} \tilde{s}_{\Lambda_1} - s_{N-\Lambda_2} \tilde{s}_{\Lambda_2} \right) \qquad (c_N \in \mathbb{F}), \tag{12}$$

where $\Theta_1,\ \Theta_2$ *are special, and* $\Lambda_1 \in \bar{C}_{\Theta_1} \cap P^+,\ \ \Lambda_2 \in \bar{C}_{\Theta_2} \cap P^+$.

Proof. To show that ϕ is well defined, it is sufficient to show, that we have $\sum_{\gamma \in K(\Theta)} c_{\gamma\Lambda} S_{\gamma+\Lambda} \in \widehat{\mathbb{F}[T]}^{\mathcal{W}}$ for all Θ special, $\Lambda \in \bar{C}_\Theta \cap P^+$, $c_{\gamma\Lambda} \in \mathbb{F}$. Due to Theorem 2.3, it is sufficient to show $K(\Theta) + \Lambda \subseteq P(\Lambda) \cap P^+$. Because Θ is relevant for Λ, this follows from Theorem 1.4 and its preceding Equation (1).

Due to Theorem 2.3, ϕ is surjective, if for every $\Lambda \in P^+$ the set $P(\Lambda) \cap P^+$ is the union of finitely many sets of the form $N + K(\Theta)$, Θ special, and $N \in \bar{C}_\Theta \cap P^+$, i.e., Θ relevant for N.

Due to Theorem 1.4 and its preceding Equation (1), the set $P(\Lambda) \cap P^+$ is the finite union of the sets

$$N + K(\Xi^\infty) \quad \text{with } N \in \Lambda + S(\Xi^0), \tag{13}$$

$$N + K(\Theta) \quad \text{with } N \in \Lambda + S(\Xi^0) + M(\Xi^\infty), \tag{14}$$

where $\Theta \subseteq \Xi^\infty$ is special, and Ξ relevant for Λ. Note that the elements N belong to $P(\Lambda) \cap P^+$.

It is easy to check, that in (13), Ξ^∞ is relevant for N, because Ξ^∞ is already relevant for Λ. In (14), $\Theta = \emptyset$ is relevant for N. Let $\Theta \neq \emptyset$, and let $\Theta_1, \ldots, \Theta_m$ be its

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connected components. For every i choose an element $\gamma_i \in K(\Theta_i)$. Due to (14) for Θ_i, we have $N+\gamma_i \in P(\Lambda) \cap P^+$. Because γ_i is an imaginary root, and $N, N+\gamma_i \in P(\Lambda) \cap P^+$, we can apply [K], Corollary 11.9, and find $(N|\gamma_i) \neq 0$. This is equivalent to the condition, that Θ_i is relevant for N. Therefore Θ is relevant for N.

It is easy to check that ϕ is a homomorphism of algebras. Obviously elements of the form (12) belong to the kernel of ϕ. Let x be an element of the kernel of ϕ. We may write x in the form

$$x = \sum_{N \in P^+} x_N, \quad \text{where} \quad x_N := \sum_{\Theta \text{ special}} \sum_{\substack{\Lambda \in E_\Theta,\, \gamma \in K(\Theta) \\ \Lambda+\gamma=N}} c^{\Theta}_{\Lambda\gamma} s_\gamma \tilde{s}_\Lambda,$$

with $E_\Theta \subseteq \bar{C}_\Theta \cap P^+$ finite. Applying ϕ, and using the s-linear independence of $(S_N)_{N\in P^+}$, we get

$$\sum_{\Theta \text{ special}} \sum_{\substack{\Lambda \in E_\Theta,\, \gamma \in K(\Theta) \\ \Lambda+\gamma=N}} c^{\Theta}_{\Lambda\gamma} = 0, \quad \text{for all } N \in P^+. \tag{15}$$

For every element $N \in P^+$ with $x_N \neq 0$, choose a special set Θ_N, elements $\Lambda_N \in E_{\Theta_N}$, $\gamma_N \in K(\Theta_N)$, such that $N = \Lambda_N + \gamma_N$. Using (15), we get

$$x_N = \sum_{\Theta \text{ special}} \sum_{\substack{\Lambda \in E_\Theta, \gamma \in K(\Theta) \\ \Lambda+\gamma=N}} c^{\Theta}_{\Lambda\gamma} (s_\gamma \tilde{s}_\Lambda - s_{\gamma_N} \tilde{s}_{\Lambda_N}).$$

By using the formula

$$\sum_{N\in P^+} \sum_{\substack{\gamma \in K(\Theta) \text{ with } \Lambda+\gamma=N \\ \gamma' \in K(\Theta') \text{ with } \Lambda'+\gamma'=N}} a_{N,\gamma,\gamma'} = \sum_{N \in (\Lambda+K(\Theta))\cap(\Lambda'+K(\Theta'))} a_{N,N-\Lambda,N-\Lambda'},$$

it is easy to see, that $x = \sum_{N \text{ with } x_N \neq 0} x_N$ can be written in the form

$$x = \sum_{\substack{\Theta', \Theta \text{ special} \\ \Lambda' \in E_{\Theta'},\, \Lambda \in E_\Theta}} \sum_{N \in (\Lambda+K(\Theta))\cap(\Lambda'+K(\Theta'))} c_{N\Theta\Theta'\Lambda\Lambda'} (s_{N-\Lambda} \tilde{s}_\Lambda - s_{N-\Lambda'} \tilde{s}_{\Lambda'}),$$

where

$$c_{N\Theta\Theta'\Lambda\Lambda'} := \begin{cases} c^{\Theta}_{\Lambda\, N-\Lambda} & \text{if} \quad x_N \neq 0,\ \Theta' = \Theta_N,\ \Lambda' = \Lambda_N. \\ 0 & \text{else} \end{cases} \qquad \square$$

The structure of the kernel, and therefore also the structure of $\widehat{\mathbb{F}[T]}$ is quite complicated, except for the following *examples* with at most two special sets:

(1) If A has only components of finite type, then $\widehat{\mathbb{F}[T]}^{\mathcal{W}} = \widehat{\mathbb{F}[T]}^{\mathcal{W}}$ is isomorphic to $\widehat{\mathbb{F}[P^+]}$.

(2) Let A be of affine type. We fix a system of fundamental dominant weights as described in [K], chapter 6.

