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MULTIPLE DISCRETE SEMIGROUPS OF OPERATIONS

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For each $t \in \{0,1,2,\dots\}$, let T_t be a set of operators in a Banach space. T_t is called a multiple semigroup of operators with respect to some operation $\otimes$ between sets of operators, if T_t satisfies the semigroup property $T_{t+s} = T_t \otimes T_s$. Two operations, $(\circ)$ and $(\cdot)$, between sets of operators are defined and properties of T_t are studied. Applications to the theory of Controlled Markov Processes are considered.

1. INTRODUCTION

Let $X(t)$ be a Markov process with values in a Banach space. Then there exists a contractive semigroup of operators T_t which defines, and is defined by, the transition probability of $X(t)$ (see, for example, Dynkin, [**1**]).

Now let, $(X(t), a(t))$ be a Controlled Markov Process (CMP), where $X(t)$ is a trajectory and $a(t)$ a control, which depends in general on the state of the process $X(t)$.

It is not in general clear how to define control $a(t)$ as we need to know $X(t)$ on the interval $[0,t)$ to define a at the moment t; on the other hand $a(t)$ influences the distribution of the state of the process at the moment t.

The control a can be introduced as a parameter in the transition probability function. This approach yields a piece-wise constant control. While this is perfectly adequate for CMPs in discrete time, in continuous time it necessitates taking limits. More on the subject can be found in Gikhman and Skorohod, [**2**].

Taking limits may be avoided if the controlling function $a(t,\ x_{[0,t]})$, which satisfies some smoothness conditions, is included in the coefficients of a stochastic equation, the sort most studied being equations of the diffusion type (see Fleming and Rishel, [**3**]).

However, this approach is bound to particular types of (CMP) and is not suitable for deriving a general definition.

Bensoussan (see [4]) considers a CMP as a collection of semigroups of operators and a probabilistic interpretation is sought under various assumptions on the semigroups. Non-linear semigroups are studied for a certain class of optimisation problems.

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Our approach has more similarities with that of Kloeden [5], whereby we consider a set of distributions of the state of a process for each moment of time. Thus we introduce an object which we shall call *a multiple semigroup of operators.*

The idea of describing a CMP using a multiple semigroup of operators goes back to 1976 (Frishling and Mirzashvili, [6] and Chitashvili *et al*, [7]), where it was tested on a simple case with discrete time and discrete phase space.

We will refer to the discrete phase space throughout the paper as all results are transparent for this case. The exposition will be in a way in reverse order, in the sense that abstract concepts of multiple semigroups of operators will be introduced and studied first, and then applied to the theory of Controlled Markov Processes.

The main purpose of the paper is to suggest a new approach to the theory of Controlled Markov Processes and to illustrate the validity of the approach.

2. Rectangles and quasirectangles of operators

We shall denote sets of operators by capital Latin letters and elements of these set by corresponding small Greek letters. The result of the action of an operator μ on a function f we denote by μf or $\mu(x, f)$, while $\mu(x, \cdot)$ will stand for the section of μ at the point x.

Let:

(1) X be a Banach space with Borel σ-algebra $\mathcal{B}$,
(2) $\mathcal{B}(X)$ be the set of all real bounded $\mathcal{B}$-measurable functions,
(3) the norm of an operator μ in X be defined in the usual way as $\|\mu\| = \sup\limits_{|f|\leqslant 1} |\mu f|$.

Definition 1: We call a set of operators M a *rectangle* if for any sequence $\{\mu^i\} \in M$ and any partition $\{X^i\}$ of X an operator μ such that

$$\mu(x, \cdot) = \mu^i(x, \cdot) \text{ for } x \in X^i$$

is an element of M.

Definition 2: We call a set of operators M a *complete rectangle* if for any parametrised set of operators $M' = \{\mu^x \in M : x \in X\}$ such that $\mu^x(x, f) \in \mathcal{B}(X)$ for any $f \in \mathcal{B}(X)$, the operator μ such that for each x

$$\mu(x, \cdot) = \mu^x(x, \cdot) \text{ and } \mu(x, f) \in \mathcal{B}(X)$$

belongs to M.