Recall that c denotes the edge of the Tits cone. We have $c \cap P = \mathbb{Z}\,\delta/a_0$. Here δ denotes the minimal positive imaginary root, and $a_0 = 1$, unless A is of type $\mathrm{A}_{2l}^{(2)}$, in which case $a_0 = 2$. Define the following $\mathbb{F}$-algebra of formal series with coefficients in $\mathbb{F}$:

$$\mathbb{F}(c \cap P) := \mathbb{F}\left(\mathbb{Z}\frac{\delta}{a_0}\right) := \left\{ \sum_{n \in \mathbb{Z}} c_n s_{n\,\delta/a_0} \,\middle|\, c_n \in \mathbb{F},\ \exists\, n_0\, \forall n \geqslant n_0 \colon c_n = 0 \right\}.$$

Using the decomposition $P^+ = P_I^+ \oplus (c \cap P)$, where $P_I^+ := P_I \cap \bar{C}$, it is easy to see, that the invariant algebra $\widehat{\mathbb{F}[T]}^{\mathcal{W}}$ is isomorphic to the subalgebra

$$\mathbb{F}[c \cap P] \oplus \mathbb{F}(c \cap P)[P_I^+ \backslash \{0\}]$$

of $\mathbb{F}(c \cap P)[P_I^+]$. This result is similar to the result of Looijenga [Loo], Theorem 4.2 (i^b), whose invariant algebra $A^{\mathcal{W}}$ is isomorphic to $\mathbb{Z}(c \cap P)[P_I^+]$.

(3) Let A be of strongly hyperbolic type, such that the following condition is satisfied: There exist $m_1, \dots, m_n \in \mathbb{N}$ such that $Q_0^- \cap \bar{C} = \oplus_{i=1}^n \mathbb{N}_0(m_i N_i)$, where $N_1, \dots, N_n$ are the fundamental dominant weights.

It is easy to check, that the elements of the kernel are finite sums of elements

$$\sum_{N \in (\alpha + (Q_0^- \cap \bar{C})) \cap (\beta + (Q_0^- \cap \bar{C}))} c_N \big(s_{N-\alpha} \tilde{s}_\alpha - s_{N-\beta} \tilde{s}_\beta \big)\, \tilde{s}_\Lambda \qquad (c_N \in \mathbb{F}),$$

with

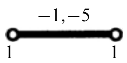
$$\alpha, \beta \in Q_0^- \cap \bar{C}, \quad \alpha \neq \beta, \quad \text{and} \quad \Lambda \in M := \left\{ \sum_{i=1}^n k_i N_i \,\middle|\, k_i = 0, \dots, (m_i - 1) \right\}.$$

Every $\Lambda \in P^+$ can be written uniquely in the form $\Lambda = \Lambda(\mathrm{mod} M) + \alpha$, with $\Lambda(\mathrm{mod} M) \in M$ and $\alpha \in Q_0^- \cap \bar{C}$. On the $\mathbb{F}$-linear space $\mathbb{F}[[Q_0^- \cap \bar{C}]][M]$ the structure of an algebra is induced by

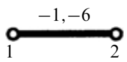
$$(s_\gamma \tilde{s}_\Lambda)(s_{\gamma'} \tilde{s}_{\Lambda'}) := s_{\gamma + \gamma' + \Lambda + \Lambda' - ((\Lambda + \Lambda') \bmod M)} \tilde{s}_{(\Lambda + \Lambda') \bmod M},$$

and this algebra is in the obvious way isomorphic to $\widehat{\mathbb{F}[T]}^{\mathcal{W}}$.

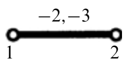
It isn't difficult to check, that the Dynkin diagrams (as defined in [K], section 4.7) of the strongly hyperbolic symmetrizable generalized Cartan matrices, which satisfy the above condition, are:



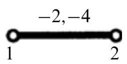
$m_1 = 1, \quad m_2 = 1,$



$m_1 = 2, \quad m_2 = 2,$



$m_1 = 2, \quad m_2 = 1,$



$m_1 = 2, \quad m_2 = 2,$

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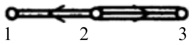
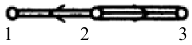


$m_1 = 2, \quad m_2 = m_3 = 1,$



$m_1 = 2, \quad m_2 = m_3 = 1.$

One of the main results of the exponential invariant theory of [Loo] is the description of a certain $m^{\mathcal{W}}$-adic completion of the algebra $A^{\mathcal{W}}$ in the irreducible, nonfinite case. In our notation, the result in the affine case is $\mathbb{Z}(c \cap P)[[P_I^+]]$, and in the indefinite case $\mathbb{Z}[c \cap P][[P_I^+]]$. (Recall that c denotes the edge of the Tits cone X, and $P_I^+ := P_I \cap \bar{C}$.)

Using Theorem 2.3 and Proposition 1.3, it is easy to derive a similar result for the invariant algebra $\widehat{\mathbb{F}[T]}^{\mathcal{W}}$, which we state as a supplement: Because c is a face of X, we get an ideal of $\widehat{\mathbb{F}[T]}^{\mathcal{W}}$ by

$$m^{\mathcal{W}} := \left\{ f \in \widehat{\mathbb{F}[T]}^{\mathcal{W}} \,\middle|\, \operatorname{supp}(f) \subseteq X \backslash c \right\}.$$

Choose a family $(S_\Lambda)_{\Lambda \in P^+}$ as above, with $S_\Lambda S_{\Lambda'} = S_{\Lambda+\Lambda'}$ for $\Lambda, \Lambda' \in P^+$. For $\lambda \in P^+$ set $h(\lambda) = \sum_{i=1}^n \lambda(h_i)$.

PROPOSITION 2.5. *Let the generalized Cartan matrix A have no component of finite type. Then the* $m^{\mathcal{W}}$*-adic completion of* $\widehat{\mathbb{F}[T]}^{\mathcal{W}}$*, described as inverse limit, is given by* $\mathbb{F}[c \cap P][[P_I^+]]$ *together with the maps*

$$\mathbb{F}[c \cap P][[P_I^+]] \to \widehat{\mathbb{F}[T]}^{\mathcal{W}} / (m^{\mathcal{W}})^p,$$
$$\sum_{\lambda_1 \lambda_2} c_{\lambda_1 \lambda_2} s_{\lambda_1} \tilde{s}_{\lambda_2} \mapsto \sum_{\lambda_1 \lambda_2,\, h(\lambda_2) < p} c_{\lambda_1 \lambda_2} S_{\lambda_1 + \lambda_2} + (m^{\mathcal{W}})^p, \quad p \in \mathbb{N}.$$

3. A Formal Chevalley Restriction Theorem

We first describe a formal completion of the algebra of strongly regular functions $\mathbb{F}[G]$ of a Kac–Moody group G. Its construction is similar, but not completely parallel to the construction of the completion in the last section:

Let $\Phi \colon \oplus_{\Lambda \in P^+} L(\Lambda) \otimes L(\Lambda) \to \mathbb{F}[G]$ be the isomorphism of the Peter–Weyl theorem. The algebra of strongly regular functions is $(X \cap P) \times (X \cap P)$-graded by

$$\mathbb{F}[G]_{\lambda\mu} := \bigoplus_{\Lambda \in P^+} \Phi\big(L(\Lambda)_\lambda \otimes L(\Lambda)_\mu\big), \tag{16}$$

$(\lambda, \mu \in X \cap P)$. For an element $f \in \prod_{\lambda,\mu \in X \cap P} \mathbb{F}[G]_{\lambda\mu}$, we denote by $f_{\lambda\mu} := pr_{\lambda\mu}(f)$ its projection onto $\mathbb{F}[G]_{\lambda\mu}$. Set

$$\operatorname{supp}(f) := \big\{ (\lambda, \mu) \in (X \cap P) \times (X \cap P) \,|\, f_{\lambda\mu} \neq 0 \big\}.$$

Denote by $pr_\Lambda(f_{\lambda\mu})$ the projection of $f_{\lambda\mu}$ onto the Λ-summand of (16). Define $pr_\Lambda(f) := \prod_{\lambda,\mu \in X \cap P} pr_\Lambda(f_{\lambda\mu})$, and set

$$\operatorname{Supp}(f) := \big\{ \Lambda \in P^+ \,|\, pr_\Lambda(f) \neq 0 \big\} \subseteq P^+.$$

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PROPOSITION 3.1.

$$\widehat{\mathbb{F}[G]} := \left\{ f \in \prod_{\lambda,\mu \in X\cap P} \mathbb{F}[G]_{\lambda\mu} \,\middle|\, \exists\, k \in \mathbb{N}, \exists \Lambda_1, \ldots, \Lambda_k \in P^+ : \mathrm{Supp}(f) \subseteq \bigcup_{i=1}^{k} P(\Lambda_i) \right\}$$

is a commutative associative algebra with unit, the multiplication given by

$$\left(f\tilde{f}\right)_{\lambda\mu} := \sum_{\substack{\lambda_1,\lambda_2 \in X\cap P,\ \lambda=\lambda_1+\lambda_2 \\ \mu_1,\mu_2 \in X\cap P,\ \mu=\mu_1+\mu_2}} f_{\lambda_1\mu_1} \tilde{f}_{\lambda_2\mu_2}. \tag{17}$$

Proof. We only show that the multiplication map (17) is well defined, all other things are obvious. Let

$$\mathrm{Supp}(f) \subseteq \bigcup_{i=1}^{k} P(\Lambda_i), \qquad \mathrm{Supp}(\tilde{f}) \subseteq \bigcup_{j=1}^{\tilde{k}} P(\tilde{\Lambda}_j).$$

Theorem 1.2(b) implies

$$\mathrm{supp}(f) \subseteq \bigcup_{i=1}^{k} P(\Lambda_i) \times P(\Lambda_i), \qquad \mathrm{supp}(\tilde{f}) \subseteq \bigcup_{j=1}^{\tilde{k}} P(\tilde{\Lambda}_j) \times P(\tilde{\Lambda}_j),$$

therefore the sum (17) is finite. To show $f\tilde{f} \in \widehat{\mathbb{F}[G]}$, we write a summand of (17) as the finite sum

$$f_{\lambda_1\mu_1}\tilde{f}_{\lambda_2\mu_2} = \sum_{\substack{\Lambda \in \mathrm{Supp}(f) \\ \tilde{\Lambda} \in \mathrm{Supp}(\tilde{f})}} pr_{\Lambda}(f_{\lambda_1\mu_1}) pr_{\tilde{\Lambda}}(\tilde{f}_{\lambda_2\mu_2}).$$

Due to Theorem 1.2(a), (b), and (c), we find

$$\left\{ M \in P^+ \middle| pr_M\left(pr_{\Lambda}(f_{\lambda_1\mu_1}) pr_{\tilde{\Lambda}}(\tilde{f}_{\lambda_2\mu_2})\right) \neq 0 \right\} \subseteq P(\Lambda + \tilde{\Lambda}) = P(\Lambda) + P(\Lambda').$$

Using Theorem 1.2(b) and (c) once more, we find $\mathrm{Supp}(f_{\lambda_1\mu_1}\tilde{f}_{\lambda_2\mu_2}) \subseteq \bigcup_{i,j} P(\Lambda_i + \tilde{\Lambda}_j)$. Therefore also $\mathrm{Supp}(f\tilde{f}) \subseteq \bigcup_{i,j} P(\Lambda_i + \tilde{\Lambda}_j)$. □

Remark. We identify $\mathbb{F}[G]$, in the obvious way, with a subalgebra of $\widehat{\mathbb{F}[G]}$. If A has only components of finite type, then $\widehat{\mathbb{F}[G]} = \mathbb{F}[G]$.

It is useful, to introduce a natural limit concept for $\widehat{\mathbb{F}[G]}$. This is done in a similar way as for $\widehat{\mathbb{F}[T]}$, but 'supp' being replaced by 'Supp'. Call a sequence $(f_i)_{i\in\mathbb{N}} \subseteq \widehat{\mathbb{F}[G]}$ convergent to $f \in \widehat{\mathbb{F}[G]}$ if:

- There exist $k \in \mathbb{N}$, $\Lambda_1, \ldots, \Lambda_k \in P^+$, such that $\mathrm{Supp}(f_i) \subseteq (P(\Lambda_1) \cup \cdots \cup P(\Lambda_k)) \cap P^+$ for all $i \in \mathbb{N}$.
- For every $\lambda, \mu \in X \cap P$, there exists an element $i_0 \in \mathbb{N}$, such that for all $i \geqslant i_0$ we have $(f_i)_{\lambda\mu} = f_{\lambda\mu}$.

It is obvious how to define 'dense' and 'continuous', also if maps between different spaces, as $\widehat{\mathbb{F}[G]}$ and $\widehat{\mathbb{F}[T]}$, are involved. Note that the addition, multiplication with a

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scalar, and the multiplication of the algebra $\widehat{\mathbb{F}[G]}$ are continuous, as well as the projections $pr_{\lambda\mu}$, pr_Λ, and that $\mathbb{F}[G]$ is dense in $\widehat{\mathbb{F}[G]}$. It is also obvious, how to define 'summable', 's-linear independent', and 's-base'.

THEOREM 3.2. (1) *The action* π *of* $G \times G$ *on* $\mathbb{F}[G]$ *extends uniquely to an action* π *on* $\widehat{\mathbb{F}[G]}$*, by continuous homomorphisms of algebras.*

(2) *The Chevalley involution* $*$ *of* $\mathbb{F}[G]$ *extends uniquely to a continuous involution* $*$ *of* $\widehat{\mathbb{F}[G]}$.