NOTE. In otherwords any section of μ belongs to M'.

Obviously, if a set of operators M is a complete rectangle it is a rectangle as well. In fact, it can be easily seen that if X is a discrete space the two concepts coincide.

Further, we shall need to consider rectangles or complete rectangles which contain a given set. In particular, we shall be interested in minimal such sets. The existence of such sets is ensured by the following lemma.

LEMMA 1. *If for each γ, M_γ, is a rectangle (complete rectangle), then $\cap M_\gamma$ is a rectangle (complete rectangle).*

PROOF: We shall prove the lemma only for complete rectangles, as the proof for rectangles is identical.

Let a set of operators μ^x be such that $\mu^x \in \cap M_\gamma$ for each $x \in X$ and let an operator μ be such that

$$\mu(x,\cdot) = \mu^x(x,\cdot) \text{ and } \mu(x,f) \in \mathcal{B}(X) \text{ for any } f \in \mathcal{B}(X).$$

As each μ^x belongs to M_γ and the latter is a complete rectangle, $\mu \in M_\gamma$, for each γ.

Thus $\mu \in \cap M_\gamma$. The Lemma is proved. □

It is not difficult to see that the set of all operators in X is a complete rectangle (and, hence, a rectangle). So the family of rectangles (complete rectangles), which contain a given set of operators is not empty.

Hence, if $\{M_\gamma\}$ is the family of al rectangles (complete rectangles), which contain a set M, then $\cap M_\gamma$ is the minimal rectangle (complete rectangle) containing M. Thus the following definition makes sense.

DEFINITION 3: The minimal rectangle $r(M)$ (complete rectangle $r^c(M)$) which contains the set M is called the *rectangular (complete rectangular)* hull of M.

NOTE 1. Trivially $r(M) \subseteq r^c(M)$.

NOTE 2. Let an operator μ be "glued" from operators from M, that is for each $x \in X$ there exists $\mu^x \in M$, such that $\mu(x,f) = \mu^x(x,f)$ for all $f \in \mathcal{B}(X)$ and $\mu(x,f) \in \mathcal{B}(X)$ for any f. Then $r^c(M)$ consists of all such "glued" operators.

DEFINITION 4: A set of operators M is called a *quasirectangle (closed quasirectangle)* if for any $f \in \mathcal{B}(X)$ and $\varepsilon \geqslant 0 (\varepsilon = 0)$ there exists $\mu^* \in M$, such that

$$\mu^*(x,f) \geqslant \sup_{\mu \in M} \mu(x,f) - \varepsilon.$$

NOTE. If M is a quasirectangle, then $\sup_{\mu \in M} \mu f \in \mathcal{B}(X)$. This is not generally true for an arbitrary M.

Generally, if a set is a rectangle, it does not follow that it is a quasirectangle and vice versa. However, if X is a discrete space it is easy to see that any rectangle is a quasirectangle as well.

For general spaces there are problems of measurability of $\sup \mu f$ (see the note above) and even if the measurability is assumed the problem of existence of μ^* such that

$$\sup_{\mu \in M} \mu f - \varepsilon \leqslant \mu^* f \leqslant \sup_{\mu \in M} \mu f$$

still remains.

These problems are related to some problems in the theory of analytical functions (see Dynkin and Yushkevich, [8] for further details), but as they lie outside the scope of this paper we shall not elaborate on them:

To conclude this section we shall point out one simple fact. If M is a quasirectangle, then $r(M)$ and $r^c(M)$ are obviously quasirectangles and

$$\sup_{\mu \in M} \mu f = \sup_{\mu \in r(M)} \mu f = \sup_{\mu \in r^c(M)} \mu f.$$

Hence the extension of M to a rectangle, or even a complete rectangle, does not increase sup.

3. Operations on the sets of operators

We need to introduce two operations between sets of operators. They will be introduced formally in this section, their probabilistic interpretation will be discused later.

DEFINITION 5: We shall call a set of operators K the *composition* of sets of operators M and N and write $K = M \cdot N$ if

$$K = \{\kappa : \kappa = \mu \cdot \nu \in M, \nu \in N\},$$

where $\mu \cdot \nu$ is the composition of the operators μ and ν.

DEFINITION 6: We shall call a set of operators L the *strong composition* of sets of operators M and N and write $L = M \circ N$ if

$$L = \{\lambda : \lambda f = \mu(x, \nu^x f)\ \mu \in M, \nu^x \in N\}.$$

While composition is a simple operation, strong composition can better be illustrated on example of finite dimensional spaces. If X is an Euclidean space then M and

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N are sets of matrices. To build the strong composition of M and N each row of a matrix $\mu \in M$ is multiplied by a matrix from N and different rows may be multiplied by different matrices from N.

It turns out that the property of a set of being a quasirectangle (rectangle) is invariant with respect to the operation of composition (strong composition). Lemmae 2 and 3 below state this precisely.

Lemma 2. *If M and N are quasirectangles (closed quasirectangles) of positive operators and N is a bounded set, then $M \cdot N$ is also a quasirectangle (closed quasirectangle).*

Proof: Let $\sup_{\nu \in N} \|\nu\| \leqslant c$, and μ^* and ν^* be such that for some $\varepsilon \geqslant 0$ and $f \in \mathcal{B}(X)$

$$\mu^* f \geqslant \sup_{\mu \in M} \mu f - \varepsilon$$

and

$$\nu^* \sup_{\mu \in M} \mu f \geqslant \sup_{\nu \in N} \nu \left(\sup_{\mu \in M} \mu f \right) - \varepsilon.$$

Then

$$\nu^* \mu^* f \geqslant \nu^* \left(\sup_{\mu \in M} \mu f \right) - c\varepsilon \geqslant \sup_{\nu \in N} \nu \left(\sup_{\mu \in M} \mu f \right) - \varepsilon - c\varepsilon.$$

For closed quasirectangles the proof is similar. □

Lemma 3. *If M is a rectangle, then $M \circ N$ is also a rectangle.*

Proof: Let $\{X_i\}$ be a partition of X and $\{\lambda_i\} \in L = M \cdot N$. Hence, there exist $\{\mu_i\}$ and $\{\nu_i\}$ such that for each i, $\lambda_i(x, \cdot) = \mu_i(x, \nu_i^x(\cdot))$. Put $\mu(x, \cdot) = \mu_i(x, \cdot)$ for $x \in X_i$. Then $\lambda(x, \cdot) = \mu(x, \nu^x(\cdot))$ where $\nu^x(\cdot) = \nu_i^x(\cdot)$ for $x \in X_i$ coincides with $\lambda_i(x, \cdot)$ for $x \in X_i$. □

Note. This lemma is not, in general, true for complete rectangles, because if $\{\lambda^y \cdot y \in L\} \subset L$ and $\lambda^y(x, \cdot) = \mu^y(x, \nu_y^x(\cdot))$, as in the lemma, and $\mu(x, \cdot) = \mu^x(x, \cdot)$; that is $\mu(x, \cdot)$ is 'glued' from $\mu^x(x, \cdot)$, it may not belong to M, as $\mu^x(x, f)$ may not be measurable for some $f \in \mathcal{B}(X)$.

The following three lemmas describe other properties of operations of the composition and strong composition.

Lemma 4. *For sets of operators M and N*

$$M \cdot N \subseteq r(M \cdot N) \subseteq r(M \circ N) = r(M) \circ N \subseteq r^c(M) \circ N \subseteq r^c(M \cdot N).$$

Proof: It is sufficient to prove the third and last relations as the others are trivial.

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If $\lambda \in r(M \cdot N)$, there exist a partition X_i and a sequence $\{\lambda_i\} \subset M \cdot N$ such that $\lambda(x, \cdot) = \lambda_i(x, \cdot)$ for $x \in X_i$. Hence, $\lambda_i(x, f) = \mu_i(x, \nu_i^x f)$ for $x \in X_i$ and $\lambda(x, f) = \mu(x, \nu^x f)$, where $\mu(x, \cdot) = \mu_i(x, \cdot)$ and $\nu^x = \nu_i^x$ for $x \in X_i$. This implies that $\lambda \in r(M) \circ N$. The converse is proved similarly.