Proof. Let $g, h \in G$ and $f \in \widehat{\mathbb{F}[G]}$. If there exist extensions with these continuity-properties, then, due to $f = \sum_{\lambda,\mu} f_{\lambda\mu}$, they are uniquely determined, and satisfy:

$$(\pi(g,h)f)_{\tilde\lambda\tilde\mu} = \sum_{\lambda,\mu} pr_{\tilde\lambda\tilde\mu}(\pi(g,h)f_{\lambda\mu}), \quad \tilde\lambda, \tilde\mu \in X \cap P, \tag{18}$$

$$(f^*)_{\lambda\mu} = (f_{\lambda\mu})^*, \quad \lambda, \mu \in X \cap P. \tag{19}$$

It's easy to see, that (19) defines a continuous involution of $\widehat{\mathbb{F}[G]}$. It remains to show, that (18) defines an action of $G \times G$ on $\widehat{\mathbb{F}[G]}$ by continuous homomorphisms.

Note that for $f \in \widehat{\mathbb{F}[G]}$ with $\mathrm{Supp}(f) \subseteq (P(\Lambda_1) \cup \cdots \cup P(\Lambda_k)) \cap P^+$, we can fix a decomposition

$$(P(\Lambda_1) \cup \cdots \cup P(\Lambda_k)) \cap P^+ = \dot{\bigcup_{i=1,\ldots,k}} C_i \quad \text{with } C_i \subseteq P(\Lambda_i),$$

and write f in the form $f = \sum_{i=1}^k f^{(i)}$ with $f^{(i)} := \sum_{\Lambda \in C_i} pr_\Lambda(f) \in \widehat{\mathbb{F}[G]}$.

To check that (18) gives a well defined continuous linear map, we therefore may restrict to elements $f \in \widehat{\mathbb{F}[G]}$ with $\mathrm{Supp}(f) \subseteq P(\Lambda)$, $\Lambda \in P^+$. Fix $\tilde\lambda$, $\tilde\mu \in X \cap P$. Write $g, h \in G$ in the form:

$$\begin{aligned} g &= u_{\alpha_1} \ldots u_{\alpha_p} t && \text{with } u_{\alpha_i} \in U_{\alpha_i},\ t \in T, \\ h &= u_{\beta_1} \ldots u_{\beta_q} \tilde t && \text{with } u_{\beta_i} \in U_{\beta_i},\ \tilde t \in T. \end{aligned}$$

For $\eta, \eta' \in P(\Lambda)$, $\alpha \in \Delta_{\mathrm{re}}^+$, denote the relation $\eta' \in \eta + \mathbb{N}_0\alpha$ by $\eta \xrightarrow{\alpha} \eta'$. For $N \in P(\Lambda) \cap P^+$ we have, due to Theorem 1.2(b), $P(N) \subseteq P(\Lambda)$. In particular, every α-string of $P(N)$ is contained in a unique α-string of $P(\Lambda)$. Therefore, $pr_{\tilde\lambda\tilde\mu}(\pi(g,h)f_{\lambda\mu}) \neq 0$ is only possible for such λ, μ, from which $\tilde\lambda, \tilde\mu$ can be reached by a sequence of directed parts of strings in $P(\Lambda)$ of the form

$$\lambda \xrightarrow{-\alpha_p} * \xrightarrow{-\alpha_{p-1}} * \cdots * \xrightarrow{-\alpha_1} \tilde\lambda, \qquad \mu \xrightarrow{\beta_q} * \xrightarrow{\beta_{q-1}} * \cdots * \xrightarrow{\beta_1} \tilde\mu.$$

This is only possible for finitely many λ, μ, because a real root string in $P(\Lambda)$ contains only finitely many elements. If we denote by $S_{\tilde\lambda}$ the set of all such λ's, and by $T_{\tilde\mu}$ the set of all such μ's, then (18) can be written as

$$(\pi(g,h)f)_{\tilde\lambda\tilde\mu} = \sum_{(\lambda,\mu) \in S_{\tilde\lambda} \times T_{\tilde\mu}} pr_{\tilde\lambda\tilde\mu}(\pi(g,h)f_{\lambda\mu}), \quad \tilde\lambda, \tilde\mu \in X \cap P. \tag{20}$$

To check the continuity of $\pi(g,h)$, let $(f_i)_{i\in\mathbb{N}}$ be a sequence convergent to f with $\mathrm{Supp}(f_i), \mathrm{Supp}(f) \subseteq P(\Lambda)$, $\Lambda \in P^+$. Applying $\pi(g,h)$ to f_i and f doesn't change their Supp-sets. Using (20), we find that $\pi(g,h)f_i$ is convergent to $\pi(g,h)f$.

To show that $\pi(g,h)$ is a homomorphism of algebras, let $f, f' \in \widehat{\mathbb{F}[G]}$. Choose sequences $(f_i)_{i\in\mathbb{N}}, (f_i')_{i\in\mathbb{N}} \subseteq \mathbb{F}[G]$ with limits f, f'. We have

$$\pi(g,h)(f_i f_i') = (\pi(g,h)f_i)(\pi(g,h)f_i') \quad (i \in \mathbb{N}).$$

Due to the continuity of $\pi(g,h)$, and of the multiplication map, we get

$$\pi(g,h)(ff') = (\pi(g,h)f)(\pi(g,h)f') \quad (i \in \mathbb{N}).$$

In a similar way, the action property of π transfers from $\mathbb{F}[G]$ to $\widehat{\mathbb{F}[G]}$. □

The adjoint action of G on $\mathbb{F}[G]$ extends uniquely to an action c on $\widehat{\mathbb{F}[G]}$ by continuous homomorphisms. It is given by $c(g) := \pi(g,g)$, $g \in G$. Next we determine the corresponding invariant algebra $\widehat{\mathbb{F}[G]}^G$.

Let $\Lambda \in P^+$. For every $\lambda \in P(\Lambda)$ choose $\langle\langle\ |\ \rangle\rangle$-dual bases

$$(a_{\lambda k})_{k=1,\ldots,m_\lambda}, \qquad (b_{\lambda k})_{k=1,\ldots,m_\lambda}$$

of $L(\Lambda)_\lambda$. The formal G-character of $L(\Lambda)$ is defined by

$$\mathrm{Tr}_\Lambda := \sum_{\lambda\in P(\Lambda)} \sum_{i=1}^{m_\lambda} f_{a_{\lambda i} b_{\lambda i}} \in \widehat{\mathbb{F}[G]}.$$

It is independent of the chosen dual bases. In the classical case, it coincides with the G-character of $L(\Lambda)$.

THEOREM 3.3. *The family* $(\mathrm{Tr}_\Lambda)_{\Lambda\in P^+}$ *is an s-base of* $\widehat{\mathbb{F}[G]}^G$. *Its s-linear combinations are given by the sums*

$$\sum_{\Lambda\in(P(\Lambda_1)\cup\cdots\cup P(\Lambda_k))\cap P^+} c_\Lambda \mathrm{Tr}_\Lambda, \tag{21}$$

with $c_\Lambda \in \mathbb{F}$, $\Lambda_1, \ldots, \Lambda_k \in P^+$, $k \in \mathbb{N}$.

Proof. We only show, that every element of $\widehat{\mathbb{F}[G]}^G$ is of the form (21). Then the rest of the theorem can be proved similar to the corresponding parts of Theorem 2.3.

For every $\Lambda \in P^+$, the projection $pr_\Lambda\colon \widehat{\mathbb{F}[G]} \to \widehat{\mathbb{F}[G]}$ is G-invariant. Since for $f \in \widehat{\mathbb{F}[G]}^G$ we have

$$f = \sum_{\Lambda\in\mathrm{Supp}(f)} pr_\Lambda(f) \quad \text{with} \quad pr_\Lambda(f) \in \left(pr_\Lambda(\widehat{\mathbb{F}[G]})\right)^G,$$

it is sufficient to show $\left(pr_\Lambda(\widehat{\mathbb{F}[G]})\right)^G = \mathbb{F}\,\mathrm{Tr}_\Lambda$.

We first show the inclusion '$\supseteq$'. Define an action c of G on $\mathrm{End}(L(\Lambda))$ by

$$c(g)\phi := g\phi g^{-1}, \quad g \in G, \ \phi \in \mathrm{End}(L(\Lambda)).$$

Note, that due to [K], Lemma 9.3, we have $(\mathrm{End}(L(\Lambda))^G = \mathbb{F}\mathrm{id}_{L(\Lambda)}$. Since $\langle\langle\,|\,\rangle\rangle$ is nondegenerate, we get an injective linear map Ψ. $\mathrm{End}(L(\Lambda)) \to pr_\Lambda(\widehat{\mathbb{F}[G]})$ by

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$$\Psi(\phi)_{\lambda\mu} := \sum_{ij} f_{a_{\lambda i} b_{\mu j}} \langle\langle a_{\mu j} | \phi b_{\lambda i} \rangle\rangle, \quad \lambda, \mu \in X \cap P, \ \phi \in \mathrm{End}(L(\Lambda)).$$

We show, that it is also G-invariant. The group G is generated by the groups $U_\alpha T$, $\alpha \in \Delta_{\mathrm{re}}$. Therefore it is sufficient to show the $U_\alpha T$-invariance for all α. Let $u \in U_\alpha T$, let $\tilde{\lambda}, \tilde{\mu} \in P(\Lambda)$, and denote by R resp. S the α-string through $\tilde{\lambda}$ resp. $\tilde{\mu}$. We have

$$(c(u)\Psi(\phi))_{\tilde{\lambda}\tilde{\mu}} = \sum_{(\lambda,\mu) \in R \times S} pr_{\tilde{\lambda}\tilde{\mu}}\big(c(u)(\Psi(\phi)_{\lambda\mu})\big).$$

Insert the definition of $\Psi(\phi)_{\lambda\mu}$ in the expression on the right. After some transformations, it is equal to

$$pr_{\tilde{\lambda}\tilde{\mu}}\left(\sum_{(\lambda,\mu) \in R \times S} \sum_{ij} f_{(u^*)^{-1} a_{\lambda i} u b_{\mu j}} \langle\langle (u^*)^{-1} a_{\mu j} \mid u\phi u^{-1} u b_{\lambda i} \rangle\rangle\right). \tag{22}$$

The pairs

$$((u^*)^{-1} a_{\lambda i})_{\lambda \in R,\ i=1,\dots,m_\lambda}, \qquad (u b_{\lambda i})_{\lambda \in R,\ i=1,\dots,m_\lambda}, \tag{23}$$

$$((u^*)^{-1} a_{\mu j})_{\mu \in S,\ j=1,\dots,m_\mu}, \qquad (u b_{\mu j})_{\mu \in S,\ j=1,\dots,m_\mu} \tag{24}$$

are pairs of $\langle\langle\ |\ \rangle\rangle$-dual bases of $\oplus_{\lambda \in R} L(\Lambda)_\lambda$ resp. $\oplus_{\mu \in S} L(\Lambda)_\mu$. Expression (22) does not change, if we use other pairs of $\langle\langle\ |\ \rangle\rangle$-dual bases. In particular, we can use the pairs of dual bases (23) and (24) with u replaced by 1, and obtain $\Psi(c(u)\phi)_{\tilde{\lambda}\tilde{\mu}}$.

We find

$$\left(pr_\Lambda(\widehat{\mathbb{F}[G]})\right)^G \supseteq \Psi((\mathrm{End}(L(\Lambda)))^G) = \mathbb{F}\Psi(\mathrm{id}_{L(\Lambda)}) = \mathbb{F}\mathrm{Tr}_\Lambda.$$

To show equality, let $f \in \left(pr_\Lambda(\widehat{\mathbb{F}[G]})\right)^G$. Because Ψ is injective and G-equivariant, it is sufficient to find an element $\phi_f \in \mathrm{End}(L(\Lambda))$, such that $f = \Psi(\phi_f)$. Define $\phi_f \in \mathrm{End}(L(\Lambda))$ by

$$\phi_f b_{\lambda i} := \sum_k c_{\lambda k i} b_{\lambda k}, \quad \lambda \in P(\Lambda), \ i = 1, \dots, m_\lambda,$$

where the coefficients $c_{\lambda k i}$ are given by $f_{\lambda\lambda} = \sum_{ij} c_{\lambda j i} f_{a_{\lambda i} b_{\lambda j}}$. Using the definition of Ψ, it is easy to check, that we have $(\Psi(\phi_f))_{\lambda\lambda} = f_{\lambda\lambda}$, and $(\Psi(\phi_f))_{\lambda\mu} = 0$ for $\lambda, \mu \in X \cap P$, $\lambda \neq \mu$.