Say now, $\lambda \in r^c(M) \circ N$. Then there exists $\mu \in r^c(M)$, such that for any $x \in X$ there exists $\mu^x \in M$ with $\mu(x, \cdot) = \mu^x(x, \cdot)$ (see Note 2 to Definition 4) and $\lambda(x, \cdot) = \mu(x, \nu^x(\cdot))$. So λ coincides with some operator from $M \cdot N$ for each $x \in X$.

As $\lambda(x, f) \in \mathcal{B}(X)$ for any $f \in \mathcal{B}(X)$ by the definition of the operation of strong composition, $\lambda \in r^c(M \cdot N)$. ∎

NOTE. The converse of the last inclusion is true for special spaces. It is true, for instance, if X is a discrete space.

LEMMA 5. *Composition is an associative operation, that is,*

$$(M \cdot N) \cdot K = M \cdot (N \cdot K).$$

PROOF: If $\lambda \in (M \cdot N) \cdot K$, then $\lambda = (\mu \cdot \nu) \cdot \kappa = \mu \cdot (\nu \cdot \kappa) \in M \cdot (N \cdot K)$. The inverse is proved similarly. ∎

LEMMA 6.

$$(M \circ N) \circ K \subseteq M \circ (N \cdot K) \subseteq r^c(M \cdot N \cdot K).$$

PROOF: Let $\lambda \in (M \circ N) \circ N$, so that $\lambda(x, \cdot) = \delta(x, \kappa^x(\cdot))$, where $\delta \in M \circ N$, and $\delta(x, \cdot) = \mu(x, \nu^x(\cdot))$. Then

$$\lambda(x, \cdot) = \mu(x, \nu^x \kappa^x(\cdot)) \in M \circ (N \cdot K).$$

The second part is a consequence of Lemma 4. ∎

The following two lemmas deal with the convex hull operation on (quasi) rectangles. The convex hull of a set of operators M is denoted by $\operatorname{co}(M)$.

LEMMA 7. $r(\operatorname{co} M) = \operatorname{co}(r(M))$.

PROOF: 1. Let M be a convex set of operators and $\mu^1, \mu^2 \in r(M)$. Then there exist sequences $\{\mu_i^1\} \subseteq M$ and $\{\mu_j^2\} \subseteq M$ and partitions of X, $\{X_i^1\}$ and $\{X_j^2\}$, such that

$$\mu^1(x, \cdot) = \mu_i^1(x, \cdot) \text{ for } x \in X_i^1,$$
$$\mu^2(x, \cdot) = \mu_j^2(x, \cdot) \text{ for } x \in X_j^2.$$

Let us define $\mu = \alpha\mu^1 + (1 - \alpha)\mu^2$ for some $0 \leqslant \alpha \leqslant 1$. Now introducing the partition $X_{i,j} = \{X_i^1 \cap X_j^2\}$, we get that

$$\mu(x, \cdot) = \alpha\mu_i^1(x, \cdot) + (1 - \alpha)\mu_j^2(x, \cdot) \text{ for } x \in X_i^1 \cap X_j^2.$$

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So the rectangular hull of a convex set is convex.

As $r(\mathrm{co}(M)) \supseteq r(M)$ for any M, $r(\mathrm{co}(M)) \supseteq \mathrm{co}(r(M))$.

2. Let $\mu', \mu'' \in \mathrm{co}(M)$, where M is a rectangle, so μ' and μ'' can be represented as $\mu' = \sum_1^I \alpha_i \mu_i'$ and $\mu'' = \sum_1^J \beta_j \mu_j''$ respectively, where $\alpha_i, \beta_j \geqslant 0$, $\sum \alpha_i = \sum \beta_j = 1$, $\mu_i', \mu_j'' \in M$.

To make the proof more transparent let us suppose that $I = J = 2$, $\alpha_1 \geqslant \beta_1$ and the partition is $\{X_1, X_2\}$.