To prove $f = \Psi(\phi_f)$, it remains to show $f_{\lambda\mu} = 0$ for $\lambda \neq \mu$. For $t \in T$ we have $c(t)f = f$. Therefore

$$e_\lambda(t^{-1}) e_\mu(t) f_{\lambda\mu} = f_{\lambda\mu}, \qquad \lambda, \mu \in P(\Lambda).$$

For $\lambda \neq \mu$ there exists an element $t \in T$, such that $e_\lambda(t) \neq e_\mu(t)$. Using the last equation, we find $f_{\lambda\mu} = 0$. □

Due to the explicit descriptions of the invariant algebras $\widehat{\mathbb{F}[T]}^{\mathcal{W}}$ and $\widehat{\mathbb{F}[G]}^G$, it is now easy to derive the formal Chevalley restriction theorem:

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COROLLARY 3.4. *The restriction map* $r\colon \mathbb{F}[G] \to \mathbb{F}[T]$ *extends uniquely to a continuous surjective homomorphism of algebras* $r\colon \widehat{\mathbb{F}[G]} \to \widehat{\mathbb{F}[T]}$. *This extension induces an isomorphism of the invariant algebras* $\widehat{\mathbb{F}[G]}^G$ *and* $\widehat{\mathbb{F}[T]}^{\mathcal{W}}$.

Proof. The restriction map $r\colon \mathbb{F}[G] \to \mathbb{F}[T]$ has been given explicitely in the proof of Proposition 2.1. It is easy to check, that the map $r\colon \widehat{\mathbb{F}[G]} \to \widehat{\mathbb{F}[T]}$ defined by $r(f) := \sum_\lambda f_{\lambda\lambda}(1)e_\lambda$, $f \in \widehat{\mathbb{F}[G]}$, is an extension with the properties stated in the corollary. It is also obvious, that $r(\mathrm{Tr}_\Lambda) = \chi_\Lambda$, where χ_Λ is the formal T-character defined in (10). Due the last theorem and Theorem 2.3, the restricted map $r\colon \widehat{\mathbb{F}[G]}^G \to \widehat{\mathbb{F}[T]}^{\mathcal{W}}$ is bijective. □

Remark. The $G \times G$-algebra $\widehat{\mathbb{F}[G]}$ has been defined using the Cartan subalgebra $\mathbf{h}$. Let $\mathbf{h}'$ be another Cartan subalgebra with corresponding $G \times G$-algebra $\widehat{\mathbb{F}[G]}'$. By using the transitivity of the adjoint action of G on the Cartan subalgebras, it is not difficult to see, that the identity map of $\mathbb{F}[G]$ can be extended uniquely to a continuous, continuously invertible isomorphism of $G \times G$-algebras between $\widehat{\mathbb{F}[G]}$ and $\widehat{\mathbb{F}[G]}'$. This isomorphism maps G-characters to G-characters. Thus, we may identify the $G \times G$-algebras belonging to different Cartan subalgebras.

4. A Convergent Chevalley Restriction Theorem in the Affine Case

In this section, we restrict to a generalized Cartan matrix of affine type, and to the ground field of complex numbers $\mathbb{F} = \mathbb{C}$.

We replace the Chevalley involution of G by the compact involution. We replace the nondegenerate contravariant symmetric bilinear forms on the modules $L(\Lambda)$, $\Lambda \in P^+$, by the contravariant positive definite Hermitian forms. But for the sake of simplicity, we shall use the same notations. Note that $\mathbb{C}[G]_{\lambda\mu}$ is also spanned by the matrix coefficients of elements $v \in L(\Lambda)_\lambda$, $w \in L(\Lambda)_\mu$, $\Lambda \in P^+$, relative to these forms.

The algebra $\widehat{\mathbb{C}[G]}$ is equipped with the adjoint action c, and with the involution $*$, induced by the compact involution of $\mathbb{C}[G]$.

Denote by G^{tr} the set of elements $g \in G$, such that for all $\Lambda \in P^+$, the linear map $\pi_\Lambda(g)$ on $L(\Lambda)$ can be extended to a trace class operator on the Hilbert space completion of $L(\Lambda)$. Note that G^{tr} is invariant under the compact involution. For a subset M of G set $M^{\mathrm{tr}} := M \cap G^{\mathrm{tr}}$.

Choose a system of fundamental dominant weights as in [K], chapter 6. Denote by δ the minimal positive imaginary root, denote by d the scaling element. Due to [B], Lemma 3 and Theorem 1, we have

$$\begin{aligned} T^{\mathrm{tr}} &= \{\exp(h) \mid h \in \mathbf{h} \text{ with } \mathrm{Re}(\delta(h)) > 0\} \\ &= \left\{ t_{h_1}(c_1)\cdots t_{h_n}(c_n)t_d(c_{n+1}) \;\middle|\; \begin{array}{l} c_1, \ldots, c_n \in \mathbb{C}^\times, \\ c_{n+1} \in \mathbb{C} \text{ with } |c_{n+1}| > 1 \end{array} \right\}, \\ G^{\mathrm{tr}} &\supseteq G'(T_{\mathrm{rest}})^{\mathrm{tr}} = U^\pm N^{\mathrm{tr}} U^\pm = K\,(T^+)^{\mathrm{tr}} U^\pm. \end{aligned} \tag{25}$$

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G. Brüchert conjectured equality in (25). Note that T^{tr} is invariant under the $\mathcal{W}$-action on T, and $G'(T_{\mathrm{rest}})^{\mathrm{tr}}$ is invariant under the conjugation action of G on itself.

For $\Lambda \in P^+$, we get an orthonormal base of $L(\Lambda)$, by choosing an orthonormal base $(v_{\lambda i})_{i=1,\ldots,m_\lambda}$ of every weight space $L(\Lambda)_\lambda$, $\lambda \in P(\Lambda)$. This base is a complete orthonormal system of the Hilbert space completion of $L(\Lambda)$. The trace function of the semigroup of trace class operators of the Hilbert space completion of $L(\Lambda)$ induces functions on T^{tr} and G^{tr}, which can be described by the following absolutely convergent series:

$$\sum_{\lambda \in P(\Lambda)} m_\lambda e_\lambda(t), \quad t \in T^{\mathrm{tr}}. \tag{26}$$

$$\sum_{\lambda \in P(\Lambda)} \sum_{i=1}^{m_\lambda} \langle\langle v_{\lambda i} \mid g\, v_{\lambda i} \rangle\rangle, \quad g \in G^{\mathrm{tr}}. \tag{27}$$

Clearly (26) is a $\mathcal{W}$-invariant function. G. Brüchert showed in [B], Theorem 3(a), that (27) restricted to $G'(T_{\mathrm{rest}})^{\mathrm{tr}}$ is a G-invariant function.

THEOREM 4.1. *In* (25) *we have equality, i.e.,*

$$G^{\mathrm{tr}} = G'(T_{\mathrm{rest}})^{\mathrm{tr}} \;=\; U^{\pm} N^{\mathrm{tr}} U^{\pm} \;=\; K(T^+)^{\mathrm{tr}} U^{\pm}.$$

Proof. Due to the Iwasawa decomposition we have $G = KT^+U$. The group K gives rise to groups of unitary operators on the Hilbert space completions of $L(\Lambda)$, $\Lambda \in P^+$. Therefore we get $G^{\mathrm{tr}} = K(T^+U)^{\mathrm{tr}}$. Due to (25) we have $(T^+)^{\mathrm{tr}} U \subseteq (T^+U)^{\mathrm{tr}}$. To show the reverse inclusion, let $t \in T^+$, $u \in U$ with $tu \in (T^+U)^{\mathrm{tr}}$. Fix an element $\Lambda \in P^+$ with $\Lambda(h_j) > 0$ for some $j \in I$, and choose an orthonormal base of $L(\Lambda)$ as above. We have

$$\infty > \sum_{\lambda \in P(\Lambda)} \sum_{i=1}^{m_\lambda} |\langle\langle v_{\lambda i} \mid tu\, v_{\lambda i} \rangle\rangle| = \sum_{\lambda \in P(\Lambda)} m_\lambda \, |e_\lambda(t)|.$$

Using [K], Proposition 11.10 and Equation (11.10.1), we find $t \in T^{\mathrm{tr}}$. □

Next we define an appropriate notion of convergence for the elements of $\widehat{\mathbb{C}[T]}$ and $\widehat{\mathbb{C}[G]}$, such that the formal characters χ_Λ and Tr_Λ are convergent, and give rise to the functions (26) and (27). (For this note, that $\mathrm{Tr}_\Lambda = \sum_{\lambda \in P(\Lambda)} \sum_{i=1}^{m_\lambda} f_{v_{\lambda i} v_{\lambda i}}$, where the matrix coefficient $f_{v_{\lambda i} v_{\lambda i}}$ is built with the Hermitian form.) As part of the following theorem, we extend Theorem 3(a) of [B] to all functions corresponding to convergent elements of $\widehat{\mathbb{C}[G]}$.

Call an element $f \in \widehat{\mathbb{C}[T]}$ convergent, if $\sum_\lambda |f_\lambda(t)| < \infty$ for all $t \in T^{\mathrm{tr}}$. Call an element $f \in \widehat{\mathbb{C}[G]}$ convergent, if

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$$\sum_{\lambda\mu} |(c(h)f)_{\lambda\mu}(g)| < \infty$$

for all $h \in G$ and $g \in G^{\mathrm{tr}}$. This notion is independent of the chosen Cartan subalgebra.

Assigning to $t \in T^{\mathrm{tr}}$ (resp. $g \in G^{\mathrm{tr}}$) the value $\sum_\lambda f_\lambda(t)$ (resp. $\sum_{\lambda\mu} f_{\lambda\mu}(g)$), we get a function on T^{tr} (resp. G^{tr}), which we also denote by f.

THEOREM 4.2. (1) *The set of convergent elements of* $\widehat{\mathbb{C}[T]}$ *forms a* $\mathcal{W}$*-invariant subalgebra of* $\widehat{\mathbb{C}[T]}$. *It can be identified with the corresponding algebra of functions on* T^{tr}*, which we denote by* $\mathbb{C}[T^{\mathrm{tr}}]$*, the* $\mathcal{W}$*-action being induced by the conjugation action of* $\mathcal{W}$ *on* T^{tr}.

(2) *The set of convergent elements of* $\widehat{\mathbb{C}[G]}$ *forms a* G *and* $*$*-invariant subalgebra of* $\widehat{\mathbb{C}[G]}$. *It can be identified with the corresponding algebra of functions on* G^{tr}*, which we denote by* $\mathbb{C}[G^{\mathrm{tr}}]$*, the* G*-action being induced by the conjugation action of* G *on* G^{tr}*, and the involution* $*$ *being induced by the compact involution of* G^{tr}.

Remark. The algebras $\mathbb{C}[T]$, $\mathbb{C}[G]$ contain only convergent elements. They can be identified with the corresponding subalgebras of $\mathbb{C}[T^{\mathrm{tr}}]$, $\mathbb{C}[G^{\mathrm{tr}}]$.

Proof. We only show (2). The arguments in the proof of (1) are easy, or similar to some arguments in the proof of (2).

The definition of 'convergent' implies, that the set of convergent elements of $\widehat{\mathbb{C}[G]}$ is a G-invariant subspace of $\widehat{\mathbb{C}[G]}$. It is $*$-invariant, due to $c(h)f^* = (c((h^*)^{-1})f)^*$.

It is also a subalgebra. Obviously the unit of $\widehat{\mathbb{C}[G]}$ is convergent. If $f_1, f_2 \in \widehat{\mathbb{C}[G]}$ are convergent, a Cauchy summation argument shows the convergence of $f_1 f_2$.

Next, we prove, that the algebra of convergent elements can be identified with its corresponding algebra of functions on G^{tr}. Let $f \in \widehat{\mathbb{C}[G]}$ be convergent, with $\sum_{\lambda\mu} f_{\lambda\mu}(g) = 0$ for all $g \in G^{\mathrm{tr}}$. Due to [K,P 2], Lemma 2.1(d), which is also valid for the algebra of strongly regular functions of the slightly bigger Kac–Moody group, the condition $f_{\lambda\mu}|_{U^-TU^+} = 0$ for all λ, μ, is sufficient for $f = 0$.

Fix an element $g \in U^-TU^+ = U^-U^+T$, and write g in the form

$$g = \exp(y_{\beta_1}) \cdots \exp(y_{\beta_p}) \exp(x_{\gamma_1}) \cdots \exp(x_{\gamma_q}) e^h$$

with $p, q \in \mathbb{N}$, $\beta_i, \gamma_j \in \Delta^+_{\mathrm{re}}$, $y_{\beta_i} \in \mathbf{g}_{-\beta_i}$, $x_{\gamma_j} \in \mathbf{g}_{\gamma_j}$, and $h \in \mathbf{h}$. To show $f_{\lambda\mu}(g) = 0$, it is sufficient to show

$$f_{\lambda\mu}\left(y_{\beta_1}^{k_1} \cdots y_{\beta_p}^{k_p} x_{\gamma_1}^{l_1} \cdots x_{\gamma_q}^{l_q}\right) = 0, \tag{28}$$

for all $k_1, \ldots, k_p, l_1, \ldots, l_q \in \mathbb{N}_0$.