Now defining

$$\alpha_1' = \beta_1, \alpha_2' = \alpha_1 - \beta_1, \alpha_3' = 1 - \alpha_1$$

and

$$\mu_1(x, \cdot) = \begin{cases} \mu_1'(x, \cdot) & \text{for } x \in X_1 \\ \mu_1''(x, \cdot) & \text{for } x \in X_2, \end{cases}$$

$$\mu_2(x, \cdot) = \begin{cases} \mu_1'(x, \cdot) & \text{for } x \in X_1 \\ \mu_2''(x, \cdot) & \text{for } x \in X_2, \end{cases}$$

$$\mu_3(x, \cdot) = \begin{cases} \mu_2'(x, \cdot) & \text{for } x \in X_1 \\ \mu_2''(x, \cdot) & \text{for } x \in X_2, \end{cases}$$

we get that the operator $\mu = \sum_{i=1}^{3} \alpha_i' \mu_i$ coincides with μ' on X_1 and with μ'' on X_2, so $\mathrm{co}(M)$ is a rectangle.

As for any M, $\mathrm{co}(r(M)) \supseteq \mathrm{co}(M)$, we have $\mathrm{co}(r(M)) \supseteq r(\mathrm{co}(M))$. ∎

Lemma 8. $r^c(\mathrm{co}(M)) \supseteq co(r^c(M))$,

Proof: The proof is the same as that of the first part of Lemma 7. ∎

4. Applications to controlled Markov processes

The definition of a Controlled Markov Process we are about to introduce is one of many possible variants. It is not new and given only for completeness.

Let the following objects be given:

(a) $(\Omega, \mathcal{F})$ - a probability space;

(b) $(X, \mathcal{B})$ and $(A, \mathcal{A})$ - measurable spaces of states of a Controlled Markov Process and its controls respectively.

(c) $\mathcal{P}_x = \{P_x^\pi : \pi \in \Pi\}$ - a family of measures on $(\Omega, \mathcal{P})$. Here Π is a parameter space the nature of which will be specified later.

We shall assume that the spaces X and A are complete, metric, separable spaces; their σ-algebras $\mathcal{B}$ and $\mathcal{A}$ are Borel σ-algebras; for all $x \in X$ the one element set

$\{x\} \in \mathcal{B}$. These conditions ensure the existence of regular variants of all conditional probabilities which appear later (see Loeve, [9]).

DEFINITION 6: We call a process $\xi_t = (x_t, a_t)$ a *homogeneous, fully observable CMP, which proceeds from* x if the following hold:

(a) the stochastic process ξ_t is defined on $(\Omega, \mathcal{F})$ and takes values in $(X \times A, \mathcal{B} \times \mathcal{A})$;

(b) $P_x^\pi(x_{t+1} \in B/x_0, a_0; \dots; x_t, a_t) = P_x^\pi(x_{t+1} \in B/x_t, a_t) - P_x^\pi$ a.s. for any $P_x^\pi \in \mathcal{P}_x$, $x_0 = x$, $B \in \mathcal{B}$, and the last probability does nt depend on t, x, π;

(c) $P_x^\pi(x_0 \in B) = I_B(x)$ for any $P_x^\pi \in \mathcal{P}_x$;

(d) All measures satisfying (b) and (c) are elements of $\mathcal{P}_x$.

The second condition expresses the Markovian property of the CMP, the last condition expresses full observability.

Because of the conditions on the spaces X and A, there exists a regular variant of the conditional probability $P_x^\pi(\cdot/x_t, a_t)$, for each $\omega \in \Omega$ there exists a function $P'(\omega, \{x_{t+1} \in B\})$ which is (x_t, a_t)-measurable. This measure induces a measure on $(X, \mathcal{B})$ in the obvious way. The function $P(\omega, B) = P'(\omega, \{x_{t+1} \in B\})$ is a.s. equal to $P_x^\pi(x_{t+1} \in B/x_t, a_t)$ with respect to the measure P_x^π reduced to the σ-algebra generated by (x_t, a_t), (see Dynkin, [1]). Thus, condition (b) implies the existence of the transition probability function for CMP. We shall denote this function $p_x^a(\cdot)$.