To abbreviate the notation, for $c \in \mathbb{C}$ set

$$\mathfrak{y}_{\beta_i}(c) := \exp(c y_{\beta_i}), \quad \mathfrak{x}_{\gamma_j}(c) := \exp(c x_{\gamma_j}).$$

Due to $U^- U^+ T^{\mathrm{tr}} \subseteq G^{\mathrm{tr}}$, and the description of T^{tr} given at the beginning of this section, we have for all $r_i, s_j \in \mathbb{C}$, $c_1, \ldots, c_n \in \mathbb{C}^\times$, $c_{n+1} \in \mathbb{C}$ with $|c_{n+1}| > 1$:

$$0 = \sum_{\lambda,\mu} f_{\lambda\mu}\Big(\mathfrak{y}_{\beta_1}(r_1) \cdots \mathfrak{y}_{\beta_p}(r_p)\mathfrak{x}_{\gamma_1}(s_1) \cdots \mathfrak{x}_{\gamma_q}(s_q) t_{h_1}(c_1) \cdots t_{h_n}(c_n) t_d(c_{n+1})\Big)$$

$$= \sum_{\mu}\left(\sum_{\lambda}\sum_{\substack{k_1,\ldots,k_p\\ l_1,\ldots,l_q}} f_{\lambda\mu}\Big(y_{\beta_1}^{k_1} \cdots y_{\beta_p}^{k_p} x_{\gamma_1}^{l_1} \cdots x_{\gamma_q}^{l_q}\Big)\frac{r_1^{k_1}}{k_1!} \cdots \frac{s_q^{l_q}}{l_q!}\right) c_1^{\mu(h_1)} \cdots c_{n+1}^{\mu(d)}.$$

This expression is a Laurent series in $c_1, \ldots, c_{n+1}$, its coefficients vanish. Due to the orthogonality of the weight spaces, Equation (28) is valid for

$$\lambda \neq \mu + l_1\gamma_1 + \cdots + l_q\gamma_q - k_1\beta_1 - \cdots - k_p\beta_p.$$

Therefore the coefficients of the Laurent series are power series in $r_1, \ldots, r_p, s_1, \ldots, s_q$. The vanishing of the coefficients of these power series proves (28) for $\lambda = \mu + l_1\gamma_1 + \cdots + l_q\gamma_q - k_1\beta_1 - \cdots - k_p\beta_p$.

Obviously, the involution on the set of convergent elements identifies with the involution induced by the compact involution on G^{tr}. To check the corresponding thing for the G-actions, we may restrict to convergent elements $f \in \widehat{\mathbb{C}[G]}$ with $\mathrm{Supp}(f) \subseteq P(\Lambda) \cap P^+$, $\Lambda \in P^+$. Note that due to Theorem 1.2(b), $\mathrm{supp}(f) \subseteq P(\Lambda) \times P(\Lambda)$.

Since G is generated by the groups $U_\alpha T$, $\alpha \in \Delta_{\mathrm{re}}^+$, it is sufficient to consider only $U_\alpha T$-actions. Let $u \in U_\alpha T$. Because f and $c(u)f$ are convergent, and G^{tr} is invariant under conjugation, we find for all $g \in G^{\mathrm{tr}}$:

$$(c(u)f)(g) = \left(\sum_{\substack{s_1,s_2\ \alpha\text{-strings}\\ \text{of } P(\Lambda)}} \sum_{\lambda_1 \in s_1,\ \lambda_2 \in s_2}\right)\left(\sum_{\mu_1 \in s_1,\ \mu_2 \in s_2} (c(u)f_{\mu_1\mu_2})\lambda_1\lambda_2(g)\right)$$

$$= \sum_{\substack{s_1,s_2\ \alpha\text{-strings}\\ \text{of } P(\Lambda)}} \sum_{\mu_1 \in s_1,\ \mu_2 \in s_2} (c(u)f_{\mu_1\mu_2})(g) = f(u^{-1}gu). \qquad \square$$

Due to the last proposition, the invariant algebras $\mathbb{C}[G^{\mathrm{tr}}]$, $\mathbb{C}[T^{\mathrm{tr}}]^{\mathcal{W}}$ consist of the functions induced by the convergent elements of $\widehat{\mathbb{C}[G]}^G$, $\widehat{\mathbb{C}[T]}^{\mathcal{W}}$. In particular, $\mathbb{C}[G^{\mathrm{tr}}]^G$ contains the functions given by the formal G-characters Tr_Λ, $\Lambda \in P^+$, and $\mathbb{C}[T^{\mathrm{tr}}]^{\mathcal{W}}$ contains the functions given by the formal T-characters χ_Λ, $\Lambda \in P^+$.

We can now formulate the convergent Chevalley restriction theorem:

THEOREM 4.3. *The restriction map* $r\colon \widehat{\mathbb{C}[G]} \to \widehat{\mathbb{C}[T]}$ *induces the restriction map of functions* $r\colon \mathbb{C}[G^{\mathrm{tr}}] \to \mathbb{C}[T^{\mathrm{tr}}]$, *which induces an injective homomorphism of the invariant algebra* $\mathbb{C}[G^{\mathrm{tr}}]^G$ *into* $\mathbb{C}[T^{\mathrm{tr}}]^{\mathcal{W}}$.

Proof. The restriction map $r\colon \widehat{\mathbb{C}[G]} \to \widehat{\mathbb{C}[T]}$ is given by

$$r(f)_\lambda = f_{\lambda\lambda}(1)e_\lambda, \quad f \in \widehat{\mathbb{C}[G]}, \quad \lambda \in X \cap P.$$

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If $f \in \widehat{\mathbb{C}[G]}$ is convergent, then for $t \in T^{\mathrm{tr}}$ we find

$$\sum_{\lambda} |r(f)_{\lambda}(t)| = \sum_{\lambda} |f_{\lambda\lambda}(t)| \leqslant \sum_{\lambda\mu} |f_{\lambda\mu}(t)| < \infty.$$

Therefore $r(f) \in \widehat{\mathbb{C}[T]}$ is also convergent.

The restriction map $r\colon \widehat{\mathbb{C}[G]} \to \widehat{\mathbb{C}[T]}$ induces the restriction map of functions $\mathbb{C}[G^{\mathrm{tr}}] \to \mathbb{C}[T^{\mathrm{tr}}]$, because due to the orthogonality of different weight spaces, we have $r(f)(t) = \sum_{\lambda} f_{\lambda\lambda}(t) = \sum_{\lambda\mu} f_{\lambda\mu}(t) = f(t)$ for all $t \in T^{\mathrm{tr}}$. The remaining statements follow easily from Corollary 3.4, using $r(\mathrm{Tr}_{\Lambda}) = \chi_{\Lambda}$, and Theorem 4.2. $\square$

Remark. It remains open if the restriction map is surjective. If not, a full analogue of the Chevalley restriction theorem can be obtained by replacing $\mathbb{C}[T^{\mathrm{tr}}]$ by the image $r(\mathbb{C}[G^{\mathrm{tr}}])$.

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