Each measure $P_x^\pi \in \mathcal{P}_x$ generates a sequence of conditional probabilities $\pi = \{\pi_0, \pi_1, \dots, \pi_t, \dots\}$, where

$$\pi_t(A) = P_x^\pi(a_t \in A/x_0, a_0; \dots; x_{t-1}, a_{t-1}; x_t)$$

and the above probability can be assumed to be a regular variant because of the conditions imposed on P_x^π.

Such a sequence of functions is called a *strategy*. A strategy is called *Markovian* if it depends only on the last state, that is if $\pi_t = \pi_t(A; x_t)$ for each t; a strategy is called *stationary* if it is Markovian and does not depend on t. The sets of all, all Markovian and all stationary strategies will be denoted by Π, Π^m and Π^s respectively.

Conversely, if π is a strategy, together with the transition function it generates a measure in $(X \times A, \mathcal{B} \times \mathcal{A})$.

For each a, $P_x^a(\cdot)$ generates a linear operator τ in $(X, \mathcal{B})$:

$$\tau f = \int f(x) P_x^a(x \in dx).$$

Let $\widetilde{T}_t$ and T_t be sets of operators generated by $\{P^\pi, \pi \in \Pi\}$ and $\{P^\pi, p \in \Pi^m\}$ respectively.

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Obviously Π is a convex set, therefore $\widetilde{T}_t$ is convex as well. Also, it can be easily checked that $\widetilde{T}_t$ is a rectangle.

The main result of this section is contained in the following:

THEOREM.

$$\widetilde{T}_{t+1} = \widetilde{T}_t \circ T_1,$$
$$T_{t+1} = \widetilde{T}_t \cdot T_1.$$

PROOF: Let $\tau_{t+1} \in \widetilde{T}_{t+1}$. Then there exists P_x^π such that

$$\tau_{t+1} f = \int f(y) P_x^\pi(x_{t+1} \in dy) = \int\int f(y) P_x^\pi(x_{t+1} \in dy/x_t) P_x^\pi(dx_t).$$

Then,

$$P_x^\pi(x_{t+1} \in dy/x_t) = \int P_x^\pi(x_{t+1} \in dy/x_t,\, a_t) P_x^\pi(da_t/x_t)$$
$$= \int P_{x_t}^{a_t}(x_{t+1} \in dy) P_x^\pi(da_t/x_t) \in T_1$$

by convexity of T_1. Thus

$$\tau_{t+1} f = \int f(y) \iint P_{x_t}^{a_t}(x_{t+1} \in dy) P_x^\pi(da_t/x_t) P_x^\pi(dx_t)$$

and, for each x,

$$\tau_{t+1}(x,\, f) = \tau_t(x,\, \tau_1^x f) \text{ and } \tau_{t+1} f \in \mathcal{B}(X).$$

This implies the relation $\tau_{t+1} \in \widetilde{T}_t \circ T_1$.

Now let $\tau_t \in \widetilde{T}_t$ and $\tau_1^x \in T_1$, for all $x \in X$. Then there exist π and π_0 such that

$$\tau_t(x,\, \tau_1^x f) = \int f(y) P_{x_t}^a(x_{t+1} \in dy) P_x^\pi(dx_t) \pi_0^x(da_t; x_t).$$

Let us construct a new strategy π':

$$\pi_i' = \pi_i \text{ for } i \neq t,$$
$$\pi_t'(A; x,\, a_0; \ldots; x_t) = \pi_0^x(A; x_t).$$

Then $\tau_t(x,\, \tau_1^x(\,)) \in \widetilde{T}_{t+1}$ and the first assertion of the theorem is proved.

The second assertion can be proved similarly. ∎

COROLLARY.

$$\widetilde{T}_{t+1} \subseteq T_1 \circ T_t \subseteq \overline{r}(T_{t+1}).$$

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PROOF: This follows immediately from Lemma 4. ◻

The Theorem and the Corollary describe the structure of the distributions of the states of a (CMP), when all strategies are used. The distribution of a state at any moment t can be obtained by using strategies which depend only on the last and the first states of the process, or in other works, if the initial point is fixed then it is sufficient to consider only Markovian strategies.

Thus the Corollary is a generalisation and elucidation of Straukh's result on semi-markovian strategies (see Straukh, [**10**]).

